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Lidskii's Theorem in the type II case

ABSTRACT

Lidskii's Theorem states that the trace of a trace class operator is the sum of its eigenvalues, counting multiplicity. We generalize this to operators $T \in L^1(M,\tau)$, where M is a W^*-algebra and τ a faithful, normal, semi-finite trace on M. There is a unique measure μ on $\sigma(T)\setminus\{0\}$ such that $\tau(\log|1 - zT|) = \int \log|1 - zw|\,d\mu(w)$ for all z. μ may be thought of as giving the multiplicity of elements in the spectrum of T. For all $0 < p < \infty$, $\int |w|^p d\mu(w) \leq \tau(|T|^p)$, and $\tau(T) = \int w\,d\mu(w)$. The proof uses representation theory for subharmonic functions.

0. INTRODUCTION

Throughout the paper M and τ will be as stated in the abstract. For $T \in M \cap L^1(M,\tau)$, let $u(z) = \tau(\log|1 - zT|)$ (the log of the Fuglede-Kadison determinant of $1 - zT$). The main elements of the proof of the results stated in the abstract are:

(i) Show u is subharmonic.

(ii) Show $\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})d\theta \leq \tau(\log_+ r|T|)$, $\forall r > 0$. The inequality (ii) is the main result needed for the validity of a representation theorem for u. (ii) also implies $\int \varphi(|w|)d\mu(w) \leq \tau(\varphi(|T|))$ for all increasing functions φ such that $\varphi(0) = 0$ and $\varphi(e^t)$ is convex. (i), (ii) and the main theorem, 3.13, are proved in §3. The results can be generalized, by using the usual convergence factors, to the case

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where $T \in M \cap L^p(M,\tau)$ for any $0 < p < \infty$, and it is actually this case which is considered in §3. Sections 1 and 2 contain preliminaries on type II operator theory and subharmonic functions. Section 4 gives some additional results, including that if $T_1, T_2 \in M$ and $T_1T_2, T_2T_1 \in L^p(M,\tau)$, then T_1T_2 and T_2T_1 have the same spectral multiplicity measures. An appendix deals with unbounded T.

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1. PRELIMINARIES ON s-NUMBERS AND THE FUGLEDE-KADISON DETERMINANT

The Fuglede-Kadison determinant was treated in (Fuglede and Kadison, 1952) for M a finite factor. It is known (see for example (Fack, 1982, 1983) and (Grothendieck, 1955)) that it can be generalized, but we have not found a reference that gives everything we need. We therefore give a fairly self-contained treatment, adequate for present purposes. Determinant theory for unbounded operators is treated in the appendix.

For $p \in (0,\infty)$ we denote by $L_{p,\infty}$ the space $M \cap L^p(M,\tau)$, with the topology defined by $\| \ \|_{p,\infty} = \| \ \|_p + \| \ \|$. We will speak of the $L_{p,\infty}$ topology on the coset $1 + L_{p,\infty}$ also. The closure of $L_{p,\infty}$ in the operator norm, $\| \ \|$, does not depend on p and will be denoted K_τ. If M is not a factor, K_τ may be smaller than the usual ideal of generalized compact operators.

The s-numbers of $T \in M$ were defined in (Fack, 1982). There is a non-increasing function $s_T : (0,\infty) \to [0,\infty)$ such that $\forall x > 0$, $|\{t : s_T(t) > x\}| = \tau(E_{(x,\infty)}(|T|))$, where $| \ |$ denotes Lebesgue measure and E_F denotes the spectral projection corresponding to the Borel set F. s_T vanishes at ∞ if and only if $T \in K_\tau$, and in this case

$\tau(E_F(|T|)) = |\{t : s_T(t) \in F\}|$, $\forall F \subset (0,\infty)$. Thus $\int f(s_T(t))dt = \tau(f(|T|))$ for non-negative Borel f with $f(0) = 0$. The classical non-increasing rearrangement of a measureable function on a σ-finite measure space is a special case. On a few occasions below we will claim certain points follow by spectral dominance arguments. The basic technique being referred to goes back to von Neumann and is well known. Differing formal definitions of spectral dominance were given, for example, in (Akemann, Anderson, and Pedersen, 1982; page 170) and (Brown and Kosaki; Definition 2, see also Lemmas 3 and 4 and Remark 5). The definition in (Brown and Kosaki) is directly related to s-numbers.

For $A \in 1 + L_{1,\infty}$, the determinant of A is denoted $\Delta(A)$ and defined by

(1) $$\log \Delta(A) = \tau(\log |A|).$$

(1) requires interpretation. If A has a non-trivial null-space (considering M as an algebra of operators on some Hilbert space H), we set $\log \Delta(A) = -\infty$, $\Delta(A) = 0$. Otherwise $\log|A|$ is an unbounded τ-measureable operator affiliated with M and $\log_+|A| \in L_{1,\infty}$. If $\log|A| \in L^1(M,\tau)$, the meaning of (1) is clear; if not, set $\log \Delta(A) = \tau(\log|A|) = -\infty$ (cf. (Brown and Kosaki; Definition 7)). For $\tau(1) < \infty$, $\log \Delta(A) = \int_0^{\tau(1)} \log s_A(t)dt$. Note that A has index 0, since $A - 1 \in K_\tau$. Hence $A = U|A|$ for some unitary $U \in M$, and $\Delta(A) = \Delta(A^*)$.

<u>1.1 Lemma.</u> Let f be a differentiable function into $1 + L_{1,\infty}$ and Log be a single-valued branch of log defined in a neighborhood of $\sigma(f(t_0))$ such that $\mathrm{Log}(1) = 0$. Then in some neighborhood of t_0, $\mathrm{Log}(f(t))$ is differentiable into $L_{1,\infty}$ and $\frac{d}{dt}\tau(\mathrm{Log}(f(t))) = \tau\left(\frac{d}{dt}\mathrm{Log}(f(t))\right) = \tau(f(t)^{-1}f'(t))$.

1.1 is well known and is proved by standard manipulation of resolvents. Note that the condition on $\sigma(f(t_0))$ is satisfied if $f(t_0)$ is invertible and positive.

<u>1.2 Proposition</u>. $\log \Delta(e^S) = \mathrm{Re}(\tau(S))$ for $S \in L_{1,\infty}$.

<u>Proof</u>. In view of the obvious fact, $\Delta(A)^2 = \Delta(|A|^2)$, it suffices to calculate $\frac{d}{dt}\, \tau(\log(e^{tS^*} e^{tS})) = 2\mathrm{Re}(\tau(S))$.

<u>1.3 Lemma</u>. $\Delta(AB) = \Delta(A)\Delta(B)$ for invertible $A, B \in 1 + L_{1,\infty}$.

<u>Proof</u>. Since $A = U|A|$, U unitary, and $\Delta(\cdot) = \Delta(|\cdot|)$, we may replace A by $|A|$. Similarly, using $B = |B^*|V$ and $\Delta(\cdot) = \Delta(\cdot^*)$, we replace B by $|B^*|$. Thus we now assume $A, B \geq 0$. Now since $\Delta(AB)^2 = \Delta(|AB|^2) = \Delta(BA^2B)$, it is sufficient to show

(2) $\qquad \tau(\log(BAB)) = 2\tau(\log B) + \tau(\log A), \quad \forall A, B$ as above.

To prove (2), write $A = e^S$ for $S = S^* \in L_{1,\infty}$ and calculate the derivative of both sides of (2) with A replaced by e^{tS}.

<u>1.4 Proposition</u>. $\log \Delta$ is real analytic in the $L_{1,\infty}$ topology (in particular continuous) when restricted to invertible elements of $1 + L_{1,\infty}$, Also if $A(\cdot)$ is an $L_{1,\infty}$-holomorphic function of a complex variable with invertible values in $1 + L_{1,\infty}$, then $\log \Delta(A(\cdot))$ is harmonic.

<u>Proof</u>. In view of 1.3 it is sufficient to consider both parts only in the neighborhood of 1. But for $\|A - 1\| < 1$, it follows from 1.2 that $\log \Delta(A) = \mathrm{Re}\, \Sigma_1^\infty \frac{(-1)^{n-1}}{n} \tau((A-1)^n)$.

<u>1.5 Remark</u>. $|A| \geq |B| \Rightarrow \Delta(A) \geq \Delta(B)$. This follows from the operator monotonicity of log, but it also follows very easily just from the

monotonicity of log, since $\log|B|$ is spectrally dominated by $\log|A|$ in the sense of (Akemann, Anderson, and Pedersen, 1982).

<u>1.6 Proposition.</u> $\log \Delta$ is upper semicontinuous in the $L_{1,\infty}$-topology.

<u>Proof.</u> For $\varepsilon > 0$ and $T = A - 1$, set $f_\varepsilon(A) = \frac{1}{2}\log \Delta(|A|^2 + \varepsilon|T|^2)$. $|A|^2 + \varepsilon|T|^2$ is invertible, since it is $|(1-\delta)^{1/2} + (1+\varepsilon)^{1/2}T|^2 + \delta$, where $1 - \delta = (1+\varepsilon)^{-1}$. Hence f_ε is continuous. Further $f_\varepsilon \geq \log \Delta$ and $f_\varepsilon(A)$ is non-decreasing in ε by 1.5. Thus to complete the proof we need only observe that $\log \Delta(A) = \lim\limits_{\varepsilon \to 0+} f_\varepsilon(A)$. An elementary way to see this is as follows: Choose any invertible $B \geq |A|^2$ such that $B - 1 \in L_{1,\infty}$. Then since $\frac{1}{2}\log \Delta(B + \varepsilon|T|^2) \geq f_\varepsilon(A)$ and $\lim\limits_{\varepsilon\to 0+} \frac{1}{2}\log \Delta(B + \varepsilon|T|^2) = \log \Delta(B^{1/2})$ (by 1.4), $\lim\limits_{\varepsilon \to 0+} f_\varepsilon(A) \leq \log \Delta(B^{1/2})$. But it is easy to see that $\inf\{\log \Delta(B^{1/2}) : B \text{ as above}\} = \log \Delta(A)$. We need only allow B to run through suitable functions of $|A|$.

