Groupe de travail de Géométrie non-linéaire des espaces de Banach

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Coarse embeddings into Banach spaces

1. Linear embeddings

- Let $1 \le p, q < \infty$. Pitt's theorem says that $\ell_p \subseteq \ell_q \Leftrightarrow p=q$
- Also $\ell_p \subseteq L_q \Leftrightarrow L_p \subseteq L_q \Leftrightarrow (2 \le q$ and $p \in \{2,q\})$ or $(q<2$ and $p \in [q,2])$

To prove this, use type, cotype, p-stable laws, Kadets-Pelczynski

- Also $L_p \subseteq \ell_q \Leftrightarrow p=q=2$

2. Lipschitz embeddings

Def: $f: (M,d) \to (N,\delta)$ is a Lipschitz embedding if

$\exists A,B>0 \; \forall x,y \in M \quad A d(x,y) \le \delta(f(x), f(y)) \le B d(x,y)$

In symbols: "$M \underset{L}{\hookrightarrow} N$"

Tools for B-spaces: . Gâteaux-diff^{ty} when X is separable and Y has RNP,
. w^*-G-diff^{ty} when X is separable and Y is a dual (or in Y^{**})

. local reflexivity

- ℓ_q $(1 \le p < \infty)$ and L_q $(1<q<\infty)$ have RNP
- L_1 has cotype 2
- If $p>2$, ℓ_p has cotype p.

This enables to get the same table of results in 2. as in 1.

One delicate point: $X \underset{L}{\hookrightarrow} L_1$ gives $X \subseteq L_1^{**}$ ...

Universal space for lip. embeddings: Aharoni: M separable metric space, $M \underset{L}{\hookrightarrow} c_0$.

Open question: does $c_0 \underset{L}{\hookrightarrow} X$ B-space imply $c_0 \subseteq X$?

3. Coarse Lip. embeddings:

Def: $f : M \to N$ is a coarse Lip embedding if

$\exists A, B, \vartheta > 0 \ \forall x, y \in M \ \ d(x,y) \geq \vartheta \Rightarrow A\, d(x,y) \leq \delta(f(x), f(y)) \leq B d(x,y)$

In symbols: $M \underset{CL}{\hookrightarrow} N$

Tools: $f : \underset{sep}{X} \underset{CL}{\hookrightarrow} Y$ defines $\Phi : X \underset{L}{\hookrightarrow} (Y)_{\mathcal{U}} \subseteq (Y_{\mathcal{U}})^{**}$, $\mathcal{U}$ ultrafilter

$x \longmapsto \left(\frac{f(nx)}{n}\right)_n$

w^*- G-dif + $(Y_{\mathcal{U}})^{**}$ finitely representable in $(Y)_{\mathcal{U}}$ and so in Y.

$F \subseteq X$, $\dim F < \infty$ implies $\exists G \subseteq Y \quad F \underset{1+\varepsilon}{\simeq} G$.

$\to$ local properties are preserved and (1) type $(X) \geq$ type (Y)
cotype $(X) \leq$ cotype (Y)

(2) kind of stability of asymptotic uniform smoothness (Kalton)
convexity

this yields: $p \neq q$ $\ell_p \underset{CL}{\not\hookrightarrow} \ell_q$

(3) $1 \leq q < \infty$, X sep., $X \underset{CL}{\hookrightarrow} L_p(\mu)$ yields $X \underset{L}{\hookrightarrow} (L_p(\mu))_{\mathcal{U}} = L_p(\Sigma)$

A separable subspace of an abstract L_p-space is isometric to a subspace of $L_p[0,1]$.

There is a coarse counterpart to the previous question:

¿ $c_0 \underset{CL}{\hookrightarrow} X \Rightarrow c_0 \subseteq X$?

4. Coarse embeddings

Def. $f : M \to N$ is a coarse embedding if $\exists \rho_1, \rho_2 : \mathbb{R}^+ \circlearrowleft$ $\lim_{+\infty} \rho_1 = +\infty$

$\forall x, y \quad \rho_1 \circ d(x,y) \leq \delta(f(x), f(y)) \leq \rho_2 \circ d(x,y)$

"$M \underset{C}{\hookrightarrow} N$"

Motivation: Coarsely embed special metric spaces like Cayley graph of finitely presented groups into Hilbert or superreflexive spaces enables to prove for them Baum Connes or Novikov conjecture

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Vague questions: does $\underset{C}{\hookrightarrow}$ superreflexive space imply $\underset{C}{\hookrightarrow} \ell_2$?

does $\underset{C}{\hookrightarrow} L_p$ imply $\underset{C}{\hookrightarrow} \ell_2$?

universal space for $\underset{C}{\hookrightarrow}$?

for separable, compact, locally finite, bounded geometry metric sp.

there is one positive result: $\boxed{\text{Th}}$ (Baudier) $\ell_\infty^n \subseteq X$, M proper metric sp. (i.e., balls are rel. compact), then $M \underset{C}{\hookrightarrow} X$ with $X = \left(\sum \ell_\infty^n\right)_{\ell_2}$

Open question: does $\dim X = \infty$ imply $\ell_2 \underset{C}{\hookrightarrow} X$?

$\boxed{\text{Th}}$: Ostrovskii: $M \subset \ell_2$, M locally finite (balls are finite), then $M \underset{L}{\hookrightarrow} X$ $(\dim X = \infty)$ [uses Dvoretzky's theorem]

Let us try to put up a table as in 1, 2 and 3.

A few positive results: $\boxed{\text{Th 1}}$ (Nowak '06) $1 \leq p < \infty$, $\ell_2 \underset{C}{\hookrightarrow} \ell_p$

$\boxed{\text{Th 1'}}$ (Ostrovskii building on Odell-Schlumprecht): If X has an unconditional basis and $\dim X = \infty$, and X has nontrivial cotype, then $\ell_2 \underset{C}{\hookrightarrow} X$

Application: X superreflexive space with an unc. basis.

$\boxed{\text{Th 2}}$ (Mendel-Naor '04) $0 < p \leq q$ $L_p \underset{C}{\hookrightarrow} L_q$, and $\ell_p \underset{C}{\hookrightarrow} \ell_q$

one has to check that the same proof works.

Obstructions to coarse embeddings

Th: Mendel-Naor ("metric cotype") to appear in Ann. Math. X, Y B. spaces with $\ell_1^n \notin Y$, then $X \underset{C}{\hookrightarrow} Y$ implies cotype $X \leq$ cotype Y

Th: Kalton '07: . $c_0 \underset{C}{\hookrightarrow} Y$ implies $\exists n$ $Y^{(n)}$ nonseparable, so Y is not reflexive.

. $X \underset{C}{\hookrightarrow} Y$ reflexive implies X has property (Q).

. $\exists Y$ Schur space $c_0 \underset{C}{\hookrightarrow} Y$ (n.b.: $c_0 \notin Y$ as Y contains ℓ_1 here!)

Table: (I) $\ell_p \underset{C}{\hookrightarrow} \ell_q \Leftrightarrow \ell_p \underset{C}{\hookrightarrow} L_q \Leftrightarrow L_p \underset{C}{\hookrightarrow} L_q \Leftrightarrow p \leq q$ or $q < p \leq 2$

(II) $p \leq 2 \Rightarrow L_p \underset{C}{\hookrightarrow} \ell_q$; $2 < q < p \Rightarrow L_p \underset{C}{\not\hookrightarrow} \ell_q$.

open question: does $2 < p \leq q$ imply $L_p \underset{C}{\hookrightarrow} \ell_q$?

in particular, ¿ $2 < p \Rightarrow L_p \underset{C}{\hookrightarrow} \ell_p$?

proof: I) $p \leq q$, $\ell_p \hookrightarrow \ell_q$, $\ell_p \subseteq L_p \hookrightarrow L_q$ (Mendel-Naor)

$q<p<2$ $\ell_p \subseteq L_p \hookrightarrow L_2$ (MN) $= \ell_2 \hookrightarrow \ell_q \subseteq L_q$

$2<p$, $q<p$: use cotype!

II) $p \leq 2$, $L_p \underset{c}{\hookrightarrow} L_2$ (MN) $\hookrightarrow \ell_q$ for all q.

If $2<q<p$, use cotype

$\boxed{\text{Th}}$ (Nowak) $\ell_2 \underset{c}{\hookrightarrow} \ell_p$ $(1 \leq p < \infty)$

$\boxed{\text{Proposition 1}}$ If X is metric space, $1 \leq p < \infty$, s.t. $\exists \delta > 0$ $\forall R > 0$ $\forall \varepsilon > 0$

$\exists \varphi : X \longrightarrow S_{\ell_p}$ (i) $\sup_{d(x,y) \leq R} \|\varphi(x)-\varphi(y)\|_p \leq \varepsilon$

(ii) $\liminf_{S \to \infty} \inf_{d(x,y) \geq S} \|\varphi(x)-\varphi(y)\| \geq \delta$

Then $X \underset{c}{\hookrightarrow} \ell_p$

This Prop. gives $\exists \delta > 0$ $\forall n$ $\exists \varphi_n : X \longrightarrow S_{\ell_p}$ $\exists s_n > 0$, $s_n \uparrow +\infty$

$d(x,y) \leq n \Rightarrow \|\varphi_n(x)-\varphi_n(y)\|_p \leq 2^{-n}$

$d(x,y) \geq s_n \Rightarrow \|\varphi_n(x)-\varphi_n(y)\|_p \geq \delta$.

Fix $x_0 \in X$ and let $\Phi : X \longrightarrow \ell_p(\ell_p)$

$x \longmapsto (\varphi_n(x)-\varphi_n(x_0))_n$

$\|\Phi(x)\|_p^p \leq \sum_{n<d(x,x_0)} \|\varphi_n(x)-\varphi_n(x_0)\|_p^p + \sum_{n \geq d(x,x_0)} 2^{-np} < \infty$

so that Φ is well-defined.

Let $x,y \in X$ and $k \in \mathbb{N}$: $(k-1)^{\frac{1}{p}} \leq d(x,y) \leq k^{\frac{1}{p}}$. Then

$\|\Phi(x)-\Phi(y)\|_p^p \leq \sum_{n=1}^{k-1} \|\varphi_n(x)-\varphi_n(y)\|_p^p + \sum_k^\infty 2^{-np}$

$\leq 2^p(k-1)+1$.

$\|\Phi(x)-\Phi(y)\|_p^p \leq 2^p(d(x,y))^p+1$ and there is k s.t. $S_k \leq d(x,y) \leq S_{k+1}$

and $\|\Phi(x)-\Phi(y)\|_p^p \geq \sum_{n=1}^{k} \|\varphi_n(x)-\varphi_n(y)\|_p^p \geq k\,\delta^p$.

Let $\rho : t \longmapsto k$ so that $s_k \leq t < s_{k-1}$: $\lim_{t \to \infty} \rho(t) = +\infty$.

and $\|\Phi(x)-\Phi(y)\| \geq \delta\, \rho^{1/p}$. ∎

$\boxed{\text{Proposition 2}}$ $1 \leq p,q < \infty$. If (N_p) are the assumptions in Prop. 1, then $(N_p) \Leftrightarrow (N_q)$

Proof: For instance, let $p>q$: consider the Mazur map

$\Phi : S_{L_p(\mu)} \longrightarrow S_{L_q(\mu)}$. Then $\Phi(S_{L_p}) = S_{L_q}$.

$f \longmapsto |f|^{p/q} \operatorname{sgn} f$

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and $C\|f-g\|^{\frac{p}{q}} \le \|\Phi(f)-\Phi(g)\|_q \le \frac{p}{q}\|f-g\|_p$.

Compose the φ's in (N_p) with Φ. ∎

It remains to prove

$\boxed{\text{Proposition 3}}$ ℓ_2 has (N_2).

Proof: Let $H=\mathbb{R} \oplus_2 \ell_2 \oplus_2 \ell_2 \otimes \ell_2 \oplus_2 \dots \oplus(\ell_2 \otimes \dots \otimes \ell_2)\oplus \dots$: $H \equiv \ell_2$.

where $\langle x\otimes x', y\otimes y'\rangle = \langle x,y\rangle\langle x',y'\rangle$.

Consider the exponential: $\ell_2 \xrightarrow{\exp} H$

$$y \longmapsto 1 \oplus y \oplus \frac{1}{\sqrt{2!}}(y\otimes y)\oplus \dots \oplus \frac{1}{\sqrt{n!}}(y\otimes \dots \otimes y)\dots$$

We have $\langle e^y, e^{y'}\rangle = e^{\langle y,y'\rangle}$. Let $\varphi(x) = e^{-t\|x\|^2} e^{\sqrt{2t}\,x}$.

then $\langle \varphi(x),\varphi(y)\rangle = e^{-t(\|x\|^2+\|y\|^2)} e^{2t\langle x,y\rangle} = e^{-t\|x-y\|^2}$ and $\|\varphi(x)\|=1$

and $\|\varphi(x)-\varphi(y)\|^2 = 2-2e^{-t\|x-y\|^2} = 2(1-e^{-t\|x-y\|^2})$.

Let $R>0$, $\varepsilon>0$. Let t small enough st we have $\|x-y\|\le R \Rightarrow \|\varphi(x)-\varphi(y)\|\le\varepsilon$.

then $\lim\limits_{\|x-y\|\to+\infty} \|\varphi(x)-\varphi(y)\| = \sqrt{2}$. there is a converse to this !

connected w. cond neg def. kernels

Addendum: Th (Mendel-Naor) $0<p\le q$ $L_p \underset{c}{\hookrightarrow} L_q$

Proof: $0<\alpha<2\beta$. Let $F(x) = \displaystyle\int_{-\infty}^{+\infty} \frac{(1-\cos tx)^\beta}{|t|^{\alpha+1}}\,dt$

$$= 2\int_0^{+\infty} \frac{(1-\cos tx)^\beta}{t^{\alpha+1}}\,dt.$$

If $x>0$, $u=tx$, $F(x) = \left(2\displaystyle\int_0^\infty \frac{1-\cos u}{u^{\alpha+1}}\,du\right)x^\alpha$.

and $|x|^\alpha = F(x)$. Let $1\le p<q<\infty$ and

define $T: L_p(\mathbb{R}) \longrightarrow L_q(\mathbb{R}\times\mathbb{R})$ by $Tf(s,t) = \dfrac{1-e^{itf(s)}}{|t|^{\frac{p+1}{q}}}$.

If $f,g\in L_p(\mathbb{R})$, $\|Tf-Tg\|_q^q = \displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|1-e^{it(f(s)-g(s))}|^q}{|t|^{p+1}}\,dt\,ds$

As $|1-e^{itx}|^2 = 2-2\cos tx$, $\|Tf-Tg\|_q^q = \displaystyle\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{[2-2\cos t(f(s)-g(s))]^{\frac{q}{2}}}{|t|^{p+1}}\,dt\,ds$

and $p<2\frac{q}{2}$, so, for $\begin{cases}\alpha=p\\ \beta=\frac{q}{2}\end{cases}$, $\|Tf-Tg\|_q^q = 2^{\frac{q}{2}}C_{p,q}\displaystyle\int_{\mathbb{R}}|f(s)-g(s)|^p\,ds = 2^{\frac{q}{2}}C_{p,q}\|f-g\|_p^p$

Sequel: extend the Mazur map to B-spaces w. an unc. basis.

Coarse embeddings of ℓ_2

Recall that • $f : M \to N$ is a coarse embedding if $\exists \rho_1, \rho_2 : \mathbb{R}_+ \circlearrowleft$ with $\rho_1 \xrightarrow[\infty]{} \infty$ such that $\rho_1 \circ d(x,y) \leq \delta(f(x), f(y)) \leq \rho_2 \circ d(x,y)$

• we proved last time (Nowak '06) $\ell_2 \hookrightarrow \ell_p$ $(1 \leq p \leq \infty)$

and (Mendel-Naor) $L_p \hookrightarrow L_q$ $(0 < p \leq q < \infty)$

Let us address today: $\dot{c}$ X B-space of ∞ dim $\Rightarrow \ell_2 \underset{c}{\hookrightarrow} X$?

<u>Th</u> (Odell-Schlumprecht '94): If X is a B-space with unconditional basis and nontrivial cotype [$\ell_\infty^n \not\subseteq X$], then $S_X \underset{UH}{\sim} S_{\ell_1}$ (and therefore $\underset{UH}{\sim} S_{\ell_2}$)

<u>Cor</u>: (Ostrovskii 2007) X with u.b. and $\ell_\infty^n \not\subseteq X$ implies $\ell_2 \underset{c}{\hookrightarrow} X$

. Here « $\ell_\infty^n \not\subseteq X$ » means « $\forall C \, \exists n \, \forall T : \ell_\infty^n \simeq G_n \subseteq X$ $\|T\| \|T^{-1}\| > C$. »

. We proved last time: $\forall R, \varepsilon \, \exists \varphi : \ell_2 \to S_{\ell_2}$ $\begin{cases} \|x-y\|_2 \leq R \Rightarrow \|\varphi(x) - \varphi(y)\|_2 < \varepsilon \\ \liminf\limits_{S \to \infty \, \|x-y\|_2 \geq S} \|\varphi(x) - \varphi(y)\|_2 = \sqrt{2}. \end{cases}$
[Dadarlat-Guentner]

<u>Proof of Cor</u>: Let (e_n) be a 1-unconditional basis of X, $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$, $N_i \cap N_j = \emptyset$ for $i \neq j$, $\# N_i = \infty$, $X_i = \overline{\operatorname{span}} \{e_n : n \in N_i\}$

O-S shows that $\exists \varphi_i : S_{X_i} \underset{UH}{\sim} S_{\ell_2}$. Then take $\varphi = \varphi_{R,\varepsilon}$ given by [D-G] and let $f_i = \varphi_i \circ \varphi$. Then $\|x-y\|_2 \leq R \Rightarrow \|f_i(x) - f_i(y)\| \leq \omega_{\varphi_i}(\varepsilon)$

$\liminf\limits_{S \to \infty \, \|x-y\|_2 \geq S} \|f_i(x) - f_i(y)\| \geq \delta_i$ if $\omega_{\varphi^{-1}}(\delta_i) < \sqrt{2}$, that is, if δ_i is small.

Let us apply this for $R = i$ and $\varepsilon > 0$ s.t. $\omega_{\varphi_i}(\varepsilon) \leq \frac{\delta_i}{i 2^i}$: this yields

(i) $\|x-y\|_2 \leq i \Rightarrow \|f_i(x) - f_i(y)\|_{X_i} \leq \frac{\delta_i}{i 2^i}$

(ii) $\liminf\limits_{S \to \infty \, \|x-y\|_2 \geq S} \|f_i(x) - f_i(y)\|_{X_i} \geq \delta_i$

Fix $x_0 \in \ell_2$ and set $f : \ell_2 \to X = \bigoplus_{i=1}^{\infty} X_i$

$$x \longmapsto \left(\frac{i}{\delta_i} (f_i(x) - f_i(x_0)) \right)_{i=1}^{\infty}$$

• $\sum_i \frac{i}{\delta_i} \|f_i(x) - f_i(x_0)\| = \sum\limits_{i < \|x - x_0\|} + \sum\limits_{i \geq \|x - x_0\|} \frac{1}{2^i}$: therefore f is well-def

• $\|f(x) - f(y)\| \leq \sum\limits_{i < \|x-y\|} \frac{i}{\delta_i} \times 2 + \sum\limits_{i \geq \|x-y\|} \frac{1}{2^i}$

$$\leq \sum_{i < \|x-y\|} \frac{2i}{\delta_i} + 2 = \rho_2(\|x-y\|).$$

Lower bound: it is enough to prove that $\forall h \in \mathbb{R}^+ \, \exists S \, \|x-y\| \geq S \Rightarrow \|f(x) - f(y)\| \geq h$.

or to prove that $\forall h \in \mathbb{R}^+ \, \exists S \, \exists i \, \|x-y\| \geq S \Rightarrow \frac{i}{\delta_i} \|f_i(x) - f_i(y)\| \geq h$: take $i \geq h$ use (ii). ■

We shall now prove a part of [O-S]: the superreflexive case.

th [partial O-S] Let X be uniformly convex and uniformly smooth with a 1-unconditional basis. Then $S_X \sim S_{\ell_1}$

Preparation: Assume X is uniformly smooth (US): $\forall x \in S_X \exists! x^* \in S_{X^*}$ $x^*(x)=1$ ($j(x)=x^*$) and j is unif cont.

If moreover X is UC (that is, X^* is US) (and X is reflexive), then $j_X : S_X \longrightarrow S_{X^*}$ and $j_{X^*}: S_{X^*} \to S_{X^{**}} = S_X$ is the inverse of j_X.

So $S_X \underset{UH}{\overset{j_X}{\sim}} S_{X^*}$

Lemma: If $\dim X = n$, if X is UC and US and if X has a 1-unc. basis, and if you look at X and X^* as sequence spaces whose canonical bases are 1-unconditional, and also as algebras for the coordinatewise multiplication, and if you let $F: S_X \longrightarrow S_{\ell_1^n}$, $x \longmapsto x\,|j(x)|$,

then (i) F is uniformly continuous and ω_F depends only on the modulus of smoothness of X.

if X is ℓ_p^n, F is the Mazur map.

(ii) F is 1-1, F^{-1} is uniformly continuous, $\omega_{F^{-1}}$ depends only on the modulus of convexity of X

(iii) F is onto.

Note that $j(|x|) = |j(x)|$: $\sum |x_i|\,|j(x)_i| = \langle |x|, j(|x|)\rangle = 1$.

(i) $\|F(x)-F(y)\|_1 = \|x|j(x)|-y|j(y)|\|_1 \le \|x(|j(x)|-|j(y)|)\|_1 + \|(x-y)|j(y)|\|_1$

$\le \|x(|j(x)|-|j(y)|)\|_1 + \|(x-y)|j(y)|\|_1$

$= \qquad\qquad + \langle |x-y|, j(|y|)\rangle$

$\le \sum |x_i|\,|j(x)_i - j(y)_i| + \qquad$

$= \langle |x|, |j(x)-j(y)|\rangle + \qquad$

$\le \|x\|_X \|j(x)-j(y)\|_{X^*} + \|x-y\|$

$= \|j(x)-j(y)\| + \|x-y\| \le \omega_j(\|x-y\|) + \|x-y\|$

(ii) Let $x, y \in S_X$, $f = F(x) = x|j(x)|$, $h = F(y) = y|j(y)|$.

Assume $\|f-h\|_1 = \varepsilon$. We want to show that $\|x-y\|_X \le \omega(\varepsilon)$.

It is enough to show that $\|\frac{x+y}{2}\| \ge 1-\lambda(\varepsilon)$, or that $\langle y, j(x)\rangle \ge 1-\mu(\varepsilon)$ ($\mu = 2\lambda$)

Let $\Lambda = \{i : \operatorname{sgn} f_i = \operatorname{sgn} h_i\}$: If $i \in \Lambda$, $\operatorname{sgn} x_i = \operatorname{sgn} y_i = \operatorname{sgn} f_i$. Wlog $\forall i \in \Lambda$ $f_i, h_i, x_i, y_i \ge 0$

Define $g : i \longmapsto \begin{cases} 0 & \text{if } i \notin \Lambda \\ \min(f_i, h_i) & \text{if } i \in \Lambda \end{cases}$. Note that $\|f\|_1 = \|h\|_1 = 1$ and $\sum |x_i|\,|j(x_i)| = \langle |x|, j(|x|)\rangle = 1$

We have $\boxed{\|g\|_1 = 1-\frac{\varepsilon}{2}}$: $\sum_{\Lambda^c} |f_i - h_i| = \varepsilon_1 = \underbrace{\sum'_{\Lambda^c} |f_i|}_{\varepsilon'_1} + \underbrace{\sum'_{\Lambda^c} |h_i|}_{\varepsilon''_1}$.

$0 = \sum'_{\Lambda^c} |g_i| = \sum'_{\Lambda^c} |f_i| - \varepsilon'_1 = \sum'_{\Lambda^c} |h_i| - \varepsilon''_1$

Let $\Lambda_f = \{i \in \Lambda : f_i \geq h_i\}$ and $\Lambda_h = \Lambda \setminus \Lambda_f$

Let $\varepsilon_2 = \sum_{\Lambda} |f_i - h_i| = \overbrace{\sum'_{\Lambda_f} f_i - h_i}^{\varepsilon'_2} + \overbrace{\sum'_{\Lambda_h} h_i - f_i}^{\varepsilon''_2}$

then $\sum_{\Lambda} g_i = \sum'_{\Lambda_f} h_i + \sum'_{\Lambda_h} f_i = \sum'_{\Lambda} f_i - \varepsilon'_2 = \sum'_{\Lambda} h_i - \varepsilon'_2$

$\|g\|_1 = \sum'_{\Lambda^c} |g_i| + \sum'_{\Lambda} g_i = 1 - (\varepsilon'_1 + \varepsilon'_2) = 1 - (\varepsilon''_1 + \varepsilon''_2)$ $\frac{\varepsilon}{2}$

and $\varepsilon_1 + \varepsilon_2 = \varepsilon$ and $\|g\|_1 = 1 - \frac{\varepsilon}{2}$.

$G = \operatorname{supp} g \subset (\Lambda \cap \operatorname{supp} x \cap \operatorname{supp} y)$. Then $\|g \frac{y}{x}\|_1 \leq \|f \frac{y}{x}\|_1 = \| |j(x)| |y| \|_1$

Thus $\|g(\frac{y}{x} + \frac{x}{y})\|_1 \leq 2$ $= \langle |y|, j(|x|)\rangle \leq 1$

Fix $\delta > 0$. Let $A = \{i \in G : \frac{y_i}{x_i} \wedge \frac{x_i}{y_i} < 1-\delta\}$: the set where x and y differ somewhat.

Note that $t < 1-\delta \Rightarrow t + \frac{1}{t} > \frac{1}{1-\delta} + 1 - \delta > 2 + \delta^2$. Let $C = G \setminus A$.

$2 \geq \|g(\frac{x}{y} + \frac{y}{x})\|_1 = \|\pi_A g(\frac{x}{y} + \frac{y}{x})\|_1 + \|\pi_C g(\frac{x}{y} + \frac{y}{x})\|_1 \geq (2+\delta^2)\|\pi_A g\|_1$
$+ 2\|\pi_C g\|_1$

so that $2 \geq 2\|g\|_1 + \delta^2 \|\pi_A g\|_1$ and $2 \geq 2 - \varepsilon + \delta^2 \|\pi_A g\|_1$.

So $\|\pi_A g\|_1 \leq \frac{\varepsilon}{\delta^2}$. Similarly, $\|\pi_C g\|_1 = \|g\|_1 - \|\pi_A g\|_1 \geq 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{\delta^2}$

and $\|\pi_C g \frac{y}{x}\|_1 \geq (1-\delta)(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{\delta^2})$ $\quad [\text{if } i \in C,\ \frac{y_i}{x_i} \geq 1-\delta]$

Let $\delta = \varepsilon^{1/3}$. Then $\langle \pi_G y, j(x)\rangle = \|\pi_G f \frac{y}{x}\|_1 \geq \|g \frac{y}{x}\|_1 \geq \|\pi_C g \frac{y}{x}\|_1$
$\geq (1 - \varepsilon^{1/3})(1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{\delta^2}) \geq 1 - 2\varepsilon^{1/3}$.

But $\langle |y|, |j(x)|\rangle \leq 1$, so $|\langle \pi_{G^c} y, j(x)\rangle| \leq 2\varepsilon^{1/3} \Rightarrow \langle y, j(x)\rangle \geq 1 - 4\varepsilon^{1/3}$.

(iii) F is a uniform homeomorphism from S_X into $S_{\ell_1^n}$. It is bound to be onto:

$S_X \underset{UH}{\sim} S_{\ell_2^n}$, $S_{\ell_1^n} \sim S_{\ell_2^n}$. But by Brouwer's theorem, $S_{\ell_2^n}$ is not contractible whereas every proper subset of $S_{\ell_2^n}$ is.

<u>Corollary</u>: <u>If</u> X is UC, US with a 1-unc. basis, <u>then</u> $S_X \underset{UH}{\sim} S_{\ell_1}$

<u>Proof</u>: Let $\{x_i\}$ be a 1-unc. basis of X and $\{e_i\}$ be the canonical basis of ℓ_1.

The lemma shows that $\forall A \subset \mathbb{N}$ finite $\exists F_A : S_{\operatorname{span}\{x_i : i \in A\}} \longrightarrow S_{\operatorname{span}\{e_i : i \in A\}}$ and

$\omega_{F_A}, \omega_{F_A^{-1}}$ do not depend on A; the F_A's are compatible

This yields a uniform homeomorphism from S_X onto S_{ℓ_1}

<u>Announcement</u>: We shall show that X with u.b. and $\ell_\infty^n \not\subset X$, then the 2-convexification $X^{(2)}$ of X is 2-convex and q-concave for some $q < \infty$: apply Th 0 to $X^{(2)}$: $S_X \underset{UH}{\sim} S_{X^{(2)}}$

Note that for X with unc. basis, $S_X \underset{UH}{\sim} S_{\ell_2} \Leftrightarrow \ell_\infty^n \not\subseteq X$

($\Rightarrow$ is due to Enflo.)

Remark: $\ell_2 \underset{L}{\hookrightarrow} c_0$

Open question: X unc. basis, $\ell_\infty^n \subseteq X \overset{?}{\Rightarrow} \ell_2 \underset{e}{\hookrightarrow} X$

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More on coarse embeddings of ℓ_2.

Th (Nowak 2006) If $1 \le p < \infty$, then $\ell_2 \underset{c}{\hookrightarrow} \ell_p$.

Oshovskii went further using the following:

Th (Odell - Schlumprecht 2004) Let X be a Bspace with an unconditional basis and nontrivial cotype (ℓ_∞^n don't unif^{ly} embed into X) "$\ell_\infty^n \not\subseteq X$"
Then $S_X \underset{UH}{\sim} S_{\ell_1}$.

Corollary: (Oshovskii) Let X satisfy the same conditions. Then $\ell_2 \underset{c}{\hookrightarrow} X$.

Th (O-S) Let X have 1-u.b. and be uniformly convex and uniformly smooth. Then $S_X \underset{UH}{\sim} S_{\ell_1}$.

p-convexity and q-concavity: For X with a 1-u.b. (e_i), we will write $(x_i) \in X$ if $\sum x_i e_i \in X$. For $t > 0$ and $x = (x_i) \in X$, $|x|^t = (|x_i|^t)$.
For $1 \le p, q \le \infty$, say X is p-convex if $\forall\ (x_j)_{j=1}^n \subseteq X$ $\|(\sum |x_j|^p)^{1/p}\| \le M(\sum \|x_j\|^p)^{1/p}$ for some $M > 0$. The smallest M is denoted $M^p(X)$.
Every X is 1-convex. Say X is q-concave if the reverse inequality holds, and $M_q(X)$ is defined similarly.
By duality, X is p-~~convex~~ iff X^* is p'-concave.

p-convexification of X: this is $X^{(p)} = \{(x_i) : \|(x_i)\|_{(p)} = \|\sum |x_i|^p e_i\|^{1/p} < \infty\}$

If X has a 1-unconditional basis (e_i), $X^{(p)}$ has also a 1-unconditional basis again denoted by (e_i).

If X is r-convex, then for $x, y \in X^{(p)}$ $\|(|x|^{pr} + |y|^{pr})^{\frac{1}{pr}}\|_{(p)}^{pr} = \|(|x|^{pr} + |y|^{pr})^{\frac{1}{r}}\|^{r}$
$\le M^r(X)^r (\|x\|^{pr} + \|y\|^{pr})$

so that $M^{pr}(X^{(p)}) \le M^r(X)^{\frac{1}{p}}$.

More generally, if X is r-convex and s-concave, $1 \le r \le s < \infty$, then $X^{(p)}$ is pr-convex and ps-concave with $M^{pr}(X^{(p)}) \le M^r(X)^{1/p}$
$M_{ps}(X^{(p)}) \le M_s(X)^{1/p}$.

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<u>Proposition 1</u>: Let X have a 1-ub and $M_q(X)=1$. Then $M^p(X^{(p)})=1=M_{pq}(X^{(p)})$.

<u>Proposition 2</u>: Let $1<q<\infty$ and X q-convex and with 1-ub (e_i). Then X admits an equivalent norm for which $M_q(X)=1$ and (e_i) is 1-ub.

<u>Proposition 3</u>: Let $1<p\leq 2\leq q<\infty$ and X with 1-ub and $M^p(X)=M_q(X)$. Then X is uniformly smooth and uniformly convex.

<u>Proof</u>: by duality, it suffices to show that X is u.c.

<u>Lemma</u>: Let $q\geq 2$. Then for any $1<p<\infty$ there exists $C=C(p,q)$ st

$$\left|\frac{s-t}{C}\right|^q+\left|\frac{s+t}{2}\right|^q\leq\left(\frac{|s|^p+|t|^p}{2}\right)^{1/p}$$

<u>Proof</u>: We may assume that $s=1>t\geq -1$. Consider $\varphi(t)=\left(\frac{1+|t|^p}{2}\right)^{\frac{q}{p}}-\left(\frac{1+t}{2}\right)^q$

φ is ≥ 0 on $[-1,1[$ and $\varphi''(1)>0$, so $\frac{\varphi(t)}{(1-t)^2}$ is bounded from below, hence so is $\frac{\varphi(t)}{(1-t)^q}$.

Fix $\varepsilon>0$ and let $x,y\in S_X$ be st $\|x-y\|=\varepsilon$. By the lemma, there is $C=C(p,q)$ s.t. $\left\|\left(\left(\frac{|x-y|}{C}\right)^q+\left(\frac{|x+y|}{2}\right)^q\right)^{1/q}\right\|\leq\left\|\left(\frac{|x|^p+|y|^p}{2}\right)^{\frac{1}{p}}\right\|$

Apply $M^p(X)=1=M_q(X)$ to obtain $\left(\left\|\frac{x-y}{C}\right\|^q+\left\|\frac{x+y}{2}\right\|^q\right)^{\frac{1}{q}}\leq\frac{(\|x\|^p+\|y\|^p)^{1/p}}{2^{1/p}}$

Thus $\frac{\varepsilon^q}{C^q}\leq 1-\left\|\frac{x+y}{2}\right\|^q\leq q\left(1-\frac{\|x+y\|}{2}\right)$. Hence $\delta_X(\varepsilon)\geq\frac{\varepsilon^q}{C^q q}$.

<u>Proposition 4</u>: Let $1<p<\infty$ and X has 1-ub (e_i). Then the 'Mazur' map

$$G_{p,X}: S_{X^{(p)}}\longrightarrow S_X$$
$$x\longmapsto\left((\operatorname{sgn} x_i)\,|x_i|^p\right)$$

is a uniform homeomorphism

<u>Proof</u>: Let $x=(x_i)$, $y=(y_i)\in S_{X^{(p)}}$. Let $I_+=\{i:\operatorname{sgn}x_i=\operatorname{sgn}y_i\}$, $I_-=\{\quad\neq\quad\}$

It suffices to show that for $\delta=\|x-y\|_{(p)}$, $2^{1-p}\delta^p\leq\|G_{p,X}(x)-G_{p,X}(y)\|\leq\delta^p\delta^{\frac{p}{2}}+(1-(1-\delta)^p)$

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$$\|G_{p,X}(x) - G_{p,X}(y)\| = \Big\|\sum{}' \big((\operatorname{sgn} x_i)|x_i|^p - (\operatorname{sgn} y_i)|y_i|^p\big) e_i\Big\|$$

$$= \Big\|\underbrace{\sum_{i\in I_+}{}' \big(|x_i|^p - |y_i|^p\big) e_i}_{=d_+} + \underbrace{\sum_{i\in I_-}{}' \big(|x_i|^p + |y_i|^p\big) e_i}_{=d_-}\Big\|$$

By 1-u. of (e_i) and $a^p - b^p \geq (a-b)^p$
and $a^p + b^p \geq 2^{1-p}(a+b)^p$ for $a \geq b \geq 0$

so that $\|d_+ + d_-\| \geq \Big\|\sum_{i\in I_+}{}' \big||x_i|-|y_i|\big|^p e_i + 2^{1-p}\sum_{i\in I_-}{}' \big(|x_i|+|y_i|\big)^p e_i\Big\|$

$\geq 2^{1-p}\Big\|\sum |x_i - y_i|^p e_i\Big\| = 2^{1-p}\|x-y\|^p_{(p)}$.

For the upper estimate, first note that $\|d_-\| \leq \Big\|\sum_{i\in I_-}{}' |x_i - y_i|^p e_i\Big\| \leq \|x-y\|^p_{(p)} \leq \delta^p$

Set $q = 1-\sqrt{\delta}$ and $c = (1-q)^{-p} = \delta^{-p/2}$

For $a,b \geq 0$ with $0 \leq b \leq qa$, we have $c(a-b)^p - (a^p - b^p) \geq c(1-q)^p a^p - a^p$
$= a^p\big(c(1-q)^p - 1\big)$
$= 0$.

Let $I'_+ = \{ i \in I_+ : |y_i| < q|x_i| \text{ or } |x_i| < q|y_i| \}$

$I''_+ = I_+ \setminus I'_+$. Then $d_+ = d'_+ + d''_+$, where $d'_+ = \sum_{i\in I'_+}{}' \big(|x_i|^p - |y_i|^p\big) e_i$.

By $(*)$, $\|d'_+\| \leq c\Big\|\sum_{i\in I'_+}{}' \big||x_i| - |y_i|\big|^p e_i\Big\| \leq \delta^{-p/2}\|x-y\|^p_{(p)} = \delta^{p/2}$

Proof of Odell-Schlumprecht: Let X have q-concave ub (e_i) and $\ell^n_\infty \not\subseteq X$

We need the following:

Th (Maurey-Pisier) Let X be a Bspace with same hypotheses:
Then X is q-concave for every $q > q'$.

By an equivalent renorming, we may assume (e_i) is monotone.

By Maurey-Pisier X is q-concave for some $q \in [2, \infty[$.

By Prop'n 2, we may assume $M_q(X) = 1$, hence $M^2(X^{(2)}) = 1 = M_{2q}(X^{(2)})$
by Prop'n 1, note also that $X^{(2)}$ has a 1-ub. Moreover by Prop'n 3,
$X^{(2)}$ is UC and US and hence $G_{2,X} : S_{X^{(2)}} \longrightarrow S_X$ is a uniform
homeomorphism (Prop'n 4).

By Odell-Schlumprecht theorem from last week, there is a uniform homeomorphism $F: S_{X(2)} \longrightarrow S_{\ell_1}$. Then $F \circ G_{2,\lambda}^{-1} : S_X \longrightarrow S_{\ell_1}$ is a uniform homeomorphism.

There is a converse: X with 1-u.b.: If $S_X \underset{UH}{\sim} S_{\ell_2}$, then $\ell_\infty^n \not\subseteq X$.

(Enflo's result.)

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Recall [Odell-Schlumprecht]: If X has an u.b. and $\ell_\infty^n \not\subseteq X$, then $S_X \underset{UH}{\sim} S_{\ell_2} \underset{UH}{\sim} S_{\ell_2}$

This theorem has a converse due to Enflo: If the ℓ_∞^n embed uniformly into X, then $S_X \not\underset{UH}{\sim} S_{\ell_2}$.

We will construct a sequence of finite metric spaces M_n that are uniformly bounded so that every sequence (T_n) of embeddings $M_n \to \ell_2$ is not equi-bi-uniform.

Recall Def: $(T_n : A_n \to B_n)$ is equi-bi-uniform if $\forall \varepsilon>0 \exists \delta>0 \forall n$ ~~$\forall x,y \in A_n$~~

$$\begin{cases} \forall x,y \in A_n \quad d(x,y)<\delta \Rightarrow d(T_n x, T_n y) < \varepsilon \\ \forall T_n x, T_n y \in B_n \quad d(T_n x, T_n y) < \delta \Rightarrow d(x,y) < \varepsilon \end{cases}$$

is this true?! [Note that • $S_X \underset{UH}{\overset{\varphi}{\sim}} S_{\ell_2} \Rightarrow B_X \underset{UH}{\overset{\tilde\varphi}{\sim}} B_{\ell_2}$, putting $\tilde\varphi(x) = \|x\| \, \varphi\left(\frac{x}{\|x\|}\right)$ (radial extension)

• $\ell_\infty^n \subseteq X \iff \exists \tilde T_n : \ell_\infty^n \xrightarrow{\text{linear}} X \quad \frac{\|x-y\|}{C} \le \|\tilde T_n x - \tilde T_n y\| \le \|x-y\|$

• for every finite metric space M, there is an isometry $I : M \to \ell_\infty^{|M|}$ called Frechet map: $I(m) = (d(m,n))_{n\in M}$

$$\max_n |d(m,n) - d(m',n)| = d(m,m').$$

We will consider $T_n = \varphi \circ \tilde T_n \circ I_n$, where $I_n : M_n \to \ell_\infty^{|M_n|}$

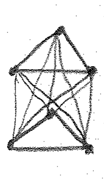
Let us consider the metric in the metric space, denoted by d.