<u>1.7 Proposition.</u> $\Delta(AB) = \Delta(A)\Delta(B)$, $\forall A,B \in 1 + L_{1,\infty}$.

<u>Proof.</u> We reduce to the case $A,B \geq 0$ as in the proof of 1.3. We then complete the proof by reducing to the case where A and B are invertible. Let $f_n(x) = \max\left(x, \frac{1}{n}\right)$. Since $\Delta(AB)^2 = \Delta(|AB|^2) = \Delta(BA^2B)$, we see that $\Delta(f_n(A)B) \geq \Delta(AB)$. Thus 1.6 implies that $\Delta(f_n(A)B) \to \Delta(AB)$. Similarly, $\Delta(f_n(A)) \to \Delta(A)$. Thus we are reduced to the case A invertible. Since also $\Delta(AB)^2 = \Delta(|(AB)^*|^2) = \Delta(AB^2A)$, a similar argument gives a reduction to the case B invertible.

<u>1.8 Proposition.</u> If P is a projection in M such that $AP = PAP$, then $\Delta(A) = \Delta_{PMP}(PAP) \cdot \Delta_{(1-P)M(1-P)}((1-P)A(1-P))$.

<u>Remark.</u> In informal notation this says

$$\Delta\left(\begin{pmatrix} A_1 & T_{12} \\ 0 & A_2 \end{pmatrix}\right) = \Delta(A_1)\Delta(A_2).$$

Proof. Since

$$\begin{pmatrix} A_1 & T_{12} \\ 0 & A_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} 1 & T_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix},$$

in view of 1.7 it is sufficient to observe

$$\Delta\begin{pmatrix} 1 & T_{12} \\ 0 & 1 \end{pmatrix} = \Delta\left(\exp\left(\begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}\right)\right) = \exp\left(\mathrm{Re}\ \tau\left(\begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}\right)\right) = 1.$$

1.9 Proposition. If $A \in 1 + L_{1,\infty}$ and B is invertible in M, then $\Delta(BAB^{-1}) = \Delta(A)$.

Proof. The result follows easily if A is a finite product of exponentials as in 1.2. If A is invertible, A is the product of two such exponentials by the polar decomposition. For general A choose invertible $A_n \to A$ in the $L_{1,\infty}$ topology such that $\Delta(A_n) \to \Delta(A)$ (for example let $A_n = Uf_n(|A|)$, f_n as in the proof of 1.7). Then since $BA_nB^{-1} \to BAB^{-1}$ in the $L_{1,\infty}$-topology, 1.6 implies $\Delta(BAB^{-1}) \geq \Delta(A)$. By symmetry, also $\Delta(A) \geq \Delta(BAB^{-1})$.

The next result concerns $\Lambda_t(T) = \int_0^t \log s_T(x)dx$. We note that if E is a projection in M and $\tau(E) = t < \infty$, then $\log \Delta_{EME}(E|T|E) \leq \Lambda_t(T)$. This can be seen from the fact that $E|T|E$ is spectrally dominated by $|T|$. If M is non-atomic, $\Lambda_t(T) = \sup\{\log \Delta_{EME}(E|T|E) : \tau(E) = t\}$; but we do not need this.

1.10 Proposition. $\Lambda_t(T_1T_2) \leq \Lambda_t(T_1) + \Lambda_t(T_2)$, $\forall T_1, T_2 \in M$, $t \in (0,\infty)$.

Proof. We first show that if E is a projection with $\tau(E) = t < \infty$, then

$$(3) \qquad \log \Delta_{EME}(E|T_1T_2|E) \leq \Lambda_t(T_1) + \Lambda_t(T_2).$$

Write $|T_1T_2| = UT_1T_2$ for $\|U\| = 1$. If the left support projection of EUT_1 or the right support projection of T_2E is $\neq E$, the left side $= -\infty$. Otherwise, let E_1 and E_2 be the right support projection of EUT_1 and the left support projection of T_2E. We can find V_i such that $V_iV_i^* = E_i$ and $V_i^*V_i = E$. $\log \Delta_{EME}(E|T_1T_2|E) =$ $= \log \Delta_{EME}(EUT_1V_1V_1^*V_2V_2^*T_2E) = \log \Delta_{EME}(EUT_1V_1) + \log \Delta_{EME}(V_1^*V_2) +$ $+ \log \Delta_{EME}(V_2^*T_2E) \leq \Lambda_t(T_1) + 0 + \Lambda_t(T_2)$, since $|EUT_1V_1|$ is spectrally dominated by $|T_1|$, $\|V_1^*V_2\| \leq 1$, and $|V_2^*T_2E|$ is spectrally dominated by $|T_2|$.

Now for arbitrary $t \in (0,\infty)$, there is a smallest $x \geq 0$ such that $\tau(E_{(x,\infty)}(|T_1T_2|)) \leq t$. Let $E_1 = E_{(x,\infty)}(|T_1T_2|)$, $E_2 = E_{[x,\infty)}(|T_1T_2|)$, and $t_i = \tau(E_i)$. If t_1 or $t_2 = t$, we are done by (3). If $t_1 < t$ and $x = 0$, the left side of 1.10 $= -\infty$. Otherwise, if $t_1 < t < t_2 < \infty$ (which will always be so if $T_1T_2 \in K_\tau$), $s_{T_1T_2} = x$ on (t_1,t_2). Then 1.10 follows from (3) for E_1 or E_2 depending on whether $x \leq s_{T_1}(t)s_{T_2}(t)$ or $x \geq s_{T_1}(t)s_{T_2}(t)$. In the remaining cases, $s_{T_1T_2} = x$ on (t_1,∞). 1.10 still follows from (3) for E_1 if $x \leq s_{T_1}(t)s_{T_2}(t)$. If $x > s_{T_1}(t)s_{T_2}(t)$, then $\forall y \in (s_{T_1}(t)s_{T_2}(t), x)$, we can find $t_2' \in (t,\infty)$ and a projection E_2' such that $\tau(E_2') = t_2'$ and $E_1 \leq E_2' \leq E_{(y,\infty)}(|T_1T_2|)$. Then (3) for E_2' implies $\Lambda_{t_1}(|T_1T_2|) + \log y \cdot (t - t_1) \leq \Lambda_t(T_1) + \Lambda_t(T_2)$.

We now let $y \to x$.

<u>1.11 Proposition.</u> Let $s_1, s_2 : (0,\infty) \to [0,\infty)$ be non-increasing, and assume $\int_0^1 \log_+ s_i(x)dx < \infty$. Then the following are equivalent:

(i) $\int_0^t \log s_1(x)dx \le \int_0^t \log s_2(x)dx, \quad \forall t \in (0,\infty)$.

(ii) $\int_0^\infty \log_+(rs_1(x))dx \le \int_0^\infty \log_+(rs_2(x))dx, \quad \forall r > 0$.

(iii) $\int_0^t \varphi(s_1(x))dx \le \int_0^t \varphi(s_2(x))dx, \quad \forall t \in (0,\infty], \ \forall$ non-decreasing functions φ on $[0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(e^y)$ is convex.

<u>Proof.</u> The hypothesis implies $\int_0^t \log_+(rs_i(x))dx < \infty, \ \forall t \in (0,\infty), r > 0$.

(i) $\Rightarrow$ (ii): Note that $\int_0^\infty \log_+(rs_i(x))dx = \sup\{\int_0^t \log(rs_i(x))dx : t \in (0,\infty)\} = \sup\{t \log r + \int_0^t \log s_i(x)dx : t \in (0,\infty)\}$.

(ii) $\Rightarrow$ (i): First assume $s_2(t) > 0$. Then for $r = \frac{1}{s_2(t)}$,

$$\int_0^t \log(rs_1(x))dx \le \int_0^t \log_+(rs_1(x))dx \le \int_0^\infty \log_+(rs_1(x))dx$$
$$\le \int_0^\infty \log_+(rs_2(x))dx = \int_0^t \log(rs_2(x))dx,$$

which gives the result for t. Since $\int_0^t \log s_i(x)dx = \lim_{t' \to t-} \int_0^{t'} \log s_i(x)dx$, all that remains is to prove it impossible that $s_2(t_1) = 0$, $s_1(t_2) > 0$, and $t_1 < t_2$. But this would contradict (ii), since $\forall r > 1$,

$$\int_0^\infty \log_+(rs_1(x))dx \ge t_2 \log r + \int_0^{t_2} \log s_1(x)dx \quad \text{and}$$

$$\int_0^\infty \log_+(rs_2(x))dx \le t_1 \log r + \int_0^{t_1} \log_+ s_2(x)dx.$$

(ii) $\Rightarrow$ (iii): If $t < \infty$, we may change $s_1(x)$ and $s_2(x)$ to 0 for $x > t$, since (i) will remain true. Thus we are reduced to the case $t = \infty$. Now if φ is continuous at 0, there are $\varphi_n \uparrow \varphi$ such that each φ_n is a linear combination of functions $\log_+(r\cdot)$ with non-negative coefficients. We then apply the monotone convergence theorem. To cover the case where φ is discontinuous at 0, we need only consider $\varphi(s) = 1$ for $s > 0$ and $\varphi(0) = 0$. (iii) for this case was dealt with in the proof of (ii) $\Rightarrow$ (i).

(iii) $\Rightarrow$ (ii) is trivial.

<u>1.12 Corollary</u>. $\forall T \in M$, φ as in 1.11, $\tau(\varphi(|T^n|)) \leq \tau(\varphi(|T|^n))$.

We remark only that it is not necessary here to assume $T \in K_\tau$, although only that case will be used below.

<u>1.13 Remark</u>. We note for later use that $\tau(\log|1 + T|) \leq \tau(\log(1 + |T|))$. This can be proved by an easy spectral dominance argument or deduced from the main result of (Akemann, Anderson, and Pedersen, 1982).

2. <u>PRELIMINARIES ON SUBHARMONIC FUNCTIONS</u>

Let u be an upper semicontinuous function defined on an open set $D \subset \mathbb{C}$ and taking values in $[-\infty,\infty)$. u is called <u>subharmonic</u> if $\forall w \in D$, $\exists \varepsilon > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{i\theta})d\theta, \qquad \forall 0 < r < \varepsilon.$$

It is usually also assumed that u is not identically $-\infty$ on any connected component of D. In this case u is locally integrable (and integrable on every circle), and $\nabla^2 u$, computed in the sense of distributions, is a non-negative measure, finite on compact sets. Any subharmonic function can be approximated from above, in the sense of everywhere pointwise convergence, by C^∞ subharmonic functions; and of course for smooth functions subharmonicity just means that the Laplacian, calculated in the usual way, is non-negative. Two subharmonic functions which agree almost everywhere must agree everywhere.

The measure $\mu_0 = \frac{1}{2\pi}\nabla^2 u$ is called the <u>Riesz measure</u> of the subharmonic function u. If K is compact and

$$v_K(z) = \int_K \log|z - w|\,d\mu_0(w), \quad \text{then} \quad v_K$$

is subharmonic and $\nabla^2 v_K = 2\pi\mu_0|_K$. It follows that $u = v_K + h$, where h is harmonic on the interior of K. In this local representation theorem the kernel $\log|z - w|$ may be replaced by others such as $\log\left|1 - \frac{z}{w}\right|$ or $\log\left|(1 - \frac{z}{w})\exp\left(\frac{z}{w} + \frac{z^2}{w^2} + \cdots + \frac{1}{m-1}\,\frac{z^{m-1}}{w^{m-1}}\right)\right|$, in which case h changes. (For the two alternate kernels given one could assume $0 \notin K$ or $0 \notin \operatorname{supp}\mu_0$, for example.) The only bar to obtaining a global representation theorem, using any of the above kernels, is the convergence of the (global) integral for w away from z. In some global representation theorems one also wants $h = 0$, which requires additional work of course.

In the results stated formally below we have tried to avoid generality unnecessary for this paper. We do not know a precise reference for 2.2; but it is certainly close to results in the literature, and in any case the proof is standard.

2.1 below is a special case of a result from (Hayman and Kennedy, 1976). (Put $\rho = 0$ in (3.7.2), page 120.) For u smooth it is a simple application of Green's Theorem.

<u>2.1 Lemma</u>. Assume u is subharmonic on $\mathbb{C}$, u is harmonic in a

neighborhood of 0, and $u(0) = 0$. Let μ_0 be the Riesz measure of u. Then

$$\int \log_+ \frac{r}{|w|}\, d\mu_0(w) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})\,d\theta, \quad \forall r > 0.$$

<u>2.2 Proposition.</u> Let u, μ_0 be as in 2.1. Let $p \in (0,\infty)$ be such that $\int \frac{1}{|w|^p}\, d\mu_0(w) < \infty$, and let $k \geq p$ be an integer. Let $u_n(z) = \Sigma_{i=0}^{n-1} u(\rho^i z)$, where ρ is a primitive n'th root of 1. Assume that u vanishes to order at least k at 0 and $u_{n+}(z) = \max(u_n(z),0) = o(|z|^n)$ as $z \to \infty$, $\forall n \geq k$. Then

$$u(z) = \int \log \left| \left(1 - \frac{z}{w}\right) \exp\left(\frac{z}{w} + \cdots + \frac{1}{k-1} \frac{z^{k-1}}{w^{k-1}} \right) \right| d\mu_0(w), \quad \forall z.$$

<u>Proof.</u> The integrand is $O\left(\left|\frac{z}{w}\right|^k\right)$ for $\frac{z}{w}$ small and $O\left(\left|\frac{z}{w}\right|^{k-1+\varepsilon}\right)$ for $\frac{z}{w}$ large. The first estimate and the hypothesis on μ_0 imply nice convergence of the integral for w away from z. Also the integrand is $\leq C\left|\frac{z}{w}\right|^k$, $C > 0$, for all z, w. If $v(z)$ is the integral, then by the facts recalled above, $u = v + h$, where h is harmonic. Since u and v both vanish to order k at 0, h does also. Also $v_+(z) = o(|z|^k)$ as $z \to \infty$. (To see this write $v(z) = \int_{|w| \leq R} + \int_{|w| > R}$, where R is chosen so that $\int_{|w|>R} \frac{1}{|w|^k}\, d\mu_0(w)$ is small. For the first integral, use the $O\left(\left|\frac{z}{w}\right|^{k-1+\varepsilon}\right)$ estimate for $z \gg R$. For the second use the "$\leq C\left|\frac{z}{w}\right|^k$" estimate.) Since

$$\int_0^{2\pi} v(re^{i\theta})\,d\theta = \int_0^{2\pi} v_+(re^{i\theta})\,d\theta - \int_0^{2\pi} v_-(re^{i\theta}) \geq v(0) = 0,$$

we see that $\int_0^{2\pi} |v(re^{i\theta})|d\theta = o(r^k)$ as $r \to \infty$. Similarly, $\int_0^{2\pi} |u_n(re^{i\theta})|d\theta = o(r^n)$, $\forall n \geq k$; and the same holds for v_n (defined similarly to u_n), since $r^k \leq r^n$ for $r \geq 1$. Now any entire harmonic function $\tilde{h}$ has an expansion

$$\sum_{m=0}^{\infty} a_m |z|^m \cos m\theta + \sum_{m=1}^{\infty} b_m |z|^m \sin m\theta,$$

where for $m > 0$, $a_m = \frac{1}{\pi r^m} \int_0^{2\pi} \tilde{h}(re^{i\theta})\cos m\theta d\theta$ and $b_m = \frac{1}{\pi r^m} \int_0^{2\pi} \tilde{h}(re^{i\theta})\sin m\theta d\theta$. For m divisible by n, $a_m(h_n) = na_m(h)$ and $b_m(h_n) = nb_m(h)$. $u = v + h$ implies $u_n = v_n + h_n$ and hence $\int_0^{2\pi} |h_n(re^{i\theta})|d\theta = o(r^n)$, as $r \to \infty$, $\forall n \geq k$. This implies $a_n(h) = b_n(h) = 0$, $\forall n \geq k$. Since h vanishes to order k at 0, $h = 0$.

3. THE MAIN THEOREM

3.1 Lemma. Let $A_1,\dots,A_n,B_1,\dots,B_n \in B(H)$ be such that $\Sigma_1^n |A_i|^2$ is invertible. Then

$$\left(\sum_1^n B_i^* A_i\right)\left(\sum_1^n |A_i|^2\right)^{-1}\left(\sum_1^n A_k^* B_k\right) \leq \sum_1^n |B_i|^2.$$

Proof. Let $\bar{A}_i = A_i(\Sigma_1^n |A_i|^2)^{-1/2}$, and let $\bar{A}, B \in B(H, \overbrace{H \oplus \cdots \oplus H}^{n \text{ times}})$ be the operators whose components are $\bar{A}_1,\dots,\bar{A}_n$ and $B_1,\dots,B_n$. Then $\bar{A}^*\bar{A} = 1$ and hence $\bar{A}\,\bar{A}^* \leq 1$. Therefore

$$(B^*\bar{A})(\bar{A}^*B) = B^*(\bar{A}\,\bar{A}^*)B \leq B^*B, \quad \text{as desired.}$$

3.2 Lemma. Let $A_1,\dots,A_n$ be holomorphic functions of a complex variable, relative to the topology of $L_{1,\infty} = M \cap L^1(M,\tau)$, such that

$A_1 - 1, A_2, \ldots, A_n \in L_{1,\infty}$. If $\Sigma_1^n |A_i(z)|^2$ is invertible $\forall z$, then $\log \Delta(\Sigma_1^n |A_i(\cdot)|^2)$ is subharmonic.

<u>Proof.</u> Let $u(z) = \log \Delta(\Sigma_1^n |A_i(z)|^2) = \tau(\log \Sigma_1^n A_i(z)^* A_i(z))$. u is C^∞ by 1.4. We calculate

$$\frac{\partial u}{\partial z} = \tau\left[\left(\sum_1^n A_i(z)^* A_i(z)\right)^{-1}\left(\sum_1^n A_j(z)^* A_j'(z)\right)\right]$$

$$\frac{1}{4}\nabla^2 u = \frac{\partial^2 u}{\partial \bar{z}\partial z} = \tau\left[\left(\sum_1^n |A_i(z)|^2\right)^{-1}\left(\sum_1^n |A_j'(z)|^2\right)\right.$$

$$\left. - \left(\sum_1^n |A_i(z)|^2\right)^{-1}\left(\sum_1^n A_j'(z)^* A_j(z)\right)\left(\sum_1^n |A_k(z)|^2\right)^{-1}\left(\sum_1^n A_\ell(z)^* A_\ell'(z)\right)\right]$$

Since $\tau[(\ )^{-1}s] = \tau[(\ )^{-1/2}s(\ )^{-1/2}]$, in order to show $\nabla^2 u \geq 0$, it is sufficient to show

$$\sum_1^n |A_i'(z)|^2 \geq \left(\sum_1^n A_i'(z)^* A_i(z)\right)\left(\sum_1^n |A_j(z)|^2\right)^{-1}\left(\sum_1^n A_k(z)^* A_k'(z)\right).$$

This last follows from 3.1 with $B_i = A_i'(z)$.

<u>3.3 Theorem.</u> Let $A_1, \ldots, A_n$ be holomorphic functions of a complex variable in the $L_{1,\infty}$-topology, such that $A_1 - 1, A_2, \ldots, A_n \in L_{1,\infty}$. Then $\log \Delta(\Sigma_1^n |A_i(\cdot)|^2)$ is subharmonic.

<u>Proof.</u> Let $u(z) = \log \Delta(\Sigma_1^n |A_i(z)|^2)$. Let $T(z) = A_1(z) - 1$, and for $\varepsilon > 0$ let $u_\varepsilon(z) = \log \Delta(\varepsilon|T(z)|^2 + \Sigma_1^n |A_i(z)|^2)$. $u_\varepsilon(z) \downarrow u(z)$ as $\varepsilon \downarrow 0$, and $\varepsilon|T(z)|^2 + \Sigma_1^n |A_i(z)|^2$ is invertible $\forall z$ by 1.5 and 1.6. By 3.2 u_ε is subharmonic. Hence u is subharmonic. (The last implication is standard and uses the definition of subharmonic and the

monotone convergence theorem.)