Let M be a metric space. A segment s in M is $s \subset M$, $\# s = 2$ (If $\# s = 1$, it is a trivial segment). The length of s is $\ell(\{a,b\}) = d(a,b)$.

A double n-simplex in M is a couple $\{A,B\}$, where $A \subset M$, $B \subset M$, $\# A \cup B = 2n$. $A = \{a_1,\dots,a_n\}$, $B = \{b_1,\dots,b_n\}$.

Their set is denoted by $\Delta_n(M)$.

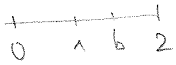
Let $D \in \Delta_n(M)$: then any segment $\{a_i,a_j\}$ or $\{b_i,b_j\}$, $i\neq j$, is called an edge of D. $\{A,B\}$ Their set is $E(D)$. Any segment $\{a_i,b_j\}$ is called a connecting line of D. Their set is $C(D)$


A
B

Def: M has generalised roundness $p \in \mathbb{R}$ [$\Theta(M) = p$] if p is the supremum of those q with the property $\sum_{c\in C(D)} \ell(c)^q > \sum_{e\in E(D)} \ell(e)^q$ for $n\in\mathbb{N}$ and $D \in \Delta_n(M)$

• For every m. sp. M we have $\Theta(M)\geq 0$ because $\#E(\Delta)=n(n-1)$, $\#C(\Delta)=n^2$.

• If the segment $[0,2]$ sits isometrically in M, then $\Theta(M)\leq 2$:



Let $\Delta_b=\{\{0,2\},\{1,b\}\}\subset[0,2]$ and define $f_p(b)=\sum'_{c\in C(\Delta)}\ell(c)^p-\sum'_{e\in E(\Delta)}\ell(e)^p$.

Then $f_p(b)\xrightarrow[b\to 1]{} 2\cdot 2\cdot 1^p-(0+2^p)=4-2^p$ which is ≥ 0 iff $p\leq 2$.

Lemma $\Theta(L_2(\mu))=2$

Proof: $\leq$ is ok. We need $\sum'_{c\in C(\Delta)}\ell(c)^2\geq\sum'_{e\in E(\Delta)}\ell(e)^2$ for $\Delta\in\Delta_n(L_2(\mu))$.

$$\int \sum'_{1\leq i,j\leq n}(a_i-b_j)^2-\sum'_{1\leq i<j\leq n}(a_i-a_j)^2-\sum'_{1\leq i<j\leq n}(b_i-b_j)^2\,d\mu\geq 0$$

because

$$=\Big(\sum'_i a_i-\sum'_j b_j\Big)^2$$

$\sum'(a_i-b_i)^2+2\sum'_{i<j}(a_i-b_i)(a_j-b_j)$

We construct (M_n) so that $\forall Y\ \Theta(Y)>0$, $\forall T_n:M_n\to Y$ (T_n) is not e-b-u.
The initial idea is to take $\Delta\in\Delta_n(M_n)$ with long edges and short connecting lines: if $T_n:M_n\to Y$ and $\Theta(Y)=p>0$, (T_n) e-b-u,
then

$$\sum'_{c\in C(\Delta)}\ell(T_nc)^p\geq\sum'_{e\in E(\Delta)}\ell(T_ne)^p\geq n(n-1)\inf_{e\in E(\Delta)}\ell(T_ne)^p\geq n(n-1)\varepsilon$$

$$n^2\sup_{c\in C(\Delta)}\ell(T_nc)^p\xrightarrow[\sup\ell(c)\to 0]{}0$$

because T_n^{-1} is u.c.

This is not possible as $\sup_{c\in C(\Delta)}\ell(c)\geq\frac{1}{2}\sup_{e\in E(\Delta)}\ell(e)$.

Possible distances in M_n: the ℓ(segments of M) may be ordered as $\ell_1^n<\ell_2^n<\dots<\ell_{2m}^n$ if we have constructed M_n in the way below.

We construct M_n so that $\lim\ell_1^n=0$.

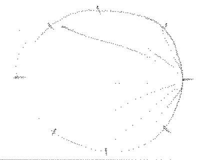
Imagine that for all $n\in\mathbb{N}$ and $i\in[1,n-1]$ there is $\Delta\in\Delta_n(M_n)$ so that $\ell(e)=\ell_{2i}^n$ for $e\in E(\Delta)$,

$\ell(c)=\ell_{2i-1}^n$ for $c\in C(\Delta)$.

The space M_n:

Let n be even and put $A_n = \{e^{\frac{2\pi i k}{2^{n+1}}} : k \in [\![0, 2^{n+1}-1]\!]\}$ with the C-distance.

$M_n = \prod_{i=1}^{n^n} A_n$ together with the sup metric.



$n=2$

If $m \in M_n$, $m = (m_1, \dots, m_{n^n})$

$\tilde{m} \in M_n$, $d(m, \tilde{m}) = \sup_{i \in [\![1, n^n]\!]} d(m_i, \tilde{m}_i)$

A segment $s \in M_n$ is called an i-segment ($s \in S_n^i$) if $\ell(s) = \ell^n_{2^{i-1}}$.

If $s = \{a, b\}$, $a = (a_1, \dots, a_{n^n})$, $b = (b_1, \dots, b_{n^n})$, we require $d(a_j, b_j) = \ell^n_{2^{i-1}}$ for $n^{(n-i)}$ indices j, and $d(a_j, b_j) = 0$ otherwise.

Call $\Delta_n^i(M_n) = \{\Delta \in \Delta_n(M_n) : \forall c \in C(\Delta)\ c \in S_n^i,\ \forall e \in E(\Delta)\ e \in S_n^{i+1}\}$

Assume that $\Delta_n^i(M_n) \neq \emptyset$ for $i \in [\![1, n-1]\!]$ and all n.

Notice that if $n \in \mathbb{N}$ and $i \in [\![1, n-1]\!]$, there is $N_C \in \mathbb{N}$ s.t. for all $s \in S_n^i$
$\#\{\Delta \in \Delta_n^i : s \in C(\Delta)\} = N_C$; whenever $s, t \in S_n^i$ there is an isometry
$f: M_n \to M_n$ s.t. $fs = t$; if $D_s = \{j : s_1(j) \neq s_2(j)\}$, then $\# D_s = n^{n-i}$, $\# D_t = n^{n-i}$

Also $\exists\ N_E \in \mathbb{N}\ \forall s \in S_n^{i+1}$ $\#\{\Delta \in \Delta_n^i : s \in E(\Delta)\} = N_E$.

If $\Delta \in \Delta_n(M_n)$, $\sum_{c \in C(\Delta)} \ell(T_n c)^p \geq \sum_{e \in E(\Delta)} \ell(T_n e)^p$

Then we sum: $\sum_{\Delta \in \Delta_n^i} \sum_{c \in C(\Delta)} \ell(T_n e)^p \geq \sum_{\Delta \in \Delta_n^i} \sum_{e \in E(\Delta)} \ell(T_n e)^p$.

$$\underbrace{N_C (\# S_n^i)}_{\# \Delta_n^i \cdot n^2} (\# S_n^i)^{-1} \sum_{s \in S_n^i} \ell(T_n s)^p \geq \underbrace{N_E (\# S_n^{i+1})}_{\# \Delta_n^i \cdot n(n-1)} (\# S_n^{i+1})^{-1} \sum_{s \in S_n^{i+1}} \ell(T_n s)^p.$$

By telescoping, $(\# S_n^1)^{-1} \sum_{s \in S_n^1} \ell(T_n s)^p \geq \left(1 - \frac{1}{n}\right)^{n-1-1} (\# S_n^n)^{-1} \sum \ell(T_n s)^p \geq \frac{1}{e} \inf_{s \in S_n^n} \ell(T_n s)^p$

$\sup_{s \in S_n} \ell(T_n s)^p \geq$

as (T_n) is an e-b-u

$\downarrow$

0 . Contradiction.

④

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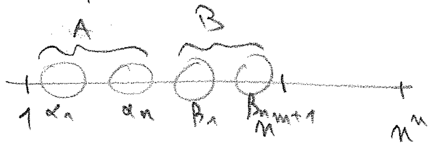
<u>Last lemma</u>: to prove that $\Delta_n^i(\Pi_n) \neq \emptyset$.

proof :



$2n$ groups of $\frac{n^m}{2}$ coordinate each

Let $\Delta = \{A, B\}$ For $a \in A$, ~~pick~~ $i \in [\![1,n]\!]$ and put $a_j = e^{\frac{\pi i}{2m}}$ for $j \in \alpha_i$.

$a = (a_1, \dots, a_{n^m})$

$b = (b_1, \dots, b_m)$ For $\ell \in [\![1,n]\!]$, $\ell \neq i$, put $a(j) = 1$ for $j \in \bigcup_{\ell \neq i} \alpha_\ell$ and $a(j) = e^{\frac{\pi i}{2m+1}}$ for the other j.

For B, to replace α by β.

Q : Does this follow from metric cotype ?

But $S_{\ell p} \underset{UH}{\cong} S_{\ell 2}$

$B_{\ell p} \underset{U}{\not\to} B_{\ell 2}$. (at least for the spaces themselves instead of their unit balls).

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Th: If X B-space, $\ell_\infty^n \subseteq X$ (uniformly $/n$), then $S_X \underset{UH}{\approx} S_{\ell_2}$

The proof is based on: ① If $S_X \underset{UH}{\approx} S_Y$, then $B_X \underset{UH}{\approx} B_Y$ (Lemma 1)
(we are not saying that $X \underset{UH}{\approx} Y$; this could contradict metric ody representation)

② $B_{c_0} \underset{unif}{\not\hookrightarrow} B_{\ell_2}$ (theorem 1, Raynaud 1983)

③ If $B_{\ell_\infty^n} \underset{equi-unif}{\hookrightarrow} \ell_2$, then $B_{c_0} \underset{unif.}{\hookrightarrow} \ell_2$ (theorem 2)

Proof of Lemma 1: put $\psi(x) = \|x\| \varphi\left(\frac{x}{\|x\|}\right)$: then $\psi^{-1}(z) = \|z\| \varphi^{-1}\left(\frac{z}{\|z\|}\right)$
Also $\|x\| = \|\psi(x)\|$, so that $\psi(B_X) = B_Y$

Let $\varepsilon > 0$ be fixed, let $x, y \in B_X$: • If $\|x\| \vee \|y\| < \frac{\varepsilon}{2}$, then $\|\psi(x) - \psi(y)\| < \varepsilon$

• Otherwise $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \leq \frac{2\|x-y\|}{\|x\| \vee \|y\|} \leq \frac{4}{\varepsilon}\|x-y\|$; choose $\delta < \frac{\varepsilon}{2}$ s.t.
$\omega_\varphi\left(\frac{4}{\varepsilon}\delta\right) < \frac{\varepsilon}{2}$; then $\underbrace{\left\|\varphi\left(\frac{x}{\|x\|}\right) - \varphi\left(\frac{y}{\|y\|}\right)\right\|}_{q} < \frac{\varepsilon}{2}$
and $\left\|\|x\| \varphi\left(\frac{x}{\|x\|}\right) - \|y\| \varphi\left(\frac{y}{\|y\|}\right)\right\| \leq \big|\|x\| - \|y\|\big| + q < \varepsilon$

We will now prove theorem 1. First define:

Def: Let (M, d) be a metric space. It is stable if for all free ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$ and all $(x_n), (y_n) \subset M$, bounded, we have
$$\lim_{\mathcal{U}, n} \lim_{\mathcal{V}, m} d(x_n, y_m) = \lim_{\mathcal{V}, m} \lim_{\mathcal{U}, n} d(x_n, y_m).$$

Examples: • ℓ_2 is stable because $\|x_n - y_m\|^2 = \|x_n\|^2 - 2\langle x_n, y_m \rangle + \|y_m\|^2$
and B_{ℓ_2} is w-cpct: $w\text{-}\lim_{\mathcal{U}, n} x_n = x \in B_{\ell_2}$, $w\text{-}\lim_{\mathcal{V}_m} y_m = y$: then
$$\lim_{\mathcal{U}} \lim_{\mathcal{V}} \|x_n - y_m\|^2 = \lim_{\mathcal{U}} \left(\|x_n\|^2 - \langle x_n, y \rangle + \lim_{\mathcal{V}} \|y_m\|^2\right)$$
$$= \lim_{\mathcal{U}} \|x_n\|^2 - \langle x, y \rangle + \lim_{\mathcal{V}} \|y_m\|^2$$

• c_0 is not stable: consider $x_n = (\overbrace{1, \dots, 1}^{n}, 0, \dots)$
$y_n = (\underbrace{0, \dots, 0, -1}_{n}, 0, \dots)$

Def: A basis (x_n) of X is <u>K-spreading</u> if $\forall N\geq 1\ \forall a_1,\dots,a_N \in \mathbb{R}\ \forall n_1<n_2<\dots<n_N$

$$\frac{1}{K}\left\|\sum_1^N a_n x_n\right\| \leq \left\|\sum_1^N a_k x_{n_k}\right\| \leq K\left\|\sum_1^N a_n x_n\right\|$$

examples: canonical basis of ℓ_p or c_0, summing basis of c_0

Def: A basis (x_n) of X is <u>bimonotone</u> if the basis projections satisfy $\|P_n\| = \|I-P_n\| = 1$.

Lemma: If (x_n) is K-spreading, X can be renormed so that (x_n) is 1-spreading and bimonotone.

Proof: exercise: consider $|||\sum_1 a_n x_n||| = \sup\|\sum' a_k x_{n_k}\|$

then $|||\sum_1 a_n x_n||| \geq |||\sum_1 a_k x_{n_k}|||$ but $\leq$ is not so clear!

Theorem: Let X be a B-space with a K-spreading basis (x_n), s.t. $B_X \underset{UH}{\sim}$ to some stable metric space. Then (x_n) is unconditional.

(If $X \overset{\varphi}{\sim} (\Omega, d)$, let us write $d(\varphi(x),\varphi(y)) = d(x,y)$.)

Proof We may suppose that (x_n) is 1-spreading and bimonotone.

Fix $m\in\mathbb{N}$, $(a_i)_{i=1}^m \subset \mathbb{R}$, $(\vartheta_i)_{i=1}^m \subset \{-1,+1\}$; let $\alpha>0$, $r>0$ be such that

$d(x,y)<\alpha$ implies $\|x-y\|<\frac{1}{2}$ (u.c. of φ^{-1})

$\|x-y\|<r$ implies $d(x,y)<\frac{\alpha}{2}$ (u.c. of φ.)

It is enough to prove that (*) $1 = \|\sum' a_i x_i\| \geq \|\sum' \vartheta_i a_i x_i\| \Rightarrow B \geq r$.

We shall prove (**): $\|\sum_{i=1}^m x_i\| \geq \|\sum' \vartheta_i x_i\|$ implies $\|\sum' \vartheta_i x_i\| \geq r\|\sum' x_i\|$

We have $1 = \left\|\sum_{i=1}^m x_i\right\| \underset{\text{1-spreading}}{=} \left\|\sum_{k=1}^m x_{n_k}\right\| \leq \Big\|\underbrace{\sum_{i\leq \ell} x_{n_i}}_{x}\Big\| + \Big\|\underbrace{\sum_{i>\ell} x_{n_i}}_{y}\Big\|$

$\leq 2\Big\|\underbrace{\sum_{i\leq\ell} x_{n_i} - \sum_{i>\ell} x_{n_i}}_{z}\Big\|$

because $\|x\| = \|P_\ell z\| \leq \|z\|$, $\|-y\| = \|(I-P_\ell)z\| \leq \|z\|$ with $\ell = \#\{i : \vartheta_i = +1\}$

Then $\|z\| \geq \frac{1}{2}$ and $d\left(\sum'_{i\leq\ell} x_{n_i}, \sum'_{i>\ell} x_{n_i}\right) \geq \alpha$ for every sequence $n_1<\dots<n_m$

Then (A) $= \lim_{\mathcal{U}_1, n_1} \lim_{\mathcal{U}_2, n_2} \dots \lim_{\mathcal{U}_m, n_m} d\left(\sum'_{i\leq\ell} x_{n_i}, \sum'_{i>\ell} x_{n_i}\right) \geq \alpha$

(B) $= \underline{\qquad\qquad\qquad} \; d\left(\sum'_{\vartheta_i=+1} x_{n_i}, \sum_{\vartheta_i=-1} x_{n_i}\right) \leq \sup\Big\{ d(x,y) : \|x-y\| \leq \Big\|\sum_{i=1}^m \vartheta_i x_i\Big\|\Big\}$

$\Big\|\sum_{\vartheta_i=1} x_{n_i} - \sum_{\vartheta_i=-1} x_{n_i}\Big\|$

Tony ③

1/12/2011

stability implies $A=B$, so that $\alpha \le \sup\{d(x,y) : \cdots\}$

If $\|\sum_i \theta_i x_i\| < r$, then $\sup\{ \quad \} \le \frac{\alpha}{2}$ by definition of r

Let us prove this stability: changing the order in which the limits are taken corresponds to mixing the x_i's affected with $+1$ and the x_i's affected with -1. (see Benyamini-Lindenstrauss for details!)

Theorem: If finite subsets of a metric space (M,d) embed equi-unif$^{\ell y}$ into a Hilbert space, then (M,d) itself does.

Cor: $B_{\ell_\infty^n} \underset{\text{equiunif}}{\not\hookrightarrow}{}^{\ell y} \ell_2$

$$\boxed{\text{Th 1: } B_{c_0} \underset{\text{unif}}{\not\hookrightarrow}{}^{\ell y} \ell_2}$$

proof: If we had $B_{c_0} \underset{\text{unif.}}{\hookrightarrow} \ell_2$, we would get a contradiction as the summing basis is 1-spreading and not unconditional.

2 options → get to studying kernels

→ give Kalton's proof (interlaced graphs as a generalisation of these sequences)

Kalton's graphs.

(1) Michal Kraus 8/12/2011

Last time: (Raynaud 1983) $B_{c_0} \not\hookrightarrow_{unif} \ell_2$. In particular, $c_0 \not\hookrightarrow_{unif} \ell_2$

<u>Theorem</u> (Kalton 2007): Let X be a B-space so that $X^{(n)}$ is separable for all n. Then c_0 does not coarsely or uniformly embed into X.

<u>Cor</u>: c_0 does not coarsely nor unif^{ly} embed into a reflexive space.

<u>Def</u>. $M \subseteq \mathbb{N}$ infinite, $k \in \mathbb{N}$, $G_k(M) = \{\sigma \subset M : |\sigma| = k\}$

$$G_0(M) = \{\emptyset\}$$

$\sigma, \tau \in G_k(M)$, $\sigma = (m_1, \dots, m_k)$ in increasing order;
$\tau = (n_1, \dots, n_k)$

σ, τ define an edge if $m_1 \le n_1 \le m_2 \le n_2 \le \dots \le m_k \le n_k$
or $n_1 \le m_1 \le n_2 \le m_2 \le \dots \le n_k \le m_k$.

Then $G_k(M)$ is a connected graph. We put a metric d on $G_k(M)$:

- any edge has length 1
- if $\sigma, \tau \in G_k(M)$, $d(\sigma, \tau)$ is the geodesic distance.

Then $\operatorname{diam} G_k(M) = k$ (For example, $d(\sigma, \tau) = k$ if $m_k < n_1$ or $n_k < m_1$.

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let X be a B-space and $f : G_k(\mathbb{N}) \xrightarrow{bdd} X$

Define $\partial f : G_{k-1}(\mathbb{N}) \longrightarrow X^{**}$

$$(m_1, \dots, m_{k-1}) \longmapsto w^* \lim_{m_k \in \mathcal{U}} f(m_1, \dots, m_k)$$ by w^*-cpacity of bdd sets in X^{**}.

By iterating, we obtain $\partial^i f : G_{k-i}(\mathbb{N}) \longrightarrow X^{(2i)}$ for $1 \le i \le k$.

$\partial^k f$ is a constant map: $\partial^k f \in X^{(2k)}$

If X is reflexive, then $\partial^i f : G_k(\mathbb{N}) \longrightarrow X$ and $\partial^k f \in X$.