3.4 Remark. Since $\log \Delta(A) = \frac{1}{2} \log \Delta(|A|^2)$, 1.1 and the case $n = 1$ of 3.3 can be combined to give that $\log \Delta$ is plurisubharmonic on its entire domain, relative to the $L_{1,\infty}$-topology. This means that it is upper semicontinuous and becomes subharmonic when composed with a holomorphic function.

We now fix $T \in L_{p,\infty} = M \cap L^p(M,\tau)$, $p \in (0,\infty)$, and an intgger $k \geq p$. Let $u(z) = \log \Delta(g_k(zT))$, where $g_k(w) = (1 - w)\exp(w + \cdots + \frac{1}{k-1} w^{k-1})$. Since $g_k - 1$ vanishes to order k at 0, 3.4 implies that u is subharmonic. Moreover, $g_k(zT)$ is invertible whenever $\frac{1}{z} \notin \sigma(T)$, and in this region u is harmonic by 1.4. In particular u is harmonic in a neighborhood of 0 and vanishes to order k at 0. Let μ_0 be the Riesz measure of u, and define μ by $d\mu(w) = d\mu_0(\frac{1}{w})$. Then μ is a non-negative measure on $\mathbb{C}\backslash\{0\}$, finite on compact sets (not containing 0), and supported on $\sigma(T)\backslash\{0\}$. If T is quasi-nilpotent, μ_0 and μ are 0.

3.5 Lemma. (i) $u_{n+}(z) = o(|z|^n)$ as $z \to \infty$, $\forall n \geq k$, where u_{n+} is as in 2.2.

(ii) $$\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta})d\theta \leq \tau(\log_+ r|T|), \quad \forall r > 0.$$

Proof. (i) Using 1.7, we see that $u_n(z) = \log \Delta(\Pi_{i=0}^{n-1} g_k(\rho^i zT)) = \log \Delta(1 - z^n T^n)$. Here we have used $\Sigma_{i=0}^{n-1} \rho^{ij} = 0$ for $0 < j < k \leq n$, where ρ is a primitive n'th root of 1. Since $T^n \in L_{1,\infty}$, we may look only at the case $n = k = 1$. Then by 1.13 $u(z) = \tau(\log|1 - zT|) \leq \tau(\log(1 + |z||T|)) = \int \log(1 + |z|x)d\lambda(x)$, where $\lambda(F) = \tau(E_F(|T|))$, for $E_F(|T|)$ the spectral projection corresponding to the Borel set F. Since

$T \in L^1(M,\tau)$, $\int x d\lambda(x) < \infty$, and $\lambda([\varepsilon,\infty)) < \infty$, $\forall \varepsilon > 0$. $\int \log(1 + rx) d\lambda(x) = o(r)$ as $r \to \infty$. To see this, write $\int = \int_0^{\varepsilon} + \int_{\varepsilon}^{\infty}$, where $\varepsilon > 0$ is chosen so that $\int_0^{\varepsilon} x d\lambda(x)$ is small. For the first integral, use $\log(1 + rx) \leq rx$. Estimate the second integral by $\log(2r)\lambda([\varepsilon,\infty)) + \int_{\varepsilon}^{\infty} \log x d\lambda(x)$, for $r \geq \frac{1}{\varepsilon}$.

(ii) $$\frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta = \frac{1}{2\pi n} \int_0^{2\pi} u_n(re^{i\theta}) d\theta$$

$$= \frac{1}{2\pi n} \int_0^{2\pi} \tau(\log|1 - r^n e^{in\theta} T^n|) d\theta$$

$$\leq \frac{1}{n} \tau(\log(1 + r^n|T^n|)) \leq \frac{1}{n} \tau(\log(1 + r^n|T|^n))$$

$$= \frac{1}{n} \int \log(1 + r^n x^n) d\lambda(x),$$

for $n \geq k$. Here we have used the arguments and notations of the proof of (i) and 1.12 applied to $\varphi(x) = \log(1 + r^n x)$, but note that now we have only $\int x^p d\lambda(x) < \infty$. For $x < \frac{1}{r}$, write $\log(1 + r^n x^n) \leq (rx)^n \leq (rx)^p$. Thus by the dominated convergence theorem $\int_0^{1/r} \log(1 + r^n x^n) d\lambda(x) \to 0$ as $n \to \infty$. For $x \geq \frac{1}{r}$, write $\log(1 + r^n x^n) \leq \log(2 r^n x^n) = \log 2 + n \log(rx)$. Thus $\frac{1}{n} \int_{1/r}^{\infty} \log(1 + r^n x^n) d\lambda(x) \leq \frac{\log 2}{n} \lambda([\frac{1}{r},\infty)) + \int_{1/r}^{\infty} \log(rx) d\lambda(x)$. The first term $\to 0$ as $n \to \infty$, and the second $= \int \log_+(rx) d\lambda(x) = \tau(\log_+ r|T|)$.

3.6 Theorem. If φ is a non-decreasing function on $[0,\infty)$, such that $\varphi(0) = 0$ and $\varphi(e^t)$ is convex, then $\int \varphi(|w|) d\mu(w) = \int \varphi\left(\frac{1}{|w|}\right) d\mu_0(w) \leq \tau(\varphi(|T|))$. Also $\int_0^t \varphi(s_1(x)) dx \leq \int_0^t \varphi(s_T(x)) dx$, where s_1 is the non-increasing rearrangement of $|w|$ relative to $d\mu(w)$.

Proof. 3.5 (ii) and 2.1 give 1.11 (ii), and 1.11 (iii) gives the desired result.

3.7 Remark. 3.6 is analogous to the Weyl inequalities (Weyl, 1949).

3.8 Corollary. (i) $\forall q \in (0,\infty)$, $\int |w|^q d\mu(w) \leq \|T\|_q^q$.

(ii) $\mu(\sigma(T)\backslash\{0\}) \leq \tau(P)$, where P = the right support projection of T = the support projection of $|T|$.

Proof. For (i) take $\varphi(x) = x^q$. For (ii) take $\varphi(0) = 0$, $\varphi(x) = 1$ for $x > 0$.

3.9 Theorem. $u(z) = \tau(\log|g_k(zT)|) = \int \log|g_k(zw)|d\mu(w)$, $\forall z \in \mathbb{C}$.

Proof. We now have all the hypotheses of 2.2 and need only observe that $\int \log|g_k(zw)|d\mu(w) = \int \log|g_k(\frac{z}{w})|d\mu_0(w)$.

3.10 Theorem. If f is homomorphic in a neighborhood of $\sigma(T) \cup \{0\}$ and f vanishes to order at least k at 0, then $\tau(f(T)) = \int f(w)d\mu(w)$.

Proof. We apply $\frac{\partial}{\partial z}$ to both sides of the equation in 3.9 for $\frac{1}{z} \notin \sigma(T)$, obtaining $\frac{1}{2}\tau\left[-\frac{T}{1-zT} + T + \cdots + z^{k-2}T^{k-1}\right] = \frac{1}{2}\int\left[-\frac{w}{1-zw} + w + \cdots + z^{k-2}w^{k-1}\right]d\mu(w)$. This simplifies to: $\tau\left(\frac{T^k}{1-zT}\right) = \int \frac{w^k}{1-zw}d\mu(w)$, for $z \neq 0$. Thus we have the theorem for $f(w) = \frac{w^k}{a-w}$, $a \notin \sigma(T) \cup \{0\}$. Now write $f(w) = w^k\tilde{f}(w)$, and write $\tilde{f} = \lim \tilde{f}_n$, uniformly in a neighborhood of $\sigma(T) \cup \{0\}$, where each $\tilde{f}_n$ is a finite linear combination of functions $\frac{1}{a - \cdot}$, $a \notin \sigma(T) \cup \{0\}$. Then 3.10 is obtained by passing to the limit in

$$\tau(T^k\tilde{f}_n(T)) = \int \tilde{f}_n(w)w^k d\mu(w).$$

3.11 Remark. For $k = 1$ $(T \in L_{1,\infty})$ and $f(w) = w$, we obtain the desired generalization of Lidskii's theorem.

3.12 Theorem. If f is holomorphic in a neighborhood of $\sigma(T) \cup \{0\}$ and $f - 1$ vanishes to order at least k at 0, then $\log \Delta(f(T)) = \tau(\log|f(T)|) = \int \log|f(w)|d\mu(w)$. (Here $-\infty$ is a possible value.)

Proof. Both sides of the equation are additive, so that if $f = f_1 f_2$, it is sufficient to prove the result for f_1 and f_2. 3.9 covers the case where $f = g_k(z\cdot)$ for some z. f has only finitely many zeroes in $\sigma(T) \cup \{0\}$, none of them at 0; and hence by removing finitely many factors $g_k(z\cdot)$ (with $\frac{1}{z} \in \sigma(T)$), we may assume f has no zeroes. Now f has a single valued logarithm (in some neighborhood of $\sigma(T) \cup \{0\}$) if and only if f is homotopic to 1 as a map of $\sigma(T) \cup \{0\}$ to $\mathbb{C}\setminus\{0\}$. The homotopy type of such maps is classified by elements of $\check{H}^1(\sigma(T) \cup \{0\})$, which is isomorphic to the free abelian group generated by the connected components of $\mathbb{C}\setminus(\sigma(T) \cup \{0\})$. Thus by removing a finite number of factors of the form $g_k(z\cdot)^{\pm 1}$ (with $\frac{1}{z} \notin \sigma(T)$), we reduce to the case $f = \exp(\tilde{f})$, where $\tilde{f}$ vanishes to order k at 0. Then the left side of 3.12 is $\operatorname{Re} \tau(\tilde{f}(T))$ and the right side is $\operatorname{Re} \int \tilde{f}(w)d\mu(w)$. Thus 3.12 has been deduced from 3.9 and 3.10.