<u>Lemma 1</u>: Let $h : G_k(\mathbb{N}) \longrightarrow \mathbb{R}$ be a bounded f^n and $\varepsilon > 0$.
Then there is an infinite $M \subset \mathbb{N}$ such that $\forall \sigma \in G_k(M)$ $|h(\sigma) - \partial^k h| < \varepsilon$.

②

Proof: by induction: Let $M = \{m_1, m_2, \dots\}$, $m_1 < m_2 < \dots$ so that

$\forall i \in \mathbb{N}$ m_i is picked so that • if $\sigma \subset \{m_1, \dots, m_i\}$, $1 \le |\sigma| \le k$, then $|\partial^{k-|\sigma|} h(\sigma) - \partial^k h| < \varepsilon$

It is possible: pick $\underline{m_1}$ so that $|\partial^{k-1} h(m_1) - \partial^k h| < \varepsilon$. (def of ∂^k)

~~if~~ $m_1, \dots, m_i$ picked, pick m_{i+1} so that ..., but we have to check the condition only for $\sigma \ni m_{i+1}$; in order to do so, fix $\sigma \subset \{m_1, \dots, m_i\}$, $1 \le |\sigma| \le k-1$, ~~and~~ then $|\partial^{k-|\sigma|-1} h(\sigma \cup \{m\}) - \partial^k h| < \varepsilon$ holds for m in an element $A_\sigma \in \mathcal{U}$, ~~choose $m_{i+1} \in A_\sigma$ with $m_{i+1} > m_i$~~

Then $A = \bigcap_{\substack{\sigma \subset \{m_1, \dots, m_i\} \\ 1 \le |\sigma| \le k-1}}$ satisfies $A \ne \emptyset$ and $A \in \mathcal{U}$.

Choose $m_{i+1} \in A$: then (1) holds for $\{m_1, \dots, m_{i+1}\}$.

Def: X B-space, $f : G_k(\mathbb{N}) \longrightarrow X$, $g : G_k(\mathbb{N}) \longrightarrow X^*$ bounded maps.

Define $f \otimes g : G_{2k}(\mathbb{N}) \longrightarrow \mathbb{R}$
$(m_1, \dots, m_{2k}) = \langle f(m_2, \dots, m_{2k}), g(m_1, \dots, m_{2k-1}) \rangle$

<u>Lemma 2</u>: $\partial^{2i}(f \otimes g) = \partial^i f \otimes \partial^i g$ for $1 \le i \le k$.

<u>Lemma 3</u>: $f : G_k(\mathbb{N}) \longrightarrow X$ bounded maps and $\varepsilon > 0$: there is an infinite $M \subset \mathbb{N}$ so that $\|f(\sigma)\| < \|\partial^k f\| + \omega_f(1) + \varepsilon$ for $\sigma \in G_k(M)$

$\omega_f(1) = \sup\{\|f(\sigma) - f(\tau)\| : d(\sigma, \tau) \le 1\}$

Proof: Define $g : G_k(\mathbb{N}) \to X^*$ so that $\|g(\sigma)\| = 1$ and $\langle f(\sigma), g(\sigma) \rangle = \|f(\sigma)\|$

By Lemma 2, we have $|\partial^{2k}(f \otimes g)| = |\partial^k(f) \otimes \partial^k(g)| = |\langle \partial^k f, \partial^k g \rangle| \le \|\partial^k f\| \underbrace{\|\partial^k g\|}_{\le 1} \le \|\partial^k f\|$

By Lemma 1, there is an infinite $A \subset \mathbb{N}$ so that ~~$f \otimes g$~~ $|f \otimes g(\sigma) - \partial^{2k}(f \otimes g)| \le \varepsilon$ for $\sigma \in G_k(A)$

Let $A = \{m_1 < n_1 < m_2 < n_2 < \dots\}$ and $M = \{m_1, m_2, \dots\}$

Then, if $\sigma \in G_k(M)$, $\sigma = (m_{u_1}, \dots, m_{u_k})$, then $\|f(\sigma)\| = \langle f(\sigma), g(\sigma) \rangle$
$= \langle f(\sigma) - f(n_{u_1}, \dots, n_{u_k}), g(\sigma) \rangle + \langle f(n_{u_1}, \dots, n_{u_k}), g(\sigma) \rangle$ $\;\; = f \otimes g(m_{u_1}, n_{u_1}, \dots, m_{u_k}, n_{u_k})$

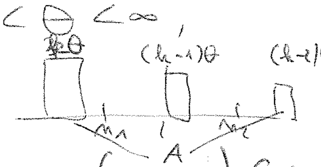
$\| \le \omega_f(1)$ since σ is at distance 1 from $(n_{u_1}, \dots, n_{u_k})$.

Lemma 4. Let $k \in \mathbb{N}$, $\varepsilon > 0$, X B-space so that $X^{(2k)}$ is separable.
If $f_i : G_k(\mathbb{N}) \longrightarrow X$ are bounded maps and $i \in I$, I uncountable, then there are $i \neq j \in I$ and an infinite $M \subset \mathbb{N}$ so that

$$\|f_i(\sigma) - f_j(\sigma)\| < \omega_{f_i}(1) + \omega_{f_j}(1) + \varepsilon \text{ for all } \sigma \in G_k(M).$$

Proof: Since $X^{(2k)}$ is separable, there are $i \neq j$ so that $\|\partial^k f_i - \partial^k f_j\| < \frac{\varepsilon}{2}$.
By Lemma 3, $\|f_i(\sigma) - f_j(\sigma)\| = \|(f_i - f_j)(\sigma)\| < \|\partial^k(f_i - f_j)\| + \omega_{f_i - f_j}(1) + \frac{\varepsilon}{2}$

Proof of the Theorem: suppose ~~on the contrary~~ that there is $h : c_0 \longrightarrow X$ bounded on bounded sets. Let $k \in \mathbb{N}$, $0 < \theta < \infty$, $A \subset \mathbb{N}$ infinite. Define $s_n(A) = \sum_{\substack{i \leq n \\ i \in A}} e_i \in c_0$.



Let $f_{\theta,A,k}(\sigma) = \theta \sum_{j=1}^{k} s_{n_j}(A) \in c_0$ for $\sigma = (n_1, \dots, n_k) \in G_k(\mathbb{N})$.
Then $h \circ f_{\theta,A,k} : G_k(\mathbb{N}) \to X$.

Fix θ, k; let $\varepsilon > 0$. By Lemma 4, (there are uncountably many infinite subsets of $\mathbb{N}$), we find $A \neq B$ and an infinite $M \subset \mathbb{N}$ so that

$$\forall \sigma \in G_k(M) \quad \|h(f_{\theta,A,k}(\sigma)) - h(f_{\theta,B,k}(\sigma))\| < \omega_{h \circ f_{\theta,A,k}}(1) + \omega_{h \circ f_{\theta,B,k}}(1) + \varepsilon$$

$$< 2\,\omega_h(\theta) + \varepsilon.$$

as f varies by θ from one vertex to the next. $\longrightarrow$ because $\omega_{f_{\theta,A,k}}(1) \leq \theta$ and the same for B.

On the other hand, since $A \neq B$, there is $\sigma \in G_k(M)$ so that
$\|f_{\theta,A,k}(\sigma) - f_{\theta,B,k}(\sigma)\| = k\theta$.

Therefore h cannot be a coarse embedding (choose $\theta, \varepsilon = 1$).
h cannot either be a uniform embedding:
if (M_1, d_1), (M_2, d_2) are metric spaces, $g : M_1 \longrightarrow M_2$,
$\varphi_g(t) = \inf\{d_2(g(x), g(y)),\ d_1(x,y) \geq t\}$, $t > 0$,
then $\varphi_g(d_1(x,y)) \leq d_2(g(x), g(y)) \leq \omega_g(d_1(x,y))$ for $x, y \in M_1$.
g is a uniform embedding if $\omega_g(t) \xrightarrow{t \to 0} 0$ and $\varphi_g(t) > 0$ for all t.
We have $\varphi_h(k\theta) \leq 2\omega_h(\theta)$; if $\omega_h(\theta) \xrightarrow{\theta \to 0} 0$, then $\varphi_h(t) = 0$ for all t.

Gilles
8/12/2011

Open questions:

Recall Ribe: $X = \left(\sum \ell_{1+\frac{1}{n}}\right)_{\ell_2}$: $\ell_1 \oplus X \underset{UH}{\simeq} X$
reflexive space : $\ell_1 \underset{u,c}{\hookrightarrow} X$

Assume $X \hookrightarrow Y$ reflexive. What other assumptions will ensure that X is reflexive?

Th: $\ell_1^n \subseteq X$ and $X \underset{c}{\hookrightarrow} Y$ reflexive $\Rightarrow$ X reflexive.
or X is an "alternate-Banach-Saks-property" space.

Counterexample: $J, J^* \underset{c}{\hookrightarrow} Y$ reflexive.

Open question: ① Is there a quasi-reflexive $(\dim X^{**}/X < \infty)$ nonreflexive B-space X such that $X \underset{c}{\hookrightarrow} Y$ reflexive?

② $X \underset{c}{\hookrightarrow} Y$ reflexive $\overset{?}{\Rightarrow}$ X is weakly sequentially complete

③ X^* separable + $X \underset{c}{\hookrightarrow} Y$ reflexive $\overset{?}{\Rightarrow}$ X reflexive.
(you can try to add some Szlenk index property.)

Michal Kraus
15/12/2012

Theorem (Albiac '08): If $1<p<q<\infty$, ℓ_p strongly embeds into ℓ_q [a strong embedding is an embedding that is coarse and uniform)

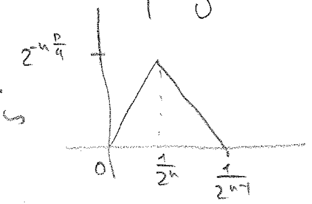
Proof: We will find an $f:\ell_p\to\ell_q$ and $A,B>0$ s.t.

$$A\|x-y\|_p^{p/q}\le\|f(x)-f(y)\|_q\le B\|x-y\|_p$$

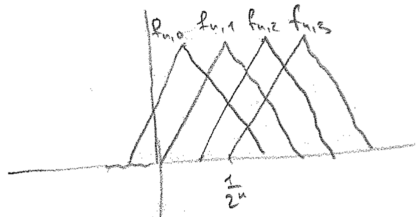
- We will work with $\ell_r(\mathbb{Z})$ instead of $\ell_r(\mathbb{N})$
- We will find $T:\ell_p\to\ell_q(\ell_q(\ell_q))\underset{\text{isom}}{\cong}\ell_q$

<u>1st step</u>: to find $f:\mathbb{R}\to\ell_q(\ell_q)$, $A,B>0$, s.t. $A^q|s-t|^p\le\|f(s)-f(t)\|_q^q\le B|s-t|^p$.

Let $n\in\mathbb{Z}$ and set $f_n(t)=\begin{cases}2^{n(1-\frac{p}{q})}t & \text{if } t\in[0,\frac{1}{2^n}]\\ -2^{n(1-\frac{p}{q})}(t-\frac{1}{2^{n-1}}) & \text{if } t\in[\frac{1}{2^n},\frac{1}{2^{n-1}}]\\ 0 & \text{otherwise.}\end{cases}$

Then the graph of f is 

and Lip $f_n=2^{n(1-\frac{p}{q})}$

Let $k\in\mathbb{Z}$ and define $f_{n,k}=f_n(t-\frac{k-1}{2^{n+1}})$, so that 

Define $f:\mathbb{R}\to\ell_q(\ell_q)$

$t\longmapsto(f_{n,k}(t))_{n,k=-\infty}^{+\infty}$

<u>Upper estimate</u>: let $s,t\in\mathbb{R}$, $N\in\mathbb{Z}$, $\frac{1}{2^{N+1}}<|s-t|\le\frac{1}{2^N}$.

For $n\le N$, $k\in\mathbb{Z}$, we have $|f_{n,k}(s)-f_{n,k}(t)|^q\le 2^{n(q-p)}|s-t|^q=2^{n(q-p)}|s-t|^p|s-t|^{q-p}$

$\le 2^{n(q-p)}|s-t|^p\frac{1}{2^{N(q-p)}}=2^{(n-N)(q-p)}|s-t|^p$

because Lip $f_{n,k}=2^{n(1-\frac{p}{q})}$

Suppose $n>N$ and $k\in\mathbb{Z}$. $|f_{n,k}(s)-f_{n,k}(t)|^q\underset{0\le f_{n,k}\le 2^{-n\frac{p}{q}}}{\le}2^{-np}=2^p2^{(N-n)p}2^{-(N+1)p}<2^p2^{(N-n)p}|s-t|^p$

Note that the estimates don't depend on k: for every $n\in\mathbb{Z}$, there are $k_1^n,\dots,k_8^n$ such that neither s nor t do belong to supp $f_{n,k}$ for other values of k.

$$\|f(s)-f(t)\|_q^q=\sum_{n,k=-\infty}^{\infty}|f_{n,k}(s)-f_{n,k}(t)|^q=\Big(\sum_{n\le N}+\sum_{n>N}\Big)\sum_{k\in\mathbb{Z}}|f_{n,k}(s)-f_{n,k}(t)|^q$$

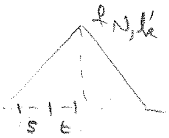
$$=\sum_{n\le N}\sum_{i=1}^{8}|f_{n,k_i^n}(s)-f_{n,k_i^n}(t)|^q+\sum_{n>N}\sum_{i=1}^{8}|f_{n,k_i^n}(s)-f_{n,k_i^n}(t)|^q$$

$$\le 8\sum_{n\le N}2^{(n-N)(q-p)}|s-t|^p+8\sum_{n>N}2^p2^{(N-n)p}|s-t|^p$$

$$=8\Big(\underbrace{\sum_{n\le N}2^{(n-N)(q-p)}+\sum_{n>N}2^{(N-n)p}}_{\frac{2^{q-p}}{2^{q-p}-1}+\frac{2^p}{2^p-1}<\infty}\Big)|s-t|^p.$$

Lower estimate: let $N \in \mathbb{Z}$ st $\frac{1}{2^{N+2}} < |s-t| \leq \frac{1}{2^{N+1}}$ and wlog $s<t$.

Let $k' \in \mathbb{Z}$ maximal st $s \in \operatorname{supp} f_{N,k'}$: then



$s \in \left[\frac{k'-1}{2^{N+1}}, \frac{k'-1}{2^{N+1}} + \frac{1}{2^{N+2}}\right[$

$t \in \left[\frac{k'-1}{2^{N+1}}, \frac{k'-1}{2^{N+1}} + \frac{1}{2^{N}}\right[$

Therefore $|f_{N,k'}(s) - f_{N,k'}(t)|^q = 2^{N(q-p)}|s-t|^q$

$$= 2^{N(q-p)}|s-t|^p|s-t|^{q-p} \geq 2^{N(q-p)}|s-t|^p \frac{1}{2^{(N+2)(q-p)}}$$

and $\|f(s)-f(t)\|_q^q = \sum_{n,k=-\infty}^{+\infty} |f_{n,k}(s) - f_{n,k}(t)|^q \qquad = \frac{1}{2^2(q-p)}|s-t|^p$

$$\geq |f_{N,k'}(s) - f_{N,k'}(t)|^q \geq \frac{1}{2^{2(q-p)}}|s-t|^p$$

So that $\quad A^q|s-t|^p \leq \|f(s)-f(t)\|_q^q \leq B^q|s-t|^p$.

2nd step: $T : \ell_p \to \ell_q(\ell_q(\ell_q))$, $(x_i) \mapsto (f(x_i))$. If $x=(x_i), y=(y_i) \in \ell_p$, then

$$\|T(x)-T(y)\|_q^q = \sum_{i\in\mathbb{Z}} \|f(x_i)-f(y_i)\|_q^q \begin{cases} \leq B^q \sum |x_i-y_i|^p = \|x-y\|_p^p \\ \geq A^q \sum |x_i-y_i|^p \end{cases}$$

Thus $A\|x-y\|_p^{\frac{p}{q}} \leq \|T(x)-T(y)\|_q \leq B\|x-y\|_p^{\frac{p}{q}}$

Theorem: $\ell_p \underset{c}{\hookrightarrow} \ell_q \iff \max(2,p) \leq \max(2,q)$

Proof: If $1 \leq p \leq 2$, $1 \leq q < \infty$, then $\ell_p \underset{c}{\hookrightarrow} \ell_2 \underset{c}{\hookrightarrow} \ell_q$ (just proved, Nowak) (Nowak)

- $2 < p \leq q < \infty$ implies $\ell_p \underset{c}{\hookrightarrow} \ell_q$ (just proved)
- $2 < p$, $p > q$ implies $\ell_p \underset{c}{\not\hookrightarrow} \ell_q$ (Mendel, Naor) [this uses metric cotype]

[If $X \underset{c}{\hookrightarrow} Y$ and Y has nontrivial type, then $\operatorname{cotype}(X) \leq \operatorname{cotype}(Y)$]

cotype $\ell_p = \max(2,p)$

Gilles Lancien ①

Nonlinear quotients

Seminal paper: Bates-Johnson-Lindenstrauss-Preiss-Schechtman 1999

Def: • (M,d) (N,δ) metric spaces. A f^n $f: M \to N$ is <u>co-uniformly continuous</u>

if $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in M\quad f(B(x,\varepsilon)) \supset B(f(x),\delta)$

If δ may be chosen a linear f^n of ε, f is <u>co-Lipschitz</u>

• N is a uniform (vs. Lipschitz) quotient of M if there is an onto f^n $f: M \to N$ that uniformly and co-uniformly continuous (vs. Lipschitz and co-Lipschitz). We write $M \overset{u}{\twoheadrightarrow} N$ (vs $M \overset{L}{\twoheadrightarrow} N$).

These def^{ns} are motivated by the open mapping theorem.

The General Question is: X, Y B-spaces (1) Do $X \overset{u\,or\,L}{\twoheadrightarrow} Y$ imply $X \twoheadrightarrow Y$

(2) Which properties are preserved by nonlinear quotients

Remarks: 1) $f: X \underset{L}{\hookrightarrow} Y \not\Rightarrow X \subseteq Y$. Assume f is G-diff^{able} at $x_0 \in X$

Then $Df(x_0)$ is a linear embedding from X into Y.

and there is a Th: X separable and Y RNP $\Rightarrow$ existence of points of G-diff^{ability}

2) $1 \le p < \infty$. There is $f: L_p \overset{L}{\twoheadrightarrow} \ell_p$ G-diff at 0, but $Df(0) = 0$.

3) If $f: X \twoheadrightarrow Y$ and f Fréchet-diff^{able} at x_0, then $Df(x_0)$ is onto

However, points of F-diff^{ability} are very rare!

$\to$ th (Preiss 90): If X is Asplund (every sep subspace has sep dual) and $f: X \to \mathbb{R}$ Lipschitz, then $\{x \in X : f \text{ F-diff at } x\}$ is dense

And there is an open question: X Asplund, $f: X \overset{Lip}{\twoheadrightarrow} \mathbb{R}^2 \overset{?}{\Rightarrow}$ existence of points of F-dif.

We therefore study weaker notions: ① $f: X \to Y$, $\varepsilon > 0$: f is ε-F-d at x_0 if

$\exists T \in B(X,Y)\ \exists r > 0\quad \|h\| \le r \Rightarrow \|f(x_0+h) - f(x_0) - Th\| \le \varepsilon\|h\|$

N.B: there are Theorems about the existence of points of ε-F-d for arbitrarily small ε's.

<u>Lemma</u>: $f: X \xrightarrow{L} Y$, $\mathrm{Lip} f = 1$, $\forall x \forall r$ $f(B(x,r)) \supset B(f(x), cr)$ $(c<1)$

Assume f is ε-F-d at x_0. Then $\exists T \in B(X,Y)$ $T: X \twoheadrightarrow Y$

Proof: We may suppose $x_0 = 0$ and $f(x_0) = 0$

If ① $\forall \rho > 0$ $f(B(0,\rho)) \supset B(0, c\rho)$

② $\exists r > 0$ $\|f(x) - Tx\| \leq \varepsilon \|x\|$ for $x \in B(0,r)$

Then $T(B(0, \cancel{(1+\varepsilon)} r)) \supset B(0, (c-\varepsilon) r)$

Let $\|y\| \leq (c-\varepsilon) r$ and x_1 with $\|x_1\| \leq \frac{c-\varepsilon}{c} r$ s.t. $y = f(x_1)$

$\|f(x_1) - Tx_1\| \leq \varepsilon \|x_1\| \leq \varepsilon \frac{c-\varepsilon}{c} r$.

Then $f(x_1) = T(x_1) + y_2$, $\|y_2\| \leq \varepsilon \frac{c-\varepsilon}{c} r$.