We note that necessarily $0 \in \sigma(T)$ if $\tau(1) = \infty$. We now consider the case $\tau(1) < \infty$ and obtain more precise results. Since $M \subset L^1(M,\tau)$, we assume $k = 1$. By 3.8 (ii), $\mu(\sigma(T)\setminus\{0\}) \leq \tau(1)$. We extend μ to a measure, still denoted μ, on $\mathbb{C}$ (supported on $\sigma(T) \cup \{0\}$) by setting $\mu(\{0\}) = \tau(1) - \mu(\sigma(T)\setminus\{0\})$. For $z \neq 0$, $1 - zT = z(\frac{1}{z} - T)$ and $1 - zw = z(\frac{1}{z} - w)$. If z is replaced by $\frac{1}{z}$, 3.9 gives $\tau(\log|z - T|) - \log|z|\tau(1) = \int \log|z - w|d\mu(w) - \log|z|\mu(\sigma(T) \cup \{0\})$, for $z \neq 0$. Since

$\mu(\sigma(T) \cup \{0\}) = \tau(1)$, we have for $z \neq 0$

$$(4) \qquad \tau(\log|z - T|) = \int \log|z - w| d\mu(w).$$

Now both sides of (4) define subharmonic functions on all of $\mathbb{C}$. (For the left side 3.4 applies. For the right side standard facts recalled in §2 suffice.) Since these subharmonic functions agree almost everywhere, they are equal; and (4) holds for all z. If $0 \notin \sigma(T)$, the left side of (4) is harmonic near 0; and this fact clearly implies that μ has no mass at 0. Thus μ is always supported on $\sigma(T)$.

Now if $0 \notin \sigma(T)$, we see that the f in 3.10 or 3.12 may as well be defined only in a neighborhood of $\sigma(T)$, since we could always extend it to $\sigma(T) \cup \{0\}$ by making it 0 in a neighborhood of 0. We claim that if $0 \in \sigma(T)$, the hypothesis on $f(0)$ (recall that $k = 1$) may be dropped from 3.10 and 3.12. For 3.10 we need only observe that $\tau(1) = \int 1 d\mu(w)$. For 3.12 we need to consider the cases $f = c \neq 0$ (to cover the case $f(0) = c \neq 0$) and $f(w) = w^m$ (to cover the case $f(0) = 0$). The first case follows from $\tau(1) = \mu(\sigma(T))$ and the second from (4) for $z = 0$.

The following summarizes the results proved:

<u>3.13 Theorem.</u> If τ is a faithful, normal, semifinite trace on a W^*-algebra $M, T \in M \cap L^p(M,\tau)$ for $p \in (0,\infty)$, and k is an integer $\geq p$, then there is a unique non-negative measure μ on $\sigma(T)\setminus\{0\}$ such that $\log \Delta(g_k(zT)) = \int \log|g_k(zw)| d\mu(w)$, $\forall z \in \mathbb{C}$, where $g_k(w) = (1 - w)\exp(w + \frac{1}{2} w^2 + \cdots + \frac{1}{k-1} w^{k-1})$. Further $\int \varphi(|w|) d\mu(w) \leq \tau(\varphi(|T|))$ for all non-decreasing φ such that $\varphi(0) = 0$ and $\varphi(e^t)$ is convex. Also

$$(5) \qquad \tau(f(T)) = \int f(w) d\mu(w),$$

for all f holomorphic in a neighborhood of $\sigma(T)$ such that f vanishes to order at least k at 0 (if $0 \in \sigma(T)$); and

$$(6) \qquad \log \Delta(f(T)) = \int \log|f(w)|\, d\mu(w),$$

for all f holomorphic in a neighborhood of $\sigma(T)$ such that $f - 1$ vanishes to order at least k at 0 (if $0 \in \sigma(T)$).

If $\tau(1) < \infty$, there is also a unique μ on $\sigma(T)$ such that $\log \Delta(z - T) = \int \log|z - w|\, d\mu(w)$, $\forall z \in \mathbb{C}$; and this μ agrees with the other on $\sigma(T)\setminus\{0\}$. For this μ, $\mu(\sigma(T)) = \tau(1)$, and (5) and (6) hold for all f holomorphic in a neighborhood of $\sigma(T)$.

The only part of 3.13 not already proved is the uniqueness of μ. But this follows from the facts that for any μ as in 3.13, $d\mu(\frac{1}{w})$ must be the Riesz measure of u for the first μ, and μ itself is the Riesz measure of $\log \Delta(\cdot - T)$ for the second μ.

3.14 Remark. Finally, we note that if k is replaced by $k + 1$, u is changed by the addition of $\frac{1}{k} \operatorname{Re} z^k \tau(T^k)$. Since this is harmonic, μ is not changed. We call μ the spectral multiplicity measure of T and denote it μ_T when clarity demands the subscript. It will not be necessary to establish separate notations for the two versions of μ when $\tau(1) < \infty$.

4. ADDITIONAL CONSIDERATIONS

In this section we prove some results intended to give further justification that μ deserves to be called the spectral multiplicity measure of T. We also give some suggestions for further work.

4.1 Theorem. Assume $T \in L_{p,\infty}$ and f is holomorphic in a neighborhood of $\sigma(T)$. If $\tau(1) = \infty$, assume $f(0) = 0$. Then $\mu_{f(T)} = f_* \mu_T$.

Proof. We recall that $f_* \mu_T$ is given by $\int g(\tilde{w}) d(f_* \mu_T)(\tilde{w}) = \int g(f(w)) d\mu_T(w)$. Thus we need only show that for $\tau(1) = \infty$

$$\log \Delta(g_k(zf(T))) = \int \log |g_k(zf(w))| \, d\mu_T(w),$$

and for $\tau(1) < \infty$

$$\log \Delta(z - f(T)) = \int \log |z - f(w)| \, d\mu_T(w).$$

These follow from (6) of 3.13.

4.2 Remark. If T is normal it is easy to see that μ_T is the obvious measure; namely, $\mu_T(F) = \tau(E_F(T))$. Thus for normal operators, the conclusion of 4.1 holds even if f is only Borel (provided $f(T) \in L_{q,\infty}$ for some $q \in (0,\infty)$). Thus, once we know that there is a normal T such that μ_T is a continuous measure, we can conclude that there are no general restrictions (other than $\int |w|^q d\mu(w) < \infty$ for some q) on what measures can occur.

4.3 Theorem. Suppose $T \in L_{p,\infty}$ is of the form $\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ as in 1.8, then $\mu_T = \mu_{T_{11}} + \mu_{T_{22}}$.

Proof. $g_k(zT) = \begin{pmatrix} g_k(zT_{11}) & * \\ 0 & g_k(zT_{22}) \end{pmatrix}$. Thus it is trivial to deduce this from 1.8 and 3.13.

Note. Under the above hypotheses $\sigma(T) = \sigma(T_{11}) \cup \sigma(T_{22})$. The proof uses index and is the same as the usual proof for compact operators in $B(H)$.

4.4 Remark. Suppose $\sigma(T)$ is not connected and P_0 is the Riesz projection for some open-closed set $C \subset \sigma(T)$. Let P be the self-adjoint projection with the same range as P_0 (P_0 is only an idempotent). Then $T_{11} = T|_{P_0H} = P_0T|_{P_0H} = T|_{PH} = PT|_{PH}$. Also T_{22} is similar to $T|_{(1-P_0)H} = (1 - P_0)T|_{(1-P_0)H}$. Thus 1.9 can be used to deduce that $\mu_{T_{22}} = \mu_{T|(1-P_0)H}$. Of course $\mu_{T_{11}} = \mu_T|_C$ and $\mu_{T_{22}} = \mu_T|_{\sigma(T)\setminus C}$. Now $\tau(P) < \infty$ if and only if $0 \notin C$ or $\tau(1) < \infty$. In this case $\tau(P) = \tau(P_0) = \mu_{T_{11}}(C) = \mu_T(C)$.

In particular if $z \neq 0$ is an isolated point of $\sigma(T)$, $\mu(\{z\}) = \tau(P_0)$. For $M = B(H)$ and τ the usual trace, this shows that our μ is the usual spectral multiplicity. Thus our results really do contain Lidskii's Theorem.

4.5 Lemma. Assume $\bar{A} - 1 \in L_{1,\infty}$ and that $\bar{A}$ has a matrix representation $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ relative to a projection $P \in M$ such that $A = P\bar{A}P$ is invertible in PMP. Then $\Delta(\bar{A}) = \Delta(A)\Delta(D - CA^{-1}B)$.

Proof. This follows from 1.8 and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D-CA^{-1}B \end{pmatrix}.$$

4.6 Theorem. If $S,T \in M$ and $ST, TS \in L^p(M,\tau)$, then $\mu_{ST} = \mu_{TS}$.

Proof. We need to show $\Delta(g_k(zST)) = \Delta(g_k(zTS))$. We may absorb z into T and consider only $\Delta(g_k(ST)) = \Delta(g_k(TS))$. We will use 4.5. We will choose an $\varepsilon > 0$ and let $E = E_{[0,\varepsilon]}(|S^*|)$, $E' = E_{[0,\varepsilon]}(|S|)$. We will use E to get a 2×2 matrix representation for $\bar{A} = g_k(ST)$ and use E' for $\overline{A'} = g_k(TS)$. ε will be chosen small enough that $A = Eg_k(ST)E$ and $A' = E'g_k(TS)E'$ are invertible, and we will show

(7) $$\Delta(A) = \Delta(A'), \text{ and}$$

(8) $$\Delta(D - CA^{-1}B) = \Delta(D' - C'(A')^{-1}B').$$

We can write $g_k(w) = 1 + \Sigma_{n=k}^{\infty} a_n w^n$, where the power series has radius of convergence $= \infty$. Since $\|ES\|, \|SE'\| \le \varepsilon$, for the invertibility of A and A' it is sufficient to have $\varepsilon \Sigma_{n=k}^{\infty} |a_n| \|S\|^{n-1} \|T\|^n < 1$. This will imply $\|A - 1\| < 1$ and hence $\log \Delta(A) = \operatorname{Re} \tau(\log A) = \operatorname{Re} \Sigma_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \cdot$ $\cdot\, \tau((A - 1)^m)$. Moreover, the series for g_k can be used to expand $\tau((A - 1)^m)$ in a series. Since a similar expansion holds for A', (7) will follow from

(9) $$\tau[E(ST)^{n_1}E(ST)^{n_2} \cdots E(ST)^{n_m}E] = \tau[E'(TS)^{n_1}E' \cdots E'(TS)^{n_m}E'],$$

for $n_1, n_2, \ldots, n_m \ge k$. To prove (9) we use Theorem 17 of (Brown and Kosaki), which asserts that $\tau(XY) = \tau(YX)$ whenever $XY, YX \in L^1(M, \tau)$. Here $X = ES$, $Y = E'T(ST)^{n_1 - 1}E(ST)^{n_2} \cdots E(ST)^{n_m}E$; and in order to show $XY =$ the left side of (9) and $YX =$ the right side of (9), we use the obvious fact

(10) $$ES = SE'.$$

(8) will follow from

(11) $$(D - CA^{-1}B)S = S(D' - C'(A')^{-1}B').$$

$(1 - E)S = S(1 - E')$ is an invertible operator from $(1 - E')H$ to $(1 - E)H$. Hence (11) implies that $(D - CA^{-1}B)$ is similar to $D' - C'(A')^{-1}B'$, and 1.9 completes the proof of (8). Now we must prove

(11), and we note that it is equivalent to

$$(12)\quad (1-E)g_k(ST)(1-E)S - (1-E)g_k(ST)(Eg_k(ST)E)^{-1}g_k(ST)(1-E)S$$
$$= S(1-E')g_k(TS)(1-E') - S(1-E')g_k(TS)(E'g_k(TS)E')^{-1}g_k(TS)(1-E').$$

Finally, (12) follows from repeated applications of (10) and

$$(13)\qquad g_k(ST)S = Sg_k(TS).$$

In particular, we note that (10) and (13) imply $(Eg_k(ST)E)^{-1}S =$ $S(E'g_k(TS)E')^{-1}$. (The inverses are taken relative to EME and $E'ME'$.)

4.7. It is trivial to check that $d\mu_{T^*}(w) = d\mu_T(\bar{w})$.

4.8. If k is the smallest integer $\geq p$, there is a universal constant Γ_p such that $\log \Delta(g_k(T)) \leq \Gamma_p \|T\|_p^p$. This is proved exactly as in (Dunford and Schwartz, 1963; page 1106). Use $\log|g_k(w)| \leq \Gamma_p|w|^p$ and 3.8.

4.9. In (Grothendieck, 1955) the problem of defining spectral multiplicity for $T \in K_\tau$ is posed. One might hope to eliminate our restriction, $T \in L_{p,\infty}$ for some $p < \infty$, by means of theorems on the continuity of $T \mapsto \mu_T$. We discuss some partial results (far from adequate for the above purpose) on continuity.

Suppose $T_n \to T_0$ in $L_{p,\infty}$. Define

$$d\bar{\mu}_n(w) = \begin{cases} |w|^p d\mu_{T_n}(w), & \tau(1) = \infty \\ d\mu_{T_n}(w), & \tau(1) < \infty. \end{cases}$$

$(\bar{\mu}_n)_{n=1}^\infty$ is bounded, and all $\bar{\mu}_n$ have support on a fixed compact set. Let $\tilde{\mu}$ be a weak* cluster point. By the upper semicontinuity of spectrum,

$\tilde{\mu}$ is supported on $\sigma(T)$. For $\tau(1) < \infty$, $\tilde{\mu}(\sigma(T)) = \tau(1)$. For $\tau(1) = \infty$, so far as we know, we cannot rule out the possibility that $\tilde{\mu}$ has mass at 0. By 3.13 (5), respectively (6), for $\tau(1) < \infty$, $\int f d\tilde{\mu} = \int f d\bar{\mu}_0$ for all f holomorphic, respectively harmonic, in a neighborhood of $\sigma(T)$. For $\tau(1) = \infty$, we assume $f(w) = \frac{g(w)}{|w|^p}$ where g is holomorphic or harmonic and vanishes to order $> p$ at 0. If $R(\sigma(T)) = C(\sigma(T))$, it follows that $\tilde{\mu} = \bar{\mu}_0$, except for mass at 0 when $\tau(1) = \infty$. Thus $\int f d\mu_{T_n} \to \int f d\mu_{T_0}$ for all continuous f such that $f(w) = o(|w|^p)$ at 0 if $\tau(1) = \infty$. This holds in particular if $\sigma(T)$ has area 0 or if $\sigma(T)$ has empty interior and $\mathbb{C}\backslash\sigma(T)$ has only finitely many components. Presumably the hypothesis $R(\sigma(T)) = C(\sigma(T))$ is too strong, since it uses only 3.13(5); but even using 3.13(6) we would still need to assume $\sigma(T)$ has empty interior. At this time we feel the main issue is whether any hypothesis on $\sigma(T)$ is really needed.

4.10. Must the support of μ be all of $\sigma(T)$ ($[\sigma(T)\backslash\{0\}]^-$ if $\tau(1) = \infty$)? In particular if μ is concentrated at 0, must T be quasi-nilpotent? By 4.4 we know μ must have some mass in every non-empty compact relatively open subset C of $\sigma(T)$ ($0 \notin C$ if $\tau(1) = \infty$). Thus the answer is "yes" if $\sigma(T)$ is totally disconnected.

However, for $\sigma(T)$ not totally disconnected the answer to both questions is "no". For example let $T = \Sigma \oplus T_n$, where T_n is a truncated shift on an n-dimensional Hilbert space H_n. Then it is well known that $\sigma(T)$ is the unit disk. If we take $M = \Sigma \oplus B(H_n)$ and choose τ so that $\tau(1) < \infty$, then $\mu_T = \Sigma\, \mu_{T_n}$ is concentrated at 0, since T_n is nilpotent. Since M can be embedded, preserving the trace, in a type II factor, the answers are also "no" when M is a type II factor. Finally, this example can be modified so that $\sigma(T)$ is any connected compact set containing 0.

APPENDIX

We indicate how to prove Lidskii's Theorem for unbounded operators in $L^1(M,\tau)$. Our results for unbounded operators are less complete than the analogous results for bounded operators. We do not know to what extent they can be improved. All operators considered will be τ-measuraable unless the contrary is stated.

A.1 Proposition. Conditions (i)-(iv) below, for an operator A, are equivalent. Moreover, if D_τ is the set of operators satisfying the conditions, then D_τ is closed under multiplication, Δ extends uniquely to a multiplicative function on D_τ, $A \in D_\tau$ and $\Delta(A) > 0$ implies $A^{-1} \in D_\tau$, and $A \in D_\tau$ if and only if the two factors of its polar decomposition belong to D_τ. In particular $A \in D_\tau$ implies index $A = 0$.

(i) $A = B^{-1}C$, $B, C \in 1 + L_{1,\infty}$, $\Delta(B) > 0$.

(ii) $A = CB^{-1}$, B, C as in (i).

(iii) $\tau(\log_+|A|) < \infty$ and $A = 1 + T_0 + T_1$ with T_0 of τ-finite rank and $T_1 \in L^1(M,\tau)$.

(iv) $\tau(\log(1 + |A - 1|)) < \infty$.

Proof. 1. (iv) $\Rightarrow$ (iii): Since $|A|$ is spectrally dominated by $1 + |A - 1|$, $\tau(\log_+|A|) \leq \tau(\log(1 + |A - 1|))$. Also, for $T = A - 1$, let $T_0 = TE$ and $T_1 = T(1 - E)$, where $E = E_{(1,\infty)}(|T|)$.

2. (iii) $\Rightarrow$ (iv): Since $\log(1 + x) \leq x$, $\tau(\log(1 + |T_1|)) \leq \|T_1\|_1 < \infty$. Using the inequality

$$(14) \quad \tau(\log(1 + |T' + T''|)) \leq \tau(\log(1 + |T'|)) + \tau(\log(1 + |T''|)),$$

((Grothendieck; Akemann, Anderson, and Pedersen, 1982; Fack, 1983; and

Brown and Kosaki for the unbounded case), we see it is sufficient to show $\tau(\log(1 + |T_0|)) < \infty$. This follows from $s_{T_0}(t) \leq 1 + s_A(\frac{t}{2}) + s_{T_1}(\frac{t}{2})$, the above, $\tau(\log_+|A|) < \infty$, and the fact that $s_{T_0}(t) = 0$ for t sufficiently large.

3. The set of operators satisfying (iii) is closed under multiplication: Let $A = 1 + T_0 + T_1$, $A' = 1 + T'_0 + T'_1$, T_i, T'_i as in (iii), and further assume T_1 bounded (the unbounded part of T_1 can be assorbed into T_0). $\tau(\log_+|AA'|) < \infty$ follows from $s_{AA'}(t) \leq s_A(\frac{t}{2})s_{A'}(\frac{t}{2})$. The second condition in (iii) follows from $AA' = 1 + (T_0 + T'_0 + T_0T'_0 + T_0T'_1 + T_1T'_0) + (T_1 + T'_1 + T_1T'_1)$.

4. If $A = U|A|$ is the polar decomposition, then A satisfies (iii) implies U, $|A|$ satisfy (iii): By the above $|A|^2$ satisfies (iii). Using (iv) $\Leftrightarrow$ (iii), we see easily that for $B \geq 0$, B satisfies (iii) if and only if $B^{1/2}$ does. If $A = 1 + T_0 + T_1$, $|A| = 1 + T'_0 + T'_1$, T_i, T'_i as in (iii), T_1, T'_1 bounded, then $1 + T_0 + T_1 = U + UT'_0 + UT'_1$. Thus $U = 1 + (T_0 - UT'_0) + T_1 - UT'_1$, where $T_1 - UT'_1 \in L_{1,\infty}$. Hence $T_0 - UT'_0$ must be bounded; and since it has τ-finite rank, it also is in $L_{1,\infty}$. (It is clear that for bounded operators (iv) is equivalent to belonging to $1 + L_{1,\infty}$.)