Let x_2 s.t. $y_2 = f(x_2)$ and $\|x_2\| \leq \frac{\varepsilon}{c} \frac{c-\varepsilon}{c} r$.

then $\|f(x_2) - Tx_2\| \leq \varepsilon \|x_2\| \leq \varepsilon \frac{\varepsilon}{c} \frac{c-\varepsilon}{c} r$

$y = T(x_1 + x_2) + y_3$, $\|y_3\| \leq \cdots$ in the end, one obtains

$y = T\left(\sum_1^\infty x_i\right)$, $\|x_i\| \leq \left(\frac{\varepsilon}{c}\right)^{i-1} \frac{c-\varepsilon}{c} r$ $\|x\| \leq (c-\varepsilon) r$.

$\|x\| \leq \frac{1}{1-\frac{\varepsilon}{c}} \frac{c-\varepsilon}{c} r = r$.

Claim: Assume only (1) and instead of (2) $\exists r > 0$ $\|f(x) - T(x)\| \leq \varepsilon r$ for $x \in B(0,r)$.
Then the same conclusion holds. (Th 11.14 in Benyamini-Lindenstrauss)
Gilles doesn't know how to work this out.

<u>Def</u>: • X, Y B-spaces. $\mathrm{Lip}(X,Y)$ has the Approximation by Affine Functions Property (AAP) if $\forall B$ ball in X $\forall f: B \xrightarrow{\mathrm{Lip}} Y$ $\forall \varepsilon > 0$ $\exists B_1 \subset B$ ball $\exists T: B_1 \xrightarrow{\text{affine}} Y$ $\forall x \in B_1$ $\|f(x) - Tx\| \leq \varepsilon r$, where r is the radius of B_1

• $\mathrm{Lip}(X,Y)$ has UAAP if moreover $\exists c(\varepsilon) > 0$ $r \geq c(\varepsilon) R$, where R is the radius of B.

Main results of BJLPS: Th 1: $\mathrm{Lip}(X,Y)$ has UAAP iff $\begin{cases} X \text{ or } Y \text{ is f. d, and} \\ X \text{ and } Y \text{ are superreflexive} \end{cases}$

Li-Naor obtained a quantitative version

Lindenstrauss-Preiss '96: X s.r., $\dim Y < \infty$, $f: X \xrightarrow{\mathrm{Lip}} Y$. Then $\forall \varepsilon > 0$ f admits points of ε-F-d.

<u>Th 2</u>: X s.r (i.e., X admits an equivalent u.c. norm).
then $X \xrightarrow{u} Y \Rightarrow Y \simeq \frac{X_{u}}{Z}$

<u>Cor</u>: If $1<p<\infty$ and $L_p \xrightarrow{u} Y$, then $L_p \twoheadrightarrow Y$
(here the authors need: $f: X$ s.r. $\xrightarrow{\mathrm{Lip}} E$ f.d. are AAP)

Th 3. $\mathrm{Lip}(X,\mathbb{R})$ has AAP iff X is Asplund

Th 4: $\mathrm{Lip}(\mathbb{R},X)$ has AAP iff there is no bounded generalised δ-separated martingale in X [which is implied by, and does not imply, RNP]

Th 5: $X \xrightarrow{L} Y$, X Asplund, implies Y Asplund

Proof of Th 5: (1) Assume X separable Asplund (i.e. X^* sep). Then Y is sep.

We need to show Y^* sep: If not, $\exists\,(y^*_\gamma)_{\gamma\in\Gamma}$, Γ uncountable, $\|y^*_\gamma\|=1$, $\|y^*_\gamma - y^*_{\gamma'}\|\geq \frac{1}{2}$ for $\gamma\neq\gamma'$

Then $f_\gamma : X \xrightarrow{f\ \mathrm{Lip}} Y \xrightarrow{y^*_\gamma} \mathbb{R}$. As X is Asplund, there are $x_\gamma\in B_X$ where f_γ is F-diff (Preiss' Thm). Let $\varepsilon>0$.

$\forall\gamma\ \exists\delta>0$ $\|h\|\leq\delta_\gamma$ $\|f(x_\gamma+h)-f(x_\gamma)-Df(x_\gamma)h\|\leq\varepsilon\|h\|$

[here one could use ε-F-d and use Lindenstrauss' Preiss thm]

By passing to an uncountable subset of Γ, we may assume $\delta=\inf_{\gamma\in\Gamma}\delta_\gamma>0$

~~then~~ and $\forall\,\gamma\neq\gamma'$ $\|x_\gamma - x_{\gamma'}\|<\varepsilon\delta$ (X is separable!)

$\|Df_\gamma(x_\gamma)-Df_{\gamma'}(x_{\gamma'})\|<\varepsilon$ (X^* is separable)

$|f_\gamma(x_\gamma)-f_{\gamma'}(x_{\gamma'})|<\varepsilon\delta$ ($\mathbb{R}$ is sep.)

Let $\|h\|\leq\delta$
$$\begin{aligned}|f_\gamma(x_\gamma+h)-f_{\gamma'}(x_{\gamma'}+h)| &\leq |f_\gamma(x_\gamma+h)-f_\gamma(x_\gamma)+Df_\gamma(x_\gamma)h|+|f_\gamma(x_\gamma)-f_{\gamma'}(x_{\gamma'})|\\ &+|Df_\gamma(x_\gamma)h-Df_{\gamma'}(x_{\gamma'})h|+|Df_{\gamma'}(x_{\gamma'})h-f_{\gamma'}(x_{\gamma'})\\ &-f_{\gamma'}(x_{\gamma'}+h)|+|f_{\gamma'}(x_{\gamma'}+h)-f_{\gamma'}(x_\gamma+h)|\\ &\leq\varepsilon\delta+\varepsilon\delta+\varepsilon\delta+\varepsilon\delta+\varepsilon\delta=5\varepsilon\delta<\tfrac{c\delta}{4}\text{ for }\varepsilon\text{ small enough.}\end{aligned}$$

But $f: X\to Y$ is a Lipschitz quotient: $\exists c>0\ \forall x\ \forall r\ f(B(x,r))\supset B(f(x),cr)$

$$\sup_{\|h\|\leq\delta}|f_\gamma(x_\gamma+h)-f_{\gamma'}(x_{\gamma'}+h)| = \sup_{\|h\|\leq\delta}|\langle y^*_\gamma - y^*_{\gamma'}, f(x_\gamma+h)\rangle|$$
$$\geq \sup_{\|y\|\leq c\delta}|\langle y^*_\gamma-y^*_{\gamma'}, f(x_\gamma)+y\rangle| \geq \|y_\gamma - y_{\gamma'}\|c\delta \geq \frac{c\delta}{2}$$

$\to$ pick y norming $y^*_\gamma - y^*_{\gamma'}$ s.t. $\mathrm{sgn}\langle y^*_\gamma - y^*_{\gamma'}, y\rangle = \mathrm{sgn}\langle y^*_\gamma - y^*_{\gamma'}, f(x_\gamma)\rangle$

This is a contradiction!

(2) General case: by separable exhaustion: $X \underset{L}{\simeq} Y$, X Asplund, pick $Y_0\subseteq Y$, Y_0 separable. We want to show Y_0^* sep. $X_0=\overline{\mathrm{span}}\, f^{-1}(Y_0)$ is sep. Let $Y_1=\overline{\mathrm{span}}(f(X_0))$ and $X_1=\overline{\mathrm{span}}\,(f^{-1}(Y_1))$. Then let $\tilde{X}=\overline{\bigcup X_n}$ and $\tilde{Y}=\overline{\bigcup Y_n}$. These are separable. $f|_{\tilde X}:\tilde{X}\to\tilde{Y}$ is a bijection; $\tilde{X}$ sep Asplund $\Rightarrow$ $\tilde{Y}$ sep Asplund $\Rightarrow$ Y_0 Asplund

(4)

Dutrieux: X^* sep., $X \xrightarrow{L} Y$. Then $\exists\, \varphi : (0,\omega_1) \to (0,\omega_1)$ $Sz\,Y \le \varphi(Sz\,X)$.

and the open question is: $\varphi = \mathrm{Id}$? This would have appl^{ns} to Sz for $C(K)$ spaces

<u>Th</u>: X Bspace then $\Leftrightarrow$ (i) $\mathrm{Lip}(X,\mathbb{R})$ has AAP

(ii) X is Asplund

(iii) $\forall\, \mathcal{U}$ open in X $\forall f : \mathcal{U} \xrightarrow{\mathrm{Lip}} \mathbb{R}$ f has a pt of F-d.

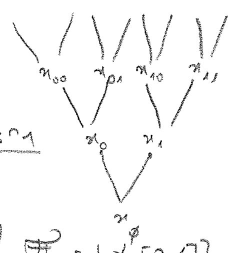
Proof: (iii) $\Rightarrow$ (i) is clear. Recall the very definition of Asplund space:

X is <u>Asplund</u> if $\forall\, \mathcal{U}$ open convex $\forall f : \mathcal{U} \xrightarrow[\text{continuous}]{\text{convex}} \mathbb{R}$ $\{x \text{ pt of F-diff}\}$ is G_δ-dense in $\mathcal{U}$

(iii) $\Rightarrow$ (ii) $f : \mathcal{U} \xrightarrow[\text{cont.}]{\text{conv}} \mathbb{R}$ is locally Lipschitz; $\{x : f \text{ F-diff at } x\}$ is G_δ

(ii) $\Rightarrow$ (iii) Preiss

(i) $\Rightarrow$ (ii) Assume X not Asplund. Then there is an equivalent norm on X which is very bad: it is rough: $\|\ \| : X \to \mathbb{R}$ is not approximable by affine functions.



<u>X-valued martingale</u>: $(x_s)_{s\in\{0,1\}^{<\omega}}$. For every s, $x_s = \dfrac{x_{s\frown 0} + x_{s\frown 1}}{2}$

<u>Generalised dyadic martingale</u>: $(\mathcal{F}_n) \uparrow$ σ-fields of $[0,1)$ with $\mathcal{F}_0 = \{\emptyset, [0,1)\}$ with $\mathcal{F}_n$ generated by 2^n atoms (intervals of the form $[a,b)$). The atoms of $\mathcal{F}_{n+1}$ are obtained by splitting the atoms of $\mathcal{F}_n$ into 2 subintervals (of $\neq$ length)

An X-valued martingale (M_n) wrt $\mathcal{F}_n$ is called is <u>generalised dyadic</u>.

It is <u>bounded</u> if $\sup \|M_n\|_\infty < \infty$ and <u>δ-separated</u> if $\|M_{n+1} - M_n\| > \delta$

We may see that as $(x_s)_{s\in\{0,1\}^{<\omega}}$. $x_s = \lambda_s x_{s\frown 0} + (1-\lambda_s) x_{s\frown 1}$; $\lambda_s \in (0,1)$.

and $\sup_s \|x_s\| < \infty$.

And δ-separation is expressed by $\|x_s - x_{s\frown 0}\| \geq \delta$ and $\|x_s - x_{s\frown 1}\| \geq \delta$

We have $\lambda_s(x_{s\frown 0} - x_{s\frown 1}) = x_s - x_{s\frown 1}$.

Assume $\|x_s\| \le 1$ δ-separated then $\lambda_s, 1-\lambda_s \geq \frac{\delta}{2}$ (1)

We are ready to prove Th 4!

Proof of Th 4 ($\Rightarrow$) Assume there is such a martingale (M_n). Let $f_n(t) = \int_0^t M_n(s)\,ds$.

Then $\mathrm{Lip}\, f_n \le 1$. If t is an end-point of an atom of $\mathcal{F}_n$, $(f_m(t))_{m \ge n}$ is constant

Such t's are dense because of (1). Suppose that $f_n \xrightarrow{u} f$, $\mathrm{Lip}\, f \le 1$.

Let $[a,c[$ atom of $\mathcal{F}_n$ and $[a,b[$, $[b,c[$ atoms of $\mathcal{F}_{n+1}$

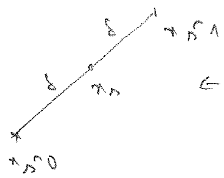
$M_{n+1} = x_{s^\frown 0}$ $\qquad M_{n+1} = x_{s^\frown 1}$

this condition is just a perturbation of the general case

$\lambda_s = \frac{b-a}{c-a} \qquad 1-\lambda_s = \frac{c-b}{c-a}$

Assume that L is affine and $L(a) = f(a)$ and $\|L(x) - f(x)\| \le \frac{\delta^2}{4}(c-a)$

then $$\|x_{s^\frown 0} - x_s\| = \left\| \frac{f(b)-f(a)}{b-a} - \frac{f(c)-f(a)}{c-a} \right\| \le \left\| \frac{L(b)-L(a)}{b-a} - \frac{L(c)-L(a)}{c-a} \right\| + \frac{\delta^2}{4}\left[\frac{c-a}{b-a} + 1\right]$$



$\leftarrow$ We have $2 \ge \|x_{s^\frown 0} - x_{s^\frown 1}\| \ge 2\delta$ and $\delta \le 1$

$$= \frac{\delta^2}{4}\left(\frac{1}{\lambda_s} + 1\right) \le \frac{\delta^2}{4}\left(\frac{2}{\delta} + 1\right) = \frac{\delta}{2} + \frac{\delta^2}{4} \le \frac{3}{4}\delta.$$

($\Leftarrow$) Assume $\mathrm{Lip}(\mathbb{R}, X)$ fails AAP: $\exists f : [0,1[\to X$, $\mathrm{Lip}\, f \le 1$, $\exists \delta > 0$

(*) $\forall [a,c[\subset [0,1[\ \exists b \in [a,c[\ \ \left\| f(b) - \underbrace{\left(\frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(c)\right)}_{L(b)} \right\| > \delta(c-a)$

$L(a) = f(a) \quad L(c) = f(c)$

Let $M_0 = f(1) - f(0)$ and assuming M_n defined, we choose for every atom $[t_n^i, t_n^{i+1}[$ of $\mathcal{F}_n$, $M_{n+1} = \frac{f(t_n^{i+1}) - f(t_n^i)}{t_n^{i+1} - t_n^i}$ on $[t_n^i, t_n^{i+1}[$

For each atom $[a,c[$ of $\mathcal{F}_n$ pick $b \in [a,c]$ given by (*)

If $[a,b[$, $[b,c[$ atoms of $\mathcal{F}_{n+1}$, $M_{n+1} = \frac{f(b)-f(a)}{b-a}$ on $[a,b[$

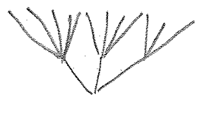
$= \frac{f(c)-f(b)}{c-b}$ on $[b,c[$

We have $\|M_n\| \le 1$ because $\mathrm{Lip}\, f \le 1$ and $\left\| \frac{f(b)-f(a)}{b-a} - \frac{f(c)-f(a)}{c-a} \right\| \ge \frac{\delta(c-a)}{b-a} \ge \delta$

$\left\| \frac{f(c)-f(b)}{c-b} - \frac{f(c)-f(a)}{c-a} \right\| \ge \frac{\delta(c-a)}{c-b} \ge \delta$

Remark on AAP $\ne$ RNP: X fails RNP iff $\exists$ bdd δ-separated martingale in X

iff $\exists$ bdd δ-separated bush in X



$x_s = \sum_{i=1}^{N_s} \lambda_s^i x_{s^\frown i}$, $\lambda_s^i \ge 0$, $\sum_i \lambda_s^i = 1$, $\|x_{s^\frown i} - x_s\| \ge \delta$

There is an X failing RNP (i.e., a bush) but not containing any generalised dyadic ($\to$ Bourgain–Rosenthal)

Gilles : quotients non linéaires (1) 7/3/2012

(1) Reminder:

X, Y B-spaces

Lemma 1 $f: X \to Y$, $\operatorname{lip} f = 1$, $f(0) = 0$. Assume $\exists\, 0<c<1$ $\forall \rho>0$ $f(\rho B_X) \supset c\rho B_X$ ~~$\forall x\in X$ $f(B(x,r)) \supset B(f(x), cr)$~~

and that (2) if $\varepsilon \in]0, c[$, $\exists r>0$ $\exists T \in B(X,Y)$ $\forall x \in rB_X$ $\|Tx - f(x)\| \le \varepsilon \|x\|$.

Then $T(rB_X) \supset (c-\varepsilon) r B_Y$.

The Dubious lemma of last time consists in replacing (2) by

(2') $\exists \varepsilon < c$ $\exists T$ $\exists r>0$ $\forall x \in rB_X$ $\|Tx - f(x)\| \le \varepsilon r$.

Then $T(rB_X) \supset (c-\varepsilon) rB_Y$ (same conclusion as before)

A weaker version of this holds true. (A. Procházka 2012)

(2'') $\exists \varepsilon < c^4$ $\exists T$ $\exists r>0$ $\forall x \in rB_X$ $\|Tx - f(x)\| \le \varepsilon r$

Then $T(rB_X) \subset c(1-c) r B_Y$.

Proof: $\forall \rho > 0$ $\rho B_Y \subset f\left(\frac{\rho}{c} B_X\right)$ $\quad f(c\rho B_X) \subset c\rho B_Y$

so that $\rho B_Y \setminus c\rho B_Y \subset f\left(\frac{\rho}{c} B_X\right) \setminus f(c\rho B_X)$

$\subset f\left(\frac{\rho}{c} B_X \setminus c\rho B_X\right)$.

Set $r_n = c^n r$, $n \ge 0$: $r_1 B_Y \setminus r_2 B_Y \subset f(r_0 B_X \setminus r_2 B_X)$

If $x \in D_n = r_n B_X \setminus r_{n+2} B_X$, define $f_n(x) = c^n f\left(\frac{x}{c^n}\right)$

Then $x \in D_n \iff \frac{x}{c^n} \in D_0$ and $r_{n+1} B_Y \setminus r_{n+2} B_Y \subset f_n(r_n B_X \setminus r_{n+2} B_X)$

* Let $y_0 \in r_1 B_Y = crB_Y$. There is $n_1 \ge 0$ s.t. $r_{n_1+2} < \|y_0\| \le r_{n_1+1} \Rightarrow \exists x_1 \in D_{n_1}$ $y_0 = f_{n_1}(x_1)$

Note that if $x_0 \in D_0$, $\|f(x) - T(x)\| \le \varepsilon r \le \frac{\varepsilon \|x\|}{c^2}$ $\quad (\|x\| \ge c^2 r)$] This is the key point, note that $\varepsilon < c^4$!

$\|f(x) - T(x)\| \le c^2 \|x\|$

$x \in D_n$ $\|f_n(x) - T(x)\| \le c^2 \|x\|$.

and $\|f_{n_1}(x_1) - T(x_1)\| \le c^2 \|x_1\| \le r_2$

Let $y_1 = y_0 - T(x_1) = f_{n_1}(x_1) - T(x_1)$: $\|y_1\| \le r_2$

Then there is $n_2 \ge 1$ s.t. $y_1 \in r_{n_2+1} B_Y \setminus r_{n_2+2} B_Y$ and $y_1 = f_{n_2}(x_2)$ and $x_2 \in r_{n_2} B_X \setminus r_{n_2+2} B_X$

We have $\|x_2\| \le r_{n_2} \le r_1$ $\quad \|\underbrace{f_{n_2}(x_2) - T(x_2)}_{y_2}\| \le c^2 \|x_2\| \le r_3$.

By induction, we get $y_0 = T(x_1 + \dots + x_n) + y_n$ $\quad \|x_n\| \le r_{n-1}$ $\quad \|y_n\| \le r_{n+1}$

Let $x = \sum x_n$ and $y = Tx$. Then $\|x\| \leq r \sum_0^\infty c^n = \frac{r}{1-c}$

$$c\, r\, B_Y \subset T\left(\frac{r}{1-c} B_X\right)$$
$$c(1-c) B_Y \subset T(B_X)$$

(2) Elementary facts on uniform quotients

Lemma 3: Let $f : X \to Y$ uniformly continuous. Then $\forall d > 0$ $\|x - x'\| > d \Rightarrow \|f(x) - f(x')\| \leq \frac{2\omega_f(d)}{d}\|x - x'\|$

Proof: x ——— x' split $[x, x']$ into $\left\lfloor \frac{\|x-x'\|}{d} \right\rfloor + 1$ segments of length $\leq d$

Then $$\|f(x) - f(x')\| \leq \left(\left\lfloor \frac{\|x-x'\|}{d} \right\rfloor + 1\right) \omega_f(d)$$
$$\leq \frac{\omega_f(d)}{d}\|x - x'\| + d\,\frac{\omega_f(d)}{d} \leq \frac{2\,\omega_f(d)}{d}\|x - x'\| \qquad (\|x - x'\| \geq d)$$

Lemma 4 $f : X \to Y$ co-uniformly continuous.