5. (iii) $\Rightarrow$ (i), (ii): Using the spectral representation of $|A|$, we see easily that (iv) for $|A|$ implies $|A| = C_0B^{-1}$, $C_0, B \in 1 + L_{1,\infty}$, $\Delta(B) > 0$. Take $C = UC_0$. To get (i), use $A = |A^*|U$.

6. (i), (ii) $\Rightarrow$ (iii): From $\Delta(B) > 0$, it follows easily that B and B^* have trivial null-spaces and $|B|^{-1}$ satisfies (iv). Thus $B^{-1} = |B|^{-1}V^*$ satisfies (iii). Apply step 3.

7. Define $\Delta(A) = \Delta(B)^{-1}\Delta(C)$, A, B, C as in (i). The fact that Δ is well-defined and multiplicative is deduced from (i) $\Leftrightarrow$ (ii) and standard

algebraic tricks (cf. the construction of the quotient division ring of a suitable non-commutative integral domain).

8. $A \in D_\tau$, $\Delta(A) > 0 \Rightarrow A^{-1} \in D_\tau$: This now follows easily from (i), (ii).

9. Finally, index A = index U = 0, since $U \in 1 + L_{1,\infty}$. We remark that since index A = 0, $A = V|A|$ for some unitary $V \in 1 + L_{1,\infty}$. Since it is easy to see how to write $|A|$ in the form (i) and $\Delta(V) = 1$, the formula (1) used to define Δ on $1 + L_{1,\infty}$ is valid also on D_τ.

Let $\delta(T) = \tau(\log(1 + |T|))$. Then $D_\tau = 1 + L_\tau$, where $L_\tau = \{T : \delta(T) < \infty\}$. It is easy to see that $\delta(AT), \delta(TA) \leq k\delta(T)$ where $k = \max(\|A\|,1)$ (use $(1 + x)^k \geq 1 + kx$, $x \geq 0$). This and (14) imply that L_τ is an ideal; i.e., L_τ is a vector space of τ-measureable operators which is closed under left and right multiplication by bounded operators. Also L_τ is a topological vector space in the metric $\delta(S - T)$. We will also speak of the L_τ-topology on the coset D_τ.

<u>A.2 Proposition.</u> Δ is upper semicontinuous in the L_τ-topology.

<u>Proof</u>. Assume $A_n \to A = B^{-1}C$, B, C as in A.1(i). We wish to show $\Delta(A) \geq \overline{\lim}\, \Delta(A_n)$. Since B is bounded, $BA_n \to C$. Therefore we are reduced to the case A bounded. Next we claim $|A_n|^2 \to |A|^2$ in D_τ. In fact, $|A_n|^2 - |A|^2 = |A_n - A|^2 + (A_n - A)^*A + A^*(A_n - A)$ implies $\delta(|A_n|^2 - |A|^2) \leq \delta(|A_n - A|^2) + 2k\delta(A_n - A)$, where $k = \max(\|A\|,1)$. Since $\delta(|T|^2) = \int \log(1 + s_T(t)^2)dt \leq \int \log(1 + 2s_T(t) + s_T(t)^2)dt = 2\delta(T)$, the claim follows.

We are now reduced to the case A and all A_n's ≥ 0. Let $A_\varepsilon = A + \varepsilon E_{[0,\varepsilon]}(A)$. Since $\Delta(A_\varepsilon) \downarrow \Delta(A)$ as $\varepsilon \downarrow 0$, we may replace A by A_ε and A_n by $A_n + \varepsilon E_{[0,\varepsilon]}(A)$. Thus we now assume A invertible. Then $A^{-1}A_n \to 1$, so that we have finally reduced to the case $A = 1$.

(A_n is no longer ≥ 0). Now write $A_n = 1 + T_n$. Then $T_n \to 0$ in L_τ. The proposition follows from $\log \Delta(A_n) = \tau(\log|1 + T_n|) \leq \tau(\log(1 + |T_n|)) \to 0$.

Now it is trivial to deduce from 1.9 that $\Delta(PAP^{-1}) = \Delta(A)$ for $A \in D_\tau$ and P bounded and invertible, but we will prove more later.

We do not know how to handle exponentials of unbounded operators (the analogue of 1.2 for infinitesimal generators should be investigated), but 1.8 is still valid on D_τ. One proves $\Delta\begin{pmatrix} 1 & T_{12} \\ 0 & 1 \end{pmatrix} = 1$ by using the fact that $\begin{pmatrix} 1 & T_{12} \\ 0 & 1 \end{pmatrix}$ is binormal (T_{12} may be regarded as a positive operator). There is no difficulty in extending 1.5, 1.10, and 1.12, to unbounded operators. In 1.10, one assumes $\int_0^1 \log_+ s_{T_i}(t)dt < \infty$.

We know no satisfactory unbounded version of 1.4. (The $L_{1,\infty}$-topology could be used even for unbounded operators, but this does not help us.) This means that we also know no satisfactory unbounded version of 3.4.

Nevertheless we can still show that for $T \in L_\tau$, $u(z) = \log \Delta(1 - zT)$ is subharmonic, and harmonic for $\frac{1}{z} \notin \sigma(T)$. For the first, use $u(z) = \log \Delta(A - zB) - \log \Delta(A)$, with $A = (1 + |T|)^{-1}$ and $B = TA$. For the second, first note that if T is unbounded, $\infty \in \sigma(T)$, so that we are not asserting u is harmonic in a neighborhood of 0. (In fact, it may be that $u(z_n) = -\infty$ for some sequence $z_n \to 0$.) Then if $z_0 \neq 0$ and $\frac{1}{z_0} \notin \sigma(T)$,

$$(15) \quad \log \Delta(1 - zT) = \log \Delta(1 - z_0T) + \log \Delta(1 - (z - z_0)(1 - z_0T)^{-1}T),$$

and $(1 - z_0T)^{-1}T \in L_{1,\infty}$.

Now 3.5, 3.6, 3,7, 3.8, and 3.9 all go through (for $k = 1$). (Also the improvement of 3.9 for $\tau(1) < \infty$.) In connection with these, note that $u(0) = 0 > -\infty$ implies $\int \log_+\left(\frac{1}{|w|}\right) d\mu_0(w) < \infty$ (in particular μ_0 has no mass at 0). In 2.1 the hypothesis that u be harmonic in a neighborhood of 0 is not necessary. The same holds for 2.2 (for $k = 1$), but other changes are needed in 2.2: "u vanishes to order at least 1 at 0" should be replaced by "$u(0) = 0$", and "$\int \frac{1}{|w|^p} d\mu_0(w) < \infty$, $0 < p \leq 1$" should be replaced by "$\int_{|w|\geq 1} \frac{1}{|w|} d\mu_0(w) < \infty$". In using 2.2, one applies 3.6 with $\varphi(x) = \log(1 + x)$ to deduce $\int \log\left(1 + \frac{1}{|w|}\right) d\mu_0(w) \leq \tau(\log(1 + |T|))$.

<u>A.3 Lemma.</u> If $T \in L^1(M,\tau)$ and $a \in (0,\infty)$, then $\log \Delta(1 - T) \leq -\mathrm{Re}\ \tau(T) + \frac{a}{2} \|T\|_1 + 2 \int_{s_T(t)>a} s_T(t)dt$.

<u>Proof.</u> Let $E_1 = E_{[0,a]}(|T|)$, $E_2 = E_{(a,\infty)}(|T|)$, and $t_0 = \tau(E_2) < \infty$. Assume $\log \Delta(1 - T) > -\infty$.

$$\begin{aligned}\log \Delta(1 - T) &= \frac{1}{2} \tau(\log(|1 - T|^2)) \\ &= \frac{1}{2} \tau(E_1 \log(1 - 2\ \mathrm{Re}\ T + |T|^2)E_1) + \frac{1}{2} \tau(E_2 \log(|1 - T|^2)E_2) \\ &\leq \frac{1}{2} \tau(E_1(-2\mathrm{Re}\ T + |T|^2)E_1) + \frac{1}{2}\int_0^{t_0} \log((1 + s_T(t))^2)dt \\ &\leq \frac{1}{2} \tau(-2E_1(\mathrm{Re}\ T)E_1 + E_1|T|^2E_1) + \int_0^{t_0} \log(1 + s_T(t))dt \\ &\leq -\mathrm{Re}\ \tau(E_1TE_1) + \frac{1}{2} \tau(E_1|T|^2) + \int_0^{t_0} s_T(t)dt.\end{aligned}$$

Here we have used the inequality $\log(1 + x) \leq x$ and spectral dominance arguments.

Now since $E_1|T|^2 \leq a|T|$, $\tau(E_1|T|^2) \leq a\|T\|_1$. Also $\int_0^{t_0} s_T(t)dt = \int_{s_T(t)>a} s_T(t)dt$. Finally, $|\mathrm{Re}\ \tau(T) - \mathrm{Re}\ \tau(E_1TE_1)| = |\tau(E_2(\mathrm{Re}\ T)E_2)| \leq \int_0^{t_0} s_T(t)dt$, by a spectral dominance argument.

A.4 Lemma. If $T \in L^1(M,\tau)$, then $u(z) \leq -\mathrm{Re}\ \tau(zT) + o(|z|)$ as $z \to 0$.

Proof. Take $a = |z|^{1/2}$ and apply A.3 to zT. $\frac{a}{2}\|zT\|_1 = \frac{1}{2}|z|^{3/2}\|T\|_1 = o(|z|)$. Also

$$\int_{s_{zT}(t)>a} s_{zT}(t)dt = |z| \int_{s_T(t)>|z|^{-1/2}} s_T(t)dt = o(|z|).$$

A.5 Proposition. If $T \in L^1(M,\tau)$, then $\lim_{r\to 0_+} \frac{1}{r} u(re^{i\theta}) = -\mathrm{Re}\ \tau(e^{i\theta}T)$ in $L^1(\frac{d\theta}{2\pi})$.

Proof. By A.4 $\frac{1}{r} u(re^{i\theta}) + \mathrm{Re}\ \tau(e^{i\theta}T) \leq f_r(\theta)$, where f_r is a non-negative function on the circle and $f_r \to 0$ uniformly as $r \to 0$. Also $\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta \geq u(0) = 0$, and $\int_0^{2\pi} \tau(e^{i\theta}T)d\theta = 0$. Thus

$$\|\tfrac{1}{r} u(re^{i\theta}) + \mathrm{Re}\ \tau(e^{i\theta}T)\|_1 \leq \|f_r(\theta) - \tfrac{1}{r} u(re^{i\theta}) - \mathrm{Re}\ \tau(e^{i\theta}T)\|_1 + \|f_r\|_1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (f_r(\theta) - \tfrac{1}{r} u(re^{i\theta}) - \mathrm{Re}\ \tau(e^{i\theta}T))d\theta + \|f_r\|_1$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} f_r(\theta)d\theta + \|f_r\|_1 = 2\|f_r\|_1 \to 0 \quad \text{as} \quad r \to 0.$$

A.6 Theorem. $\tau(T) = \int w\,d\mu_T(w)$ for $T \in L^1(M,\tau)$.

Proof. By the unbounded version of 3.8, $\int |w|\,d\mu(w) < \infty$, so that the operator of multiplication by w is in $L^1(\tilde{M},\tau)$, where $\tilde{M} = L^\infty(d\mu(w))$. Thus A.5 for $\tilde{M}$ gives $\lim_{r\to 0+} \frac{1}{r} u(re^{i\theta}) = -\mathrm{Re} \int e^{i\theta} w\,d\mu(w)$ in L^1. Comparing this equation with A.5 for T, we see that $\mathrm{Re}(e^{i\theta}\tau(T)) = \mathrm{Re}(e^{i\theta}\int w\,d\mu(w))$ for almost every θ. This gives the theorem.

We are now able to extend 3.10, 3.12, and 4.1 to unbounded operators. Instead of using holomorphic functions, we use meromorphic functions (meromorphic at ∞ also, if T is unbounded); and we always assume that if f has a pole at a finite point $\frac{1}{z_0} \in \sigma(T)$, then $\Delta(g_k(z_0 T)) > 0$. (Here, either $T \in L_\tau$ and $k = 1$, or $T \in L_{p,\infty}$.) If f has a pole at 0 (which we allow to occur only if $\tau(1) < \infty$), we also assume $\Delta(T) > 0$. The extension of 3.12 to meromorphic functions when $T \in L_{p,\infty}$ is now trivial. If $T \in L_\tau$ and has non-empty resolvent set, (15) shows that μ_T is the appropriate möbius transform of μ_S, where $S = (1 - z_0 T)^{-1} T \in L_{1,\infty}$, so that 3.12 for T follows from 3.12 for S. Otherwise f has to be rational and 3.12 follows trivially from 3.9. The extension of 4.1 is:

(16) If $T \in L_{p,\infty}$ and f satisfies the above assumptions, and has a 0 of order at least k at 0 if $\tau(1) = \infty$ then

$$\mu_{f(T)} = f_* \mu_T.$$

(17) If $T \in L_\tau$ and f satisfies the same assumptions $(k = 1)$, then $\mu_{f(T)} = f_* \mu_T$.

For $T \in L_{p,\infty}$, the condition on the order of f at 0 is stricter for f meromorphic than for f holomorphic, since we do not know how to construct μ_T for unbounded $T \in L^p(M,\tau)$.

Now the unbounded version of 3.10 is clear:

(18) If f satisfies the usual conditions and $f(T) \in L^1(M,\tau)$, then $\tau(f(T)) = \int f(w)\,d\mu_T(w)$.

This follows from A.6, (16), and (17).

Finally we wish to prove $\mu_{ST} = \mu_{TS}$ in the unbounded case. We note that it is no longer true that $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$. For example take $S = \begin{pmatrix} 0 & X \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & Y \\ 0 & 0 \end{pmatrix}$. $ST = 0$, but if $X + Y$ is unbounded, TS has empty resolvent set. This gives additional insight on why the support of μ need not be the whole spectrum.

A.7 Lemma. Assume $A, B \in D_\tau$, $P \in M$, $PA = BP$, and P is 1-1 and has dense range. Then $\Delta(A) = \Delta(B)$.

Proof. We may assume $P \geq 0$. Let $E_n = E_{(\frac{1}{n},\infty)}(P)$, $F_n = E_{[0,\frac{1}{n}]}(P)$, $P_n = E_n P + \frac{1}{n} F_n$, and $Q_n = E_n + nF_n P$. Then $P = P_n Q_n = Q_n P_n$ and P_n is invertible. Therefore $\Delta(A) = \Delta(P_n A P_n^{-1})$. We claim $P_n A P_n^{-1} \to B$ in the L_τ-topology. In fact $P_n A P_n^{-1} - B = F_n(A - B)F_n + E_n B F_n Q_n - E_n B F_n + F_n A E_n \frac{1}{n} P_n^{-1} - F_n B E_n$. Each of the five terms $\to 0$ in L_τ since $\|Q_n\| \leq 1$, $\|\frac{1}{n} P_n^{-1}\| \leq 1$, and $F_n T$, $TF_n \to 0$, $\forall T \in L_\tau$. The last assertion follows from the fact that $\forall \varepsilon > 0$, $T = T_0 + T_1$ with $\delta(T_0) < \varepsilon$ and $T_1 \in L^1(M,\tau)$.

Thus A.2 implies $\Delta(B) \geq \Delta(A)$. Similarly, $\Delta(A) \geq \Delta(B)$.

A.8 Definition. Let A, B be τ-measureable and P a closed densely defined operator affiliated with M (not necessarily τ-measureable). We say that P intertwines A to B if the graph of P is invariant under $A \oplus B$. Here $A \oplus B$ is a measureable operator affiliated with the W^*-algebra $\bar{M} = M_2 \otimes M \subset B(H \oplus H)$. If E is the projection on the graph of P, then $E \in \bar{M}$; and $E(H \oplus H)$ invariant under $A \oplus B$ means $E(A \oplus B)x = (A \oplus B)x$, $\forall x \in \mathcal{D}(A \oplus B) \cap E(H \oplus H)$. (Explicitly, $y \in \mathcal{D}(P) \cap \mathcal{D}(A)$ and $Py \in \mathcal{D}(B)$ implies $Ay \in \mathcal{D}(P)$ and $PAy = BPy$.) However, it is enough to verify $E(A \oplus B)x = (A \oplus B)x$, $\forall x \in V$, where V is any subspace of $\mathcal{D}(A \oplus B) \cap E(H \oplus H)$ such that $\forall \varepsilon > 0$ $\exists$ a projection $F \leq E$ with $\tau(F) < \varepsilon$ and $(E - F)(H \oplus H) \subset V$. In particular if P is τ-measureable, P intertwines A to B if and only $PA = BP$, as τ-measureable operators.

A.9 Proposition. If $A, B \in D_\tau$ and P is a closed densely defined operator affiliated with M (not necessarily τ-measureable) such that P

is 1-1 and has dense range and P intertwines A to B, then $\Delta(A) = \Delta(B)$.

<u>Proof</u>. Let $\bar{A} = E(A \oplus B)E \in E\bar{M}E$, where E and $\bar{M}$ are as in A.8. Then the two projections of $H \oplus H$ onto H, restricted to $E(H \oplus H)$, give bounded intertwining operators from $\bar{A}$ to A and B. Thus A.7 implies $\Delta(\bar{A}) = \Delta(A)$ and $\Delta(\bar{A}) = \Delta(B)$.

<u>A.10 Theorem</u>. Assume P is as in A.9 and P intertwines S to T where either $S,T \in L_\tau$ or $S,T \in L_{p,\infty}$. Then $\mu_S = \mu_T$. If $S,T \in L^1(M,\tau)$, then $\tau(S) = \tau(T)$.

<u>Proof</u>. $\forall z$, P intertwines $g_k(zS)$ to $g_k(zT)$. (Note that S and T are bounded if $k \neq 1$.) Thus $\Delta(g_k(zS)) = \Delta(g_k(zT))$, which implies $\mu_S = \mu_T$. The last sentence follows from A.6.

<u>A.11 Theorem</u>. If S and T are τ-measureable and either $ST, TS \in L_\tau$ or $ST, TS \in L_{p,\infty}$, then $\mu_{ST} = \mu_{TS}$.

<u>Proof</u>. Let E and F be the left and right projections of S, and let $U \in M$ be such that $U^*U = F$, $UU^* = E$. Then by triangularity, $\mu_{ST} = \mu_{ESTE}$ and $\mu_{TS} = \mu_{FTSF} = \mu_{UTSU^*}$. Thus, upon replacing S, T by ESU^*, $UTE \in EME$, we are reduced to the case where S is 1-1 and has dense range.

Now the theorem follows from A.10 and $S(TS) = (ST)S$.

<u>A.12 Remark</u>. It is clear that the τ-measureability assumption on S, T in A.11 is too strong, since A.10 should be A.11 applied to P and SP^{-1}. However, in the notation of A.11, it may be awkward to phrase sufficient conditions more general than τ-measureability.

A.13 Corollary. If $\tau(1) < \infty$ and $ST, TS \in D_\tau$, then $\Delta(ST) = \Delta(TS)$.

Proof. $D_\tau = L_\tau$ in this case, so that $\mu_{ST} = \mu_{TS}$. Since $\log \Delta(A) = \int \log|w| d\mu_A(w)$ when $\tau(1) < \infty$, we are done.

A.14 Remark. A.13 is also valid when M is finite, since in this case M is a direct sum of W^*-algebras satisfying the hypothesis of A.13.

But the result is false if M is infinite, since $\Delta(U^*U) = 1$, $\Delta(UU^*) = 0$ for U an appropriate non-unitary isometry.

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