Then $\forall d > 0\ \exists c > 0\ r \geq d \Rightarrow \forall x \in X\ f(B(x, r)) \supset B(f(x), cr)$

Proof: Let $\delta = \delta(d)$ given by the def of co-uniform continuity

$(\forall x \in X,\ f(B(x, d)) \supset B(f(x), \delta))$

Let $r \geq d$. Then $r = nd + d'$, $n \geq 1$, $d' < d$.

Fix $x \in X$, $y \in X$, $\|y - f(x)\| \leq n\delta$; split $\underset{f(x)}{y_0}\ y_1 \cdots \underset{y}{y_n}$ $\|y_{i+1} - y_i\| \leq \delta$

An easy induction yields x_n s.t. $y = f(x_n)$ $\|x_n - x\| \leq nd' \leq r$

$$B(f(x), n\delta) \subset f(B(x, r))$$
$$\underset{\text{so that } \frac{r}{2d} \leq n}{r \leq 2nd} \quad B\left(f(x), \frac{\delta}{2d} r\right) \subset f(B(x, r))$$ ∎

Ultrapowers: $\mathcal{U}$ free ultrafilter on I and $\ell_\infty^I(X) = \{x = (x_i)_{i \in I},\ (x_i) \text{ bounded in } X\}$

$\mathcal{N} = \{x \in \ell_\infty^I(X) : \lim_{\mathcal{U}} \|x_i\|_X = 0\}$.

Then $X_{\mathcal{U}} = \ell_\infty^I(X)/\mathcal{N}$ equipped with $\|\tilde{x}\| = \lim_{\mathcal{U}} \|x_i\|$ if $x = (x_i)_{i \in I} \in \tilde{x}$.

Prop 5: If $X \xrightarrow{u} Y$, then $X_{\mathcal{U}} \xrightarrow{L} Y_{\mathcal{U}}$

Proof: • $\exists f : X \to Y\ \exists C \geq 1\ \forall r \geq 1\ \forall x \in X\ B\left(f(x), \frac{r}{C}\right) \subset f(B(x, r)) \subset B(f(x), Cr)$

• $f_n(x) = \frac{f(nx)}{n}$. $\forall r \geq \frac{1}{n}\ \forall x\ B\left(f_n(x), \frac{r}{C}\right) \subset f_n(B(x, r)) \subset B(f_n(x), Cr)$

$$\tilde{f} : X_{\mathcal{U}} \to Y_{\mathcal{U}} \qquad \mathcal{U} \text{ free ultrafilter on } \mathbb{N}.$$
$$\tilde{x} = (x_n) \mapsto \left(\frac{f(nx_n)}{n}\right)_n$$

Then $\forall r > 0\ \forall \tilde{x}\ B\left(\tilde{f}(\tilde{x}), \frac{r}{C}\right) \subset \tilde{f}(B(\tilde{x}, r)) \subset B(\tilde{f}(\tilde{x}), Cr)$

This means that $\tilde{f}$ is a Lipschitz quotient. ∎

(3) UAAP (Uniform affine AP), superreflexivity and applications.

Def: Lip(X,Y) has UAAP if $\forall \varepsilon>0\ \exists\, c(\varepsilon)>0\ \forall$ ball $B\subset X\ \forall f: B \xrightarrow{\text{Lip}} Y$
$\exists B_1 \subset B$ ball $\text{rad}(B_1)\geq c(\varepsilon)\,\text{rad}(B)$
$\exists T: B_1 \longrightarrow Y$ affine continuous. $\forall x\in B_1\ \|f(x)-Tx\|\leq \varepsilon\, \text{rad}(B_1)\, \text{Lip}(f)$

Prop 6 If Lip$(X,\mathbb{R})$ has UAAP, then, for any f.d. Y, Lip(X,Y) has UAAP.

Proof: We will only prove that Lip(X,ℓ_∞^2) has UAAP. Thence it suffices to conduct an induction. If $f=(f_1,f_2): B\longrightarrow(\mathbb{R}^2,\|\cdot\|_\infty)$, $\text{Lip}(f)\leq 1 \iff \text{Lip} f_i\leq 1$ $(i=1,2)$ (rad $B=1$).

Fix $\varepsilon>0$. There is $B_1\subset B$ with $\text{rad}(B_1)\geq c(\varepsilon c(\varepsilon))$

Then there is $g_1: B_1\longrightarrow\mathbb{R}$ st $|f_1-g_1|\leq \varepsilon c(\varepsilon)\,\text{rad}(B_1)$ on B_1.

There is $B_2\subset B_1$ with rad $B_2\geq c(\varepsilon)$ rad (B_1)

There is $g_2: B_2\longrightarrow\mathbb{R}$ st $|f_2-g_2|\leq \varepsilon\,\text{rad}(B_2)$ on B_2.

Then on $B_2\subset B_1$, $|f_1-g_1|\leq \varepsilon c(\varepsilon)\,\text{rad}(B_1)\leq \varepsilon\,\text{rad}(B_2)$

If $g=(g_1,g_2)$, $\|f-g\|_{\ell_\infty^2}\leq\varepsilon\,\text{rad}(B_2)$ on B_2

Then rad $B_2\geq c(\varepsilon)\,c(\varepsilon c(\varepsilon))=c_2(\varepsilon)>0$.

Here the U in UAAP is crucial!

Theorem 7 If X is superreflexive ($X_{\mathcal{U}}$ reflexive), then Lip$(X,\mathbb{R})$ has UAAP.

Theorem 8: If $X\xrightarrow[u]{} Y$, then $\exists\lambda\geq 1\ \forall G\subseteq Y^*$ dim $G<\infty\ \exists F\subseteq X^*$ $F\underset{\lambda}{\simeq} G$.

It follows that $\exists Z\subseteq X_{\mathcal{U}}$ $Y\simeq X_{\mathcal{U}}/Z$

Corollary 9 : $1<p<\infty$, $L_p=L_p(\mathbb{R})$. $L_p \xrightarrow[u]{} Y \iff L_p \twoheadrightarrow Y$

Proof of this Cor: th 8 implies there is $\mathcal{U}$ st $Y\simeq (L_p)_{\mathcal{U}}/Z$ and there is (Σ,μ) such that $(L_p)_{\mathcal{U}}\equiv L_p(\Sigma,\mu)$ but this a nonseparable space. and there is a quotient map $Q: L_p(\Sigma,\mu)\twoheadrightarrow Y$ (because $L_p\twoheadrightarrow Y$ implies Y is separable) and there is $s: Y\longrightarrow L_p(\Sigma,\mu)$ continuous selection s.t. $Q\circ s=\text{Id}_Y$ (Bartle-Graves)

Then $E=\overline{\text{span}}\, s(Y)$ is separable because s is continuous and Y separable.

$E\subseteq L_p(\Sigma,\mu)$.

Let (f_n) be dense in E and $\mathcal{T}=\sigma$-field generated by the f_n's

then $F=L_p(\mathcal{T},\mu)$ is a separable space! and it contains E.

then $F\equiv L_p$, ℓ_p or ℓ_p^n. But $Q|_F: F\longrightarrow Y$ is onto and $Y\simeq F/W$.

cf Chapter 1 of Wojtaszczyk, Banach spaces for analysts

Proof of Th 8: Suppose $X \twoheadrightarrow Y$. Then $X_\mathcal{U} \underset{L}{\twoheadrightarrow} Y_\mathcal{U}$ with $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$.

We want to show that for $G \subseteq Y^*$ there is $F \subseteq X^*$ with $G \underset{\lambda}{\simeq} F$.

It is enough to find $F \subset (X^*)_\mathcal{U}$. (finite dimensionality of F).

But $(X^*)_\mathcal{U} = (X_\mathcal{U})^*$ because X is superreflexive. Similarly, we can work with $G \subseteq Y^* \subseteq (Y^*)_\mathcal{U} \subseteq (Y_\mathcal{U})^*$, G f.d.

Since $X_\mathcal{U}$ is superreflexive, we may assume $f : X \underset{L}{\twoheadrightarrow} Y$, $\mathrm{Lip}\, f = 1$, replacing X, Y by $X_\mathcal{U}, Y_\mathcal{U}$.

we also have $\forall x \in X$ $f(B(x,r)) \supset B(f(f(x)), cr)$, $0 < c < 1$.

If $G \subset Y^*$, $\dim G < \infty$, let $G_\perp = \{y \in Y : \forall g \in G \ \langle y, g \rangle = 0\}$ and $Q : Y \longrightarrow Y/_{G_\perp}$

the canonical quotient $Q \circ f : X \longrightarrow Y/_{G_\perp}$ f.d. and $\mathrm{Lip}(Q \circ f) \leq 1$

and $\forall x \in X$ $Q \circ f\,(B(x,r)) \supset B(Q \circ f(x), cr)$

X is superreflexive and $\dim Y/_{G_\perp} < \infty$: thus $(X, Y/_{G_\perp})$ has UAAP.

For $\varepsilon < c^4$, there is $B_1 = B(x_0, r)$ and $T : B_1 \longrightarrow Y/_{G_\perp}$ affine continuous such that $\forall x \in B_1$ $\|Q \circ f(x) - T(x)\| \leq \varepsilon \operatorname{rad}(B_1)$ $(Q \circ f)(x_0) = T(x_0)$

Let $\tilde{T}$ be the linear part of T: $\|\tilde{T}\| \leq 1 + \varepsilon$

Lemma 2 yields $\tilde{T}(B_X) \supset c(1-c)\, B_{Y/G_\perp}$

Consider $\tilde{T}^* : (Y/_{G_\perp})^* \cong G_\perp^\perp = G \longrightarrow X^*$. Then $\tilde{T}^*$ is a $\underbrace{\frac{1+\varepsilon}{c(1-c)}}$-embedding

* Classical results permit to deduce that $Y^* \underset{\simeq}{\subseteq} (X^*)_\mathcal{V}$ [Y^* is finitely crudely representable in X^*]

As X is superreflexive, X^* is superreflexive, $(X^*)_\mathcal{V}$ is reflexive, Y^* is reflexive and thus Y is reflexive. Then $Y = Y^{**} \simeq (X^*)^*_\mathcal{V}/_Z \simeq \underbrace{(X^{**})_\mathcal{V}/_Z}_{\text{because } X \text{ is superreflexive}} \simeq X_\mathcal{V}/_Z$

Th 7 Assume X is superreflexive: $\mathrm{Lip}(X, \mathbb{R})$ has UAAP

Proof We know that X admits an equivalent uniformly smooth norm: we shall assume that $\|\cdot\|_X$ is u.s. Consider $f : B \longrightarrow \mathbb{R}$, B ball of X of radius 1 and $\mathrm{Lip}\, f = 1$. Fix $\varepsilon > 0$. Let $\mathrm{Lip}_t f = \sup\limits_{\|x-y\| \geq t} \frac{\|f(x) - f(y)\|}{\|x - y\|}$. Case 1. Assume $\mathrm{Lip}_1 f \leq \varepsilon$. Then f has a good approximation by a constant f^u:

$x_0 - u$ x_0 x $x_0 + u$ $\|u\| = 1$

$\|f(x_0) - f(x_0 - u)\| \leq \varepsilon$
$\|f(x_0) - f(x_0 - u)\| \leq 2\varepsilon$ and $|f(x) - f(x_0)| \leq 3\varepsilon$

Case 2: $\mathrm{Lip}_1 f \geq \varepsilon$. $\|\ \|_X$ u.s. implies that $\exists \delta(\varepsilon) > 0$ $\forall x \in S_X$ $\forall y \in \delta B_X$ $\|x + y\| = 1 + y_x^*(y) + r(y)$ where y_x^* is the unique norming functional of x in S_{X^*}. $|r(y)| \leq \varepsilon\|y\|$. Let k be the smallest integer s.t. $(1 + \delta\varepsilon)^k \varepsilon > 0$: $k = k(\varepsilon)$. Then $\varepsilon \leq \mathrm{Lip}_1(f) \leq \mathrm{Lip}_{4^{-1}}(f) \leq \ldots \leq \mathrm{Lip}_{4^{-k}}(f) \leq 1$.

Therefore there is $j \le k$ s.t. $\mathrm{Lip}_{4^{-j}}(f) < (1+\delta\varepsilon)\,\mathrm{Lip}_{4^{-j+1}}(f)$

Denote $\Delta = 2\cdot 4^{-j}$. $\mathrm{Lip}_{\Delta/2}(f) < (1+\delta\varepsilon)\,\mathrm{Lip}_{2\Delta}(f)$

There is $d \ge \Delta$, $y, z \in B$ with $\|y-z\| = 2d$ s.t. $\mathrm{Lip}_{\frac{d}{2}}(f) \le \mathrm{Lip}_{\frac{\Delta}{2}}(f) < (1+\delta\varepsilon)\frac{\|f(y)-f(z)\|}{2d}$

We may assume $y = -z$ and $f(z) = -f(-z) > 0$.

(*) Then $\exists\, d \ge \Delta$ $\|z\| = d$ $\mathrm{Lip}_{\frac{d}{2}} f < (1+\delta\varepsilon)\frac{f(z)}{d}$

As $-z, z \in B \Rightarrow 0 \in B$ $\quad \underset{0}{\times}\overset{\frac{\delta d}{2}}{\text{---}}\underset{u}{\times}\text{---}\underset{x_0}{\times}$ $\quad \frac{\delta d}{2} < \frac{1}{4}$. $B(u, \frac{\delta d}{2}) \subset (B \cap \delta d B_X)$

Note that $\frac{\delta d}{2} \ge \psi(\varepsilon) > 0$ for some function ψ

Let z^* be the norming functional of $\frac{z}{d}$.

$\|w+z\|, \|w-z\| \ge \frac{d}{2}$ because $\|z\| = d$, $\delta < \frac{1}{2}$.

Consider
$$
\begin{aligned}
f(w) = f(w) - f(-z) - f(z) &\le \mathrm{Lip}_{\frac{d}{2}}(f)\,\|w+z\| - f(z) \\
&\le (1+\delta\varepsilon)\, f(z) \left\| \frac{z}{d} + \frac{w}{d} \right\| - f(z) \\
&\le (1+\delta\varepsilon)\, f(z) \left[1 + z^*\left(\frac{w}{d}\right) + \varepsilon \left\|\frac{w}{d}\right\|\right] - f(z) \\
&\qquad\qquad \left(\left\|\frac{w}{d}\right\| \le \delta\right) \\
&\le \frac{f(z)}{d} z^*(w) + [\varepsilon\delta + \varepsilon\delta + \delta^2\varepsilon + \delta^2\varepsilon^2]
\end{aligned}
$$

Thus $f(w) \le \frac{f(z)}{d} z^*(w) + 3\varepsilon\delta$

* Similarly, $-f(w) = f(z) - f(w) - f(z) \le \mathrm{Lip}_{\frac{d}{2}} \|z-w\| - f(z)$
$$\le \frac{f(z)}{d} z^*(-w) + 3\varepsilon\delta$$

so that $f(w) \ge \frac{f(z)}{d} z^*(w) - 3\varepsilon\delta$.

Thus $f(w) - \frac{f(z)}{d} z^*(w) \le 3\varepsilon\delta$ on a ball of radius $\ge \frac{\delta d}{2}$. ☒

One last comment on the current paper. It states the following result

Prop: $\mathrm{Lip}(X,Y)$ has UAAP $\Leftrightarrow$ X superreflexive and $\dim Y < \infty$
or Y ————————— X —

Another result we could discuss (1) If $\ell_1 \subseteq X$, then $X \twoheadrightarrow \ell_1$ (JLPS'02: Lipschitz quotient of metric trees) This yields separable Lip quotients which are not quotients

(2) If $1 \le q < p < \infty$, then $\ell_p \not\twoheadrightarrow_{L} \ell_q$ (another JLPS'02)

(3) If $1 < p < q < \infty$, $\ell_p \not\twoheadrightarrow_{u} \ell_q$ (Lima-Randrianarivony '11)

Last remark: If you have a uniform quotient $L_1 \xrightarrow{u} Y$, then $Y \simeq (L_1) u/z$ and $Y \simeq L_1/W$.

Following JLPS 2002:

<u>Motivation</u>: X super R and Y a B.sp. s.t. $X \overset{L}{\twoheadrightarrow} Y$
Then Y^* crudely f.r. in X^*. (M)

- is true even when X ⟨AUS⟩, $X = C(K)$, K a countable compact
- <u>Question</u>: is (M) true for any Banach space X?
- Answer: no
- <u>Theorem 1</u>: Let X be separable B.sp., $\ell_1 \underset{\sim}{\subseteq} X$, $\varepsilon > 0$.
  Then $\forall$ Y sep.:
  $X \xrightarrow{(1+\varepsilon)} Y$

In particular: $\ell_1 \subseteq C[0,1]$ so $\exists$ $C[0,1] \overset{L}{\twoheadrightarrow} \ell_1$
but $\ell_1^* = \ell_\infty$ is not crudely f.r. in $C[0,1]^*$
it is because cotype $\ell_\infty = \infty$ and cotype $C[0,1]^* = 2$

Remark: • $\forall$ Y sep. Banach $\ell_1 \twoheadrightarrow Y$

Proposition 2: $\left.\begin{array}{l} \ell_1 \underset{\sim}{\subseteq} X \\ X \text{ separable} \\ \varepsilon > 0 \end{array}\right] \Rightarrow X \overset{1+\varepsilon}{\twoheadrightarrow} C(\Delta)$, Δ the Cantor set

<u>Proof</u>: James: $\ell_1 \underset{\sim}{\subseteq} X \Rightarrow \ell_1 \overset{1+\varepsilon}{\underset{\sim}{\subseteq}} X$ (if $\{x_i\}$ is equiv to ℓ_1 ubv in the 1st ℓ_1, take its block basis that starts sufficiently far)
(James's ℓ_1-distortion theorem, see Albiac-Kalton)

$$\begin{array}{lccc} \text{let} & T : Z \overset{1+\varepsilon}{\sim} \ell_1 & \twoheadrightarrow & C(\Delta) \\ & \cap & & \cap \\ & S : X & \longrightarrow & \ell_\infty(\Delta) \end{array}$$

extend linearly with the same $\|\cdot\|$

in fact $\overline{S(X)} =: Y \subset \ell_\infty(\Delta)$ is separable
Pelczynski $\exists$ 1-complemented copy of $C(\Delta) = E$ in $C(\Delta) \subset Y$ (0 dim; sep.)
denote $P : Y \to E$ a projection $\|P\| = 1$

$P \circ S$ is a quotient (onto), $\|P \circ S\| \le 1+\varepsilon$ $\square$ ②

~~Prop~~ <u>Definition</u>: a metric space X is <u>metrically convex</u> if

$\forall\, x, y \in X \ \forall \lambda \in [0,1] \ \exists z$ s.t. $d(x,z) = \lambda d(x,y)$ & $d(y,z) = 1-\lambda\, d(x,y)$

Proposition 3: $\forall Y$ sep., complete, metr. convex m.s. : $C(\Delta) \xrightarrow{1-L} Y$

<u>Def.</u>: • Let $(X, d_X), (Y, d_Y)$ be m.s. s.t. $X \cap Y = \{p\}$ for some p

we put <u>ℓ_1-union</u> $X \cup_1 Y = (X \cup Y, d)$ where

$$d(x, y) = \begin{cases} d_X(x,y), & x, y \in X \\ d_Y(x,y), & x, y \in Y \\ d_X(x,p) + d_Y(p,y), & x \in X,\ y \in Y \end{cases}$$

• a m.s. (T, d) is called SMT if $\exists \{I_n\}_{n=1}^{\infty} \subset 2^T$ s.t.

• $\forall n$ (I_n, d) is isometric to a closed interval or a closed ray

• if we call $T_n = \bigcup_{i=1}^{n} I_n$, then $T_n \cap I_{n+1} = \{p_n\}$

where p_n is an end point of I_{n+1}

and $T_{n+1} = T_n \cup_1 I_{n+1}$

• T is the completion of $\bigcup_{n=1}^{\infty} T_n$

• an SMT T is called an ℓ_1-tree if $\exists \{I_n\}$ as above

+ all I_n are rays

+ $\{p_n\}_{n=1}^{\infty}$ is dense in T

- any SMT T is isometric to a subset of ℓ_1:
- $f(p_0) = 0$, $f(I_1) \subset \mathbb{R}^+ e_1$ isometrically
- if $f(x)$ has been defined for $x \in T_n$
- map I_{n+1} isometrically into $f(p_n) + \mathbb{R}^+ e_{n+1}$
- make a completion

Recall: $f : X \to Y$ is 1-Lip. quotient if $\operatorname{Lip} f \cdot \operatorname{coLip} f = 1$

$$\operatorname{Lip} f = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)}$$

$$\operatorname{coLip} f = \inf \{c : f(B(x,r)) \supset B(f(x), \tfrac{r}{c}) \ \forall x \in X, \forall r > 0\}$$

my question: is the inf in fact min? no let $X^* \ni f$ for a non-reflexive X s.t. $\max f$ is not attained

Lemma 4: Let $f : X \to Y$ be 1-Lipschitz, $X_0 \subset X$ dense s.t. $\forall x \in X_0$ $\forall r \in \mathbb{Q}^+$

$$B_Y^0(f(x), r) \subset \overline{f(B_X(x,r))}$$

then $\forall x \in X, \forall r > 0 : B_Y^0(f(x), r) \subset f(B_X^0(x,r))$

Proof: Let $d(f(x), y) < r - \varepsilon$ for some $\varepsilon > 0$.

say $< r - \varepsilon - \eta$ $\quad$ x_a, $d(x_a, x) < \frac{\eta}{2} \Rightarrow d(f(x_a), f(x)) < \frac{\eta}{2}$ $\Rightarrow d(y, f(x_a)) < r - \varepsilon - \frac{\eta}{2}$

$\exists\, x_1 \in X$

$d(x_1, x) \leq r - \varepsilon$ $\quad$ (as $d(x_1, x_a) \leq r - \varepsilon - \frac{\eta}{2}$ & $d(x_a, x) < \frac{\eta}{2}$)

$$d(f(x_1), y) < \frac{\varepsilon}{2}$$

$\exists\, x_2 \in X \quad d(x_2, x_1) < \frac{\varepsilon}{2}$ & $d(f(x_2), y) < \frac{\varepsilon}{2^2}$

$\vdots$

$\exists\, x_{n+1} \in X \quad d(x_{n+1}, x_n) < \frac{\varepsilon}{2^n}$ & $d(f(x_{n+1}), y_0) < \frac{\varepsilon}{2^{n+1}}$

$\Rightarrow x_n \to \tilde{x}$: $\|x - \tilde{x}\| < \underbrace{r - \varepsilon}_{d(x_1, x)} + \underbrace{\varepsilon}_{\sum_{i=1}^{\infty} d(x_i, x_{i+1})}$

$f(x_n) \to y = f(\tilde{x})$ $\quad \square$

$\textcircled{4}$

Proposition 5: Let T be an $\mathbb{R}$-tree, Y separable, complete, metrically convex metric space $\Rightarrow \exists\, f : T \xrightarrow{1-L} Y$.

Proof.: Let $Y_0 = \{y_n\}_{n=1}^{\infty}$ be dense in Y.

Since $\{p_n\}_{n=0}^{\infty}$ is dense & T is connected

$\forall m\ \exists\, \varphi_m : \mathbb{N} \to \mathbb{N}$ s.t. $\{\varphi_n(k)\}_k \cap \{\varphi_m(k)\}_k = \emptyset,\ n \neq m$ and $\bigcup_n \{\varphi_n(k)\}_k = \mathbb{N}$ $\Big\}$ $(n,k) \mapsto \varphi_n(k)$ is bijection $\mathbb{N}^2 \to \mathbb{N}$

and $p_{\varphi_n(k)} \to p_n$ as $k \to \infty$ $\forall n \in \mathbb{N}$

let $(\varphi_1(k))_{k=1}^{\infty} \subset \mathbb{N}$ s.t. $p_{\varphi_1(k)} \to p_1,\ k \to \infty$
if $(\varphi_1, \dots, \varphi_n)$ known disjoint $\varphi_{n+1} \subset \mathbb{N} \setminus \bigcup_{m \le n} \{\varphi_m\}$
s.t. $p_{\varphi_{n+1}(k)} \to p_{n+1}$, if afterall there are still some points (z_n) put $\varphi_n(0) = z_n$

- $f_0(p_0)$ arbitrary $\in Y$

suppose f_n has been defined, 1-Lipschitz

$f_n : T_n \to Y \Rightarrow \exists!\ (m,k)$ s.t. $n = \varphi_m(k)$

let $f_{n+1} : T_{n+1} \to Y$ be the extension of f_n s.t.

- the set $\{x \in I_{n+1} : d(p_n, x) \le d(f(p_n), y_k)\}$ is mapped onto a geodesic arc between $f(p_n)$ and y_k isometrically

- the set $\{x \in I_{n+1} : d(p_n, x) > d(f(p_n), y_k)\}$ is mapped onto y_k

then f_{n+1} is 1-Lip.

let f be the unique 1-Lip. $: T \to Y$ which extends $f_n\ \forall n$

---

we need to show that for every $p_N \in \overline{\{p_n\}_{n=1}^{\infty}}$, $r \in \mathbb{Q}^+$ and

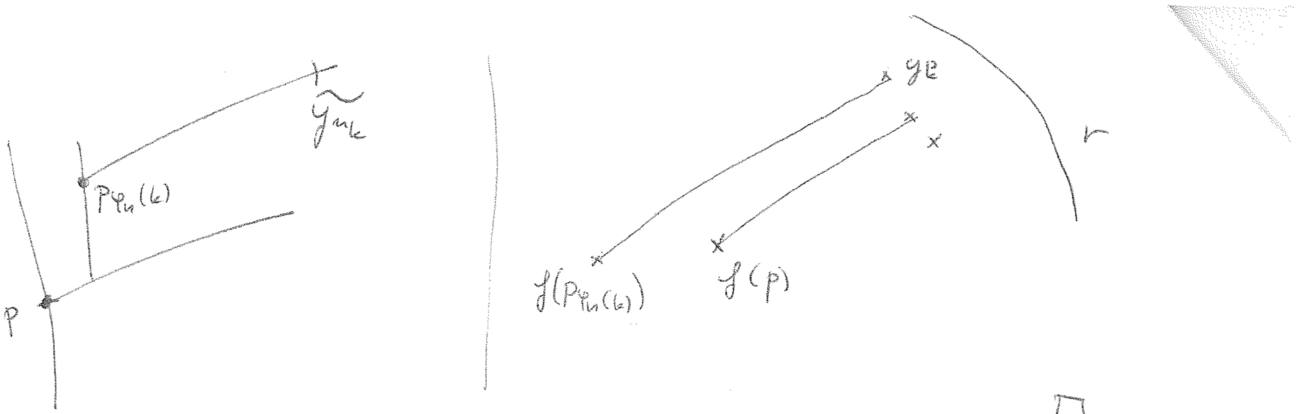
$x \in B^0_Y(f(p_N), r)$ $\Rightarrow x \in f(B_T(p_n, r))$

let $\varepsilon > 0$ be arbitrary s.t. $d(f(p), x) < r - \varepsilon$

$\exists k$ s.t. $d(p_{\varphi_N(k)}, p) < \frac{\varepsilon}{4}$ & $d(x, y_k) < \frac{\varepsilon}{2}$

$\Rightarrow d(f(p), y_k) < r - \frac{\varepsilon}{2}$ & $d(f(p_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}$

$\Rightarrow d(p, \tilde{y}_k) \le \underbrace{d(p, p_{\varphi_N(k)})}_{< \varepsilon/4} + \underbrace{d(p_{\varphi_N(k)}, \tilde{y}_k)}_{= d(f(p_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}} < r$



$\square$

Proposition 6: For Every SMT T : $C(\Delta) \xrightarrow{1-L} T$.

Lemma 7: T is a 1-absolute Lipschitz retract

Proof:
Observation: X, Y 1-abs Lip. retracts $\Rightarrow X \cup_1 Y$ is 1-a.L.r.
$|X \cap Y| = 1$

Recall: X is 1-a.L.r. if $\forall Y$ m.s. s.t. $X \subset Y$
$\exists R : Y \to X$, 1-Lip. s.t. $\forall x \in X$: $Rx = x$.

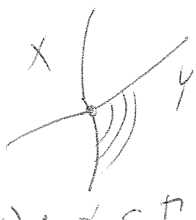
$*$ classical: X is 1-a.L.r. $\Leftrightarrow$ X is metrically convex
and every collection of mutually intersecting balls has a common point

let $\{B(x_\alpha, r_\alpha)\}_{\alpha \in \Gamma_1} \cup \{B(y_\beta, \rho_\beta)\}_{\beta \in \Gamma_2}$ be mutually intersecting balls
in $X \cup_1 Y$

$\Rightarrow \forall \alpha, \beta \quad r_\alpha + \rho_\beta \geq d(x_\alpha, p) + d(y_\beta, p)$

- if $r_\alpha \geq d(x_\alpha, p)$ & $\rho_\beta \geq d(y_\beta, p)$ $\forall \alpha, \beta \Rightarrow p$ is common point

- if not



$\exists \alpha_0 \in \Gamma_1 : r_{\alpha_0} < d(x_{\alpha_0}, p)$

$\Rightarrow \{B(x_\alpha, r_\alpha) : \alpha \in \Gamma_1\} \cup \{B(p, \rho_\beta - d(y_\beta, p)) : \beta \in \Gamma_2\}$
are mutually intersecting in $X \Rightarrow \exists$ point in common

$\square$

Let $T \subset S$. Each T_n is 1-a.L.r. $P_n : S \longrightarrow T_n$ (6)

$\forall n \in \mathbb{N}$ $P_n \in \prod_{s \in S} \left(B_{\ell_1}(0, d(s, p_0)), w^*\right)$ (better ~~$P_n \in \prod (T \cap B_{\ell_1} \dots$~~)

as if 1-Lip map $f : S \to T$ satisfies $f(p_0) = p_0 \Rightarrow d(f(s), p_0) \le d(s, p_0)$ $\forall s$

$\Rightarrow \exists$ a cluster point P, necessarily 1-Lip, retraction

(if $\|Ps - Pt\|_{\ell_1} > d(s,t) \Rightarrow \exists x \in c_0 : P \in \underbrace{\{Q : |(Qs)x - (Qt)x| > d(t,s)\}}_{\text{open}} \ni P_n$

(retract : - $\forall t \in \bigcup_{n=1}^{\infty} T_n$ $\exists$ only finitely many n s.t. $P_n(t) \neq t$

- $\forall t \in \overline{\bigcup^{\infty} T_n}$ ... lipschitzness of P

- $\forall s \in S$ ~~$\exists$~~ $Ps \in T$ as T is w^*-closed

because $T = \bigcap_{m \ge 1} \left(P_m(T_n) \times \overline{\operatorname{span}}\{e_k : k > m\}\right)$

1) $\forall m \ge n$ $T_m \subset P_n(T_n) \times \overline{\operatorname{span}}\{e_k : k > n\}$

$\Rightarrow \bigcup_{m \ge n} T_m \subset$ ———

$\Rightarrow \underbrace{\bigcap_{n \ge 1} \bigcup_{m \ge n} T_m}_{\bigcup_{m=1}^{\infty} T_m} \subset \underbrace{\bigcap_{n \ge 1} P_n(T_n) \times \overline{\operatorname{span}}\{e_k : k > n\}}_{w^*\text{-closed}} = F$

~~[illegible]~~ $\forall x \in F \Rightarrow x \in T$

indeed as $\forall n$: $P_n x \in P_n(T_n) \Rightarrow \underbrace{P_n x \times (0, 0, \dots)}_{y_n} \in T_n$

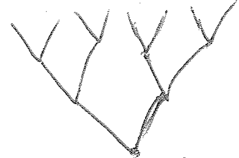
$\|x - y_n\| = \sum_{m > n}^{\infty} x_m \longrightarrow 0$

$\square \leftarrow$ end of proof of L7.

Let $r_n : \Delta \longrightarrow \{-1,1\}$ be the n-th coordinate projection

$r_n \in C(\Delta)$, $\Delta = \{-1,1\}^{\mathbb{N}}$

Let $E_n \subset C(\Delta)$ be the space of functions that depend on the 1st n coordinates ($\dim E_n = 2^n$)



Observe that $\overline{\bigcup E_n} = C(\Delta)$

By Stone–Weierstrass (xy, $x+y$, λx, $\mathbb{1}$, separates points)

let us observe that for $x \in E_n$, $m > n$, $\forall t \in \mathbb{R}$:

$$\|x + t r_m\| = \|x\| + |t| \quad (*)$$

Let $T \sim \{T_n\}$, $(x_n) \subset \bigcup E_n$ dense s.t. $x_n \in E_n$ $\forall n$

let $\{y_k\} \subset \bigcup T_n$ s.t. $\{y_k\}_k \cap T_n$ is dense in T_n $\forall n$ and $y_k \in T_{k+1}$

$\forall n$ $\exists \varphi_n : \mathbb{N} \to \mathbb{N}$ s.t. $(n,k) \mapsto \varphi_n(k)$ is a bijection $\mathbb{N}^2 \to \mathbb{N}$

and $x_{\varphi_n(k)} \longrightarrow x_n$ as $k \to \infty$ $\forall n \in \mathbb{N}$

- and MOREOVER $k \le \varphi_n(k)$ $\forall n,k$ .. achieved by insisting that $\forall n$ $(\varphi_n(k))_{k=1}^{\infty} \nearrow$

  if some points are left out after all $\Rightarrow$ just forget them
- let $f_0(x_0) = p_0$

  ass $x_0 = 0$ and $E_0 := \{0\}$ (and we constants)

Suppose f_n has been defined, 1-Lip.

$$f_n : E_n \longrightarrow T_n$$

$$\exists! \ (m,k) \ : \ \varphi_m(k) = n$$

extend f_n to $\tilde{f}_n : E_n \cup (x_n + \mathbb{R}^+ r_{n+1}) \longrightarrow T_{n+1}$

by mapping $\{x_n + t r_{n+1} : 0 \le t \le d(f_n(x_n), y_k)\}$ onto (isometrically)

the geodesic arc between $f_n(x_n)$ and y_k

put $\tilde{f}_n(x) = y_k$ if $x = x_n + t r_{n+1}$ with $t > d(f_n(x_n), y_k)$

then $\tilde{f}_n$ is 1-Lip thanks to $(*)$

T_{n+1} is 1-a.L.r. (8)

$$\tilde{f}_n : \underbrace{E_n \cup (x_n + \mathbb{R}^+ r_{m+1})}_{U} \longrightarrow T_{m+1}$$ can be extended

$$f_{n+1} : E_{n+1} \longrightarrow T_{m+1} \text{, 1-Lip.}$$

f the unique 1-Lip extension of all f_n

We need to verify that $\forall x_N \in \{x_n\}_{n=1}^{\infty}$, $r \in \mathbb{Q}^+$ and

$y \in B_T^O(f(x_N), r)$: $y \in f(B_{C(\Delta)}(x_n, r))$

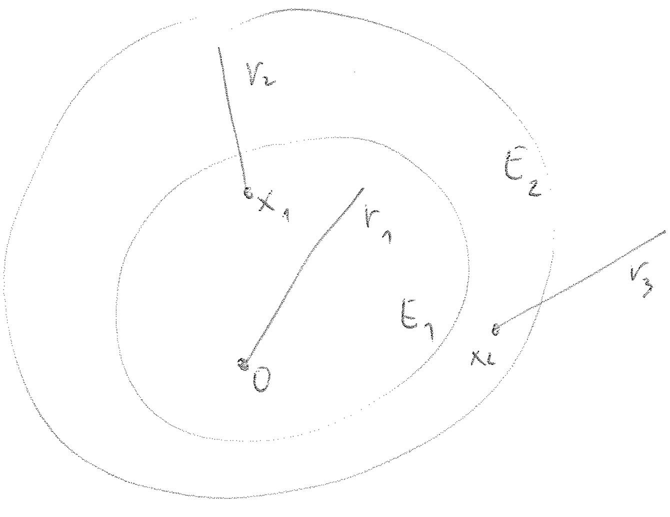
let $\varepsilon > 0$ be arbitrary s.t. $d(f(x_N), y) < r - \varepsilon$

$\exists k$ s.t. $d(x_{\varphi_N(k)}, x_N) < \frac{\varepsilon}{4}$ & $d(y, y_k) < \frac{\varepsilon}{2}$

$\Rightarrow d(f(x_N), y_k) < r - \frac{\varepsilon}{2}$ & $d(f(x_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}$

$$\Rightarrow d(x_N, \tilde{y}_k) \leq \underbrace{d(x_N, x_{\varphi_N(k)})}_{< \frac{\varepsilon}{4}} + \underbrace{d(x_{\varphi_N(k)}, \tilde{y}_k)}_{= d(f(x_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}} < r$$

Finish by L4 $\square$



guixiang

Stochastic metric decompositions D'après Lee-Naor

Déf. 3.1 : Let (Y,d) be a metric space and X a subspace of Y. Then $(\Omega, \mu, \{\Pi^i(\cdot), \gamma^i(\cdot)\}_{i\in I})$ is a stochastic decomposition of Y wrt X if I is some index set, (Ω,μ) is a prob. space, and $\{\Pi^i(\omega)\}_{i\in I}$ is a partition of Y into Borel sets for every $\omega\in\Omega$, and $\{i\in I : \exists \omega\in\Omega \ x\in \Pi^i(\omega)\}$ is countable for every $x\in Y$, and $\gamma^i:\Omega\longrightarrow X$ is a Borel measurable function for every $i\in I$ st. $d(\gamma^i(\omega), \Pi^i(\omega))\leq 2\, d(X, \Gamma^i(\omega))$ for every $\omega\in\Omega$, and $\{\omega\in\Omega : x\in\Gamma^i(\omega)\}$ is measurable.

Def 3.2 : (bounded S.D.) : Let $\Delta>0$. We say that the SD is bounded by Δ if $\forall\omega\in\Omega\,\forall i\in I$ $\operatorname{diam}(\Pi^i(\omega))<\Delta$.

Def 3.3 : (padded decomposition) We say a SD is (ε,δ)-padded if $\omega\mapsto d(x, Y\setminus\Pi^i(\omega))$ for every $x\in Y$ and $i\in I$ is measurable and, if $d(x,X)\leq\varepsilon\Delta$, then $\mu\left(\bigcup_{i\in I}\{\omega : d(x, Y\setminus\Gamma^i(\omega))\geq\varepsilon\Delta\}\right)\geq\delta$.

$\rightarrow$ fix ω and consider the partition $\{\Pi^i(\omega)\}_{i\in I}$ of Y: $\forall x$ $\exists!\, i_0\in I$ $x\in\Gamma^{i_0}(\omega)$, so that $d(x, Y\setminus\Gamma^i(\omega)) = d(x, \Gamma^{i_0}(\omega)^c)$ and we want that $B(x,\varepsilon\Delta)\subset\Pi^{i_0}(\omega)$

Def 3.4 : (thick decomposition) : a SD is (ε,δ)-thick if $\forall x\in X\,\forall i\in I$ $\omega\mapsto d(x, Y\setminus\Gamma^i(\omega))$ is measurable and if $d(x,X)\leq\varepsilon\delta \Rightarrow \int_\Omega\sum_{i\in I}\min\{d(x, Y\setminus\Gamma^i(\omega)), \Delta\}\, d\mu(\omega)\geq\delta\Delta$

¿ Why take the minimum here ?

N.B. : (ε,δ)-padded $\Rightarrow(\varepsilon,\varepsilon\delta)$-thick : $\int_\Omega \frac{\sum_{i\in I} d(X, Y\setminus\Gamma_i(\omega))}{\varepsilon\Delta}\, d\mu(\omega)\geq\mu\left(\bigcup_{i\in I}\{\omega : d(x, Y\setminus\Gamma^i(\omega))\geq\varepsilon\delta\}\right)\geq\delta\Delta$

Def 3.5 : (separating decomposition) A SD is (ε,δ)-separated if $\forall x,y\in Y$ with $d(\{x,y\},X)\leq\varepsilon\Delta$ one has $\int_\Omega\sum_{i\in I}\left|1_{\Gamma^i(\omega)}(x) - 1_{\Gamma^i(\omega)}(y)\right| d\mu(\omega)\leq\frac{2d(x,y)}{\delta\Delta}$

§3.2 : Construction

Let (X,d) be a metric space and $R>r>0$. Let $C(X,R,r)$ be the largest cardinality of a set $N\subset X$ satisfying $\forall x\neq y\in N$ $r\leq d(x,y)\leq R$

Lemma 3.11 : Let $\Delta>0$. Every (X,d) admits an $(\varepsilon,\frac{1}{2})$-padded Δ-bounded finitely supported decomposition of X wrt itself, with $\varepsilon = \frac{1}{256\log C(X, 2\Delta, \frac{\Delta}{4})}$

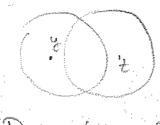
②

Proof: Consider a 4Δ-padded decomposition; assume N is a Δ-net for X.
Firstly we need to introduce a distribution over partial orders $<$ on N s.t. for every ball $B \subset X$ of radius 3Δ, $<$ is a uniformly total order on $B \cap N$. ($|B \cap N|$ is finite)

We consider the infinite graph $G = (N, E)$, where $\{x,y\} \in E$ iff $d(x,y) \leq 3\Delta$.
Then $\deg G \leq C(X, 6\Delta, \Delta) \leq C(X, 8\Delta, \Delta) < \infty$, therefore G admits a proper colouring using a finite number $M \leq C(X,6\Delta,\Delta)+1$ of colouring classes.
These classes $1, 2, \ldots, M$. Let $\chi : N \to \{1, \ldots, M\}$ be a proper colouring and let σ be a random permutation on $\{1, \ldots, M\}$. Define $x \,\underline{<}\, y$ by $\sigma(\chi(x)) \underline{<} \sigma(\chi(y))$

It is easy to see that for every ball of radius 3Δ, every point in $B \cap N$ receives a unique colour and σ induces a uniform r.p. on B.
Let us choose $R \in [\Delta, 2\Delta]$ uniformly at random, and for each $y \in N$ define a cluster

$C_y = \{x \in X : x \in B(y,R)$ and if $x \in B(z,R)$, then $y < z\}$



$C_y \cap C_z = \emptyset$ if $y < z$

Let $P = \{C_y\}_{y \in N}$ be a partition of X and $\operatorname{diam}(C_y) \leq 2R \leq 4\Delta$.
For $y \in N$ let γ_y be the minimal element of N in C_y. Now fix $t \in [0, \Delta]$ and $x \in X$. Note that $\mathbb{P}[B(x,t) \subset \Gamma(w,x)] = 1 - \mathbb{P}[B(x,t) \not\subset \Gamma(w,x)]$ (*)

$\Gamma(w,x)$ cuts $B(x,t)$.
There is $z \in N$ s.t. $C_z \cap B(x,t) \neq \emptyset$, let y be an element of it. Then $d(z,x) \leq d(x,y) + d(z,y) \leq t + f \leq t + 2\Delta < 3\Delta$. Then $N \cap B(x, 3\Delta)$...
Let $W = B(x, 2\Delta + t) \cap N$ and $m = |W| \leq C(X, 6\Delta, \Delta)$. Arrange the points $w_1, \ldots, w_m \in W$ in order of increasing distance from x. Let $I_k = [d(x,w_k) - t, d(x,w_k) + t]$
Now let $\mathcal{E}_k$ be the event that w_k is a minimal element in W (for $<$) for which C_{w_k} cuts $B(x,t)$: $\forall\, 1 \leq k \leq m \quad \mathbb{P}[\mathcal{E}_k \mid R \in I_{k}] \leq \frac{1}{k}$
?
The event $\mathcal{E}_k$ implies $R \in I_k$
N.B. Consider $1, \ldots, m$. w_k appears before $w_1, \ldots, w_{k-1}$
By contradiction, if we assume $\exists\, 1 \leq i \leq k-1$ s.t. $w_i < w_k$, then by minimality of w_k C_{w_i} should not cut $B(x,t)$: (i) $C_{w_i} \supset B(x,t)$
(ii) $C_{w_i} \cap B(x,t) = \emptyset$

(*) continues into $P[B(x,t)\text{ cut }\Gamma(\omega,x)] \le \sum_{k=1}^{m} P[\mathcal{E}_k] \le \sum_{k=1}^{m} P[R\in I_k]\,P[\mathcal{E}_k \mid R\in I_k]$

$$\le \sum_{k=1}^{m} \frac{2t}{\Delta}\frac{1}{k} \le \frac{2t}{\Delta}(1+\log m) \le \frac{\Delta t}{8}\log C(X;6\Delta,\Delta)$$

Another decomposition in the case of compact X.

Th 3.17: If σ is a nondegenerate Borel measure on X (ie $\sigma(B(x,r)\cap X)>0$), then $\forall \Delta>0$ $\exists$ Δ-decomposition $\int_\Omega \sum_{i\in I} |1_{\Gamma^i(\omega)}(x) - 1_{\Gamma^i(\omega)}(y)|\, d\mu(\omega)$

$$\le \frac{2d(x,y)}{\Delta}\left[1+\log\frac{\sigma(B_X(x,5\Delta))}{\sigma(B_X(x,\Delta))}\right]$$

Proof: Let $\Omega' = \left(\prod_{i=1}^{\infty} X, \mu'\right)$ and $\Omega = \Omega'\times[2\Delta,4\Delta]$ and $P=\mu'\times\lambda$.

Consider $\Omega'\ni\omega=(x_1,x_2,x_3,\dots)$ and $R\in[2\Delta,4\Delta]$. Let us construct a seq of subsets of Y, $\{\Gamma^i(\omega,h) : i\in\mathbb{N}\}$, $\Gamma^k(\omega,k) = B(x_k,h)\setminus\bigcup_{i=1}^{k-1}\Gamma^i(\omega,h)$.

<u>Claim</u>: $\{\Gamma^i(\omega,k)\cap S\}_{i\in\mathbb{N}}$ is a partition of $S=\{y\in Y : d(X,y)<\frac{\Delta}{2}\}$

For every $x\in X$ let $j_R(x) = \inf\{j : d(x_j,x)\le R\}$: $j_R(x)$ is finite for all $x\in B(X,2\Delta)$ with probability 1: $\mu'\{j_R(x)=\infty\}=0 = \prod_{i\in\mathbb{N}} \underbrace{\sigma(d(x_i,x)>R)}_{\text{uniformly}<1} = 0$.

Fix $x\in Y$ st $d(x,X)<\frac{\Delta}{2}$ and denote ν the distribution $\nu([\alpha,\beta]) = \sigma(z\in X : \alpha\le d(z,x)\le\beta)$

Fix $t\in\Delta$ and let D_R be the set of all $z\in X$ st $B(x,t)$ is cut by $B(z,R)$.

Finally, let $\Omega_{i,R} = \{\omega\in\Omega' : \omega_i\in D_R\}\setminus\bigcup_{j=1}^{i-1}\{\omega\in\Omega' : \omega_j\in D_R\}$

Consider $\Omega'\ni\omega\in\Omega_{i,R}$ then $\underline{R\in[d(x,\omega_i)-t, d(x,\omega_i)+t]}$.

[Let $y\in B(\omega_i,R)\cap B(x,t)$: $\Rightarrow$ $d(x,\omega_i)\le d(x,y)+d(y,\omega_i)\le t+R$

[Let $y\in B(x,t)\setminus B(\omega_i,R)$: $\Rightarrow$ $d(x,\omega_i)\ge d(y,\omega_i)-d(x,y)\ge R-t$.

We have $\underline{\Delta < d(\omega_i,x)\le 5\Delta.}$

[If $d(\omega_i,x)\le\Delta$, i.e. $\omega_i\in B(x,\Delta)$, then $B(x,t)\subset B(x,\Delta)\subset B(\omega_i,R)$ because $t\le\Delta$ and $R\ge 2\Delta$, which is in contradiction with our def$^{\underline{n}}$ of «cuts»

then $d(\omega_i,x)\le R+t\le 4\Delta+\Delta=5\Delta$.

If $\omega\in\Omega_{i,R}$, $d(\omega_j,x)>d(\omega_i,x)$ for all $j<i$.

For all $l>0$ $\mu'(\Omega_{i,R}\mid d(x,\omega_i)=l)\le 1_{\{R\in[l-t,l+t]\}}\cdot 1_{\{\Delta<l\le 5\Delta\}}\ \mu'\{\omega : 1\le j<i : d(\omega_j,x)\ldots\}$

$$\le \qquad [1-\nu[0,l]]^{i-1}$$

(9)

Hence $\mathbb{P}[B(x,t)$ is cut$] = \frac{1}{2\Delta}\int_{2\Delta}^{4\Delta}\left(\sum_{i=1}^{\infty}\mu'(\Omega_{i,R})\right)dR$

$$= \frac{1}{2\Delta}\int_{2\Delta}^{4\Delta}\sum_{i=1}^{\infty}\int_0^{\infty}\mu'(\Omega_{i,R}(d(x,\omega_i)=\rho))\,d\nu(\rho)\,dR$$

$$< \frac{1}{2\Delta}\int_{2\Delta}^{4\Delta}\sum_{i=1}^{\infty}\int_{[\Delta,5\Delta]}\mathbb{1}_{\{R\in[\rho-t,\rho+t]\}}\,[1-\nu([0,\rho])]^{i-1}\,d\nu(\rho)\,]dR$$

$$\leq \frac{t}{\Delta}\int_{\Delta}^{5\Delta}\frac{d\nu(\rho)}{\nu[0,\rho]} \leq \frac{t}{\Delta} \underset{?}{} \frac{\nu(0,5\Delta)}{\nu(0,\Delta]}$$

if $d(x,y) < \Delta$ and $t=d[x,y)$?

Def: (Y,d) metric space, $X\subset Y$ subspace, $(\Omega,\mathcal{B},\mu)$ measure space.
Given $K>0$, $\psi:\Omega\times Y\to[0,\infty[$ is a K-gentle partition of unity wrt X if

① $\forall x\in Y\setminus\overline{X}$ $\omega\longmapsto\psi(\omega,x)$ is measurable and $\int_\Omega\psi(\omega,x)\,d\mu(\omega)=1$.

② $\forall\omega\in\Omega$ $\forall x\in X$ $\psi(\omega,x)=0$

③ $\exists\gamma:\Omega\to\overline{X}$ $\forall x,y\in Y$ $\int_\Omega d(\gamma(\omega),x)\,|\psi(\omega,x)-\psi(\omega,y)|\,d\mu(\omega)\leq K\,d(x,y)$.

Th 4.1: $\exists C>0$ $\forall Y>X$ ① Let $n\in\mathbb{Z}$. If Y is an (ε,δ)-thick 2^n-bounded SD wrt X, then Y is a $\frac{C}{\varepsilon\delta}$-gentle partition of unity wrt X.

② Idem for (ε,δ)-padded SD

③ Idem for (ε,δ)-separating SD with $\frac{C}{\varepsilon\delta}$ replaced by $\frac{C(\delta+\varepsilon)}{\varepsilon\delta}$

Proof: Let φ be a 2 Lipschitz map with supp $\varphi\subset[\frac{1}{2},1]$ and $\varphi=1$ on $(1,2)$.
Let $\varphi_n(x)=\varphi\left(\frac{d(x,X)}{\varepsilon 2^{n-3}}\right)$. Let (Ω,μ) be the disjoint union of $\{I\times\Omega_n\}_{n\in\mathbb{Z}}$.

Let $\psi(i,\omega,x)=\frac{1}{s(x)}\,\vartheta^n_\omega(x)\,\varphi_n(x)\,\mathbb{1}_{\Gamma^i_n(\omega)}(x)$, where $\forall n\in\mathbb{Z}$ $\forall x\in Y$ $\omega\longmapsto\vartheta^n_\omega(x)\in(0,\infty)$ is μ_n-integrable,
and $S(x)=\sum_{n\in\mathbb{Z}}\sum_{i\in I}\int_{\Omega_n}\vartheta^n_\omega(x)\,\varphi_n(x)\,\mathbb{1}_{\Gamma^i_n(\omega)}(x)\,d\mu_n(\omega)$, and $\psi(\omega,X)=0$ if $x\in\overline{X}$.

We want to compute $\sum_{n\in\mathbb{Z}}\sum_{i\in I}\int_{\Omega_n}d(\gamma(i,\omega),x)\,|\psi(i,\omega,x)-\psi(i,\omega,y)|\,d\mu(\omega)\leq\begin{cases}\frac{C}{\varepsilon\delta}d(x,y) & \text{for ①②}\\ \frac{C(\varepsilon+\delta)}{\varepsilon\delta}d(x,y) & \text{in ③}\end{cases}$

Claim 4.2: Fix $\omega\in\Omega_n$ and assume that $\psi(i,\omega,x)\neq\psi(i,\omega,y)$. Then $d(\gamma(i,\omega),x)\leq b(x,y)+$

Proof: $\psi(i,\omega,x)>0$ or $\psi(i,\omega,y)>0$. $d(\gamma(i,\omega),x)\leq d(\gamma(i,\omega),\Gamma^i_n(\omega))+\mathrm{diam}(\Gamma^i_n(\omega))\ \frac{18}{\varepsilon}\max\{d(x,X),d(y,X)\}$
$\leq 2d(X,\Gamma^i_n(\omega))+2^n\leq 2d(x,X)+2^n$.

$\varphi_n(x)>0$, $\frac{d(x,X)}{\varepsilon 2^{n-3}}\geq\frac{1}{2}$, so that $d(x,X)\geq\varepsilon 2^{n-4}$ and $2^n\leq\frac{16}{\varepsilon}d(x,X)$

As $0<\varepsilon\leq 1$, $2d(x,X)\leq\frac{2}{\varepsilon}d(x,X)$

ⓐ $\psi(i,\omega,y)>0$ for $y\in\Gamma^i(\omega)$.

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21.5.12

Thus $d(\gamma(i,\omega),x) \le d(\gamma(i,\omega),y) + d(X,y)$

$\le d(\gamma(i,\omega),\Gamma_n^i(\omega)) + \mathrm{diam}\,(\Gamma_n^i(\omega)) + d(x,y)$

$\le 2\,d(y,X) + 2^{7}\phi(x,y)$

and $I \le d(x,y) \sum_n' \sum_i' \int_{\Omega_n} [\psi(i,\omega,x) + \psi(i,\omega,y)]\, d\mu_n(\omega)$

$+ \frac{18}{\varepsilon} \max\{d(x,X), d(y,X)\} \sum_n' \sum_i' \int_{\Omega_n} |\psi(i,\omega,x) - \psi(i,\omega,y)|\, d\mu_n(\omega)$

$\le 2\,d(x,y) + \ldots$

$d(x,y) < d(\{x,y\},X)$; $d(x,X) \geq d(y,X)$.

$I \le \frac{2}{S(x)} \sum_n' \int_{\Omega_n} \sum_i |\vartheta_\omega^n(x)\varphi_n(x)\mathbb{1}_{\Gamma_n^i}(x) - \vartheta_\omega^n(y)\varphi_n(y)\mathbb{1}_{\Gamma_n^i}(y)|\, d\mu_n(\omega)$.

1st Case: $\vartheta_\omega^n(x) = \Pi_\omega^n(x)$ with $\Pi_\omega^n(x) = \sum_{i\in I} \min\{\phi(x), \gamma(\Gamma_n^i(\omega)), 2^n\}$

Let n_0 s.t. $\frac{d(x,X)}{\varepsilon 2^{n_0-3}} \in [1,2]$. Then $\rho(x) = \sum_{n\in\mathbb{Z}}' \varphi_n(x) \int_{\Omega_n} \Pi_\omega^n(x)\, d\mu_n(\omega)$

2nd case: $\vartheta_\omega^n(x) = g\left(\frac{\Pi_\omega^n(x)}{\varepsilon 2^{n-1}}\right)$ with $g = \begin{cases} 1 & \text{if } x \geq 2 \\ x-1 & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 0 \le x \le 1 \end{cases}$

3rd case: $\vartheta_\omega^n(x) = 1$

1

Petr Hajek
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"Smoothness and weak continuity"

(we will stick to the real case)

Introduce the calculus in Banach space theory.

Let X, Y be Banach spaces and $T: X \longrightarrow Y$.
We will try to define that T is C^k-smooth. This is parallel to the theory of uniform and Lipschitz maps, but different.

Def: $X_1, \dots, X_n, Y$ Banach spaces. $M: X_1 \times \dots \times X_n$ is n-linear if

$$M(\lambda_1 x_1 + \lambda_2 x_2, x_2', \dots, x_n') = \lambda_1 M(x_1, x_2', \dots, x_n') + \lambda_2 M(x_2, x_2', \dots, x_n')$$

and the same for all entries. We write $M \in L(X_1, \dots, X_n; Y)$ and set

$\|M\| = \sup_{\|x_i\| \leq 1} \|M(x_1, \dots, x_n)\|$ and this makes a Banach space $\mathcal{L}(X_1, \dots, X_n; Y)$.

In the special case $X_1, \dots, X_n = X$, we write $\mathcal{L}(^nX; Y)$

Def: Let X, Y B-spaces (Banach?). Then $P: X \longrightarrow Y$ is an n-homogeneous polynomial if $\exists M \in L(^nX; Y)$ $P(x) = M(\underbrace{x, \dots, x}_{n \text{ times}})$. We say P is bounded if $\|P\| = \sup_{\|x\| \leq 1} \|P(x)\|_Y < \infty$
$\rightarrow \mathcal{P}(^nX; Y) \ni P = \hat{M}$ and n is the degree of the polynomial

Th: If $M \in L(^nX; Y)$, TFAE: (1) M is bdd (2) M is continuous (3) M is Borel measurable (4) M is separately continuous (5) M is bounded on some open set (6) M is Lipschitz on every bounded set.

Fact: P is bounded iff M is bounded

Def: We say that $M \in \mathcal{L}(^nX, Y)$ is symmetric if $M(x_1, \dots, x_n) = M(x_{\pi(1)}, \dots, x_{\pi(n)})$ for every $\pi \in \mathfrak{S}_n$.

Note: There is a natural mapping $M \longrightarrow M^s$, $M^s(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} M(x_{\pi(1)}, \dots, x_{\pi(n)})$

In fact, this mapping is a bounded projection with image $\mathcal{L}^s(^nX; Y)$

Fact: $P(x) = M(x, \dots, x) = M^s(x, \dots, x)$

Polarisation formula: For every $P \in \mathcal{P}(^nX; Y)$, there is a unique n-linear and symmetric mapping $M = \check{P}$ such that $P = \hat{M}$: $M(x_1, \dots, x_n) = \frac{1}{n! 2^n} \sum_{\varepsilon \in \{-1,1\}^n} P(\sum \varepsilon_i x_i) \varepsilon_1 \dots \varepsilon_n$

Moreover, $\|P\| \leq \|\check{P}\| \leq \frac{n^n}{n!} \|P\|$

This means that $P \longrightarrow \check{P}$ is an isomorphism $\mathcal{P}(^nX; Y) \longrightarrow \mathcal{L}^s(^nX; Y)$.

(2)

Remark: • If $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$, then $\mathcal{P}({}^kX,Y)$ is indeed the "usual polynomials".

• $\mathcal{L}(X_1,\mathcal{L}(X_2,\dots,X_n;Y)) = \mathcal{L}(X_1,\dots,X_n;Y)$

This will enable us to define higher derivatives!

Def: P is a polynomial of degree $\leq n$ if $P = \underset{\substack{f: X\to Y}}{P_0} + \dots + P_n$ with $P_j \in \mathcal{P}({}^jX;Y)$.

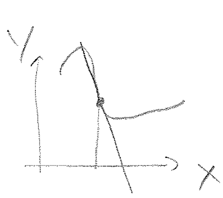
Fact: The P_j's are uniquely determined by ______

Moreover, $\mathcal{P}^n(X;Y) = \{P : \partial^\circ P \leq n\}$ endowed with $\|P\| = \sup\limits_{x\in B_X} \|P(x)\|_Y$

and $\mathcal{P}^n(X;Y) \cong \mathcal{P}({}^0X;Y)\oplus\dots\oplus\mathcal{P}({}^nX;Y)$ is a B-space isomorphism

Def: $\mathcal{U}\subset X$ open, Y B-space, $f:\mathcal{U}\to Y$, $x\in\mathcal{U}$. We say that $L\in\mathcal{L}(X;Y)$ is a Fréchet derivative at x if $\lim\limits_{\|h\|\to 0} \dfrac{\|f(x+h)-f(x)-L(h)\|}{\|h\|} = 0$

We write $L = Df(x)$ and $L(h) = Df(x)[h]$



Def: Suppose $\mathcal{U}\subset X$ is open, Y is a Banach space and $f:\mathcal{U}\to Y$ has k^{th} derivative in $\mathcal{U}$ $D^kf: \mathcal{U}\to\mathcal{L}({}^kX;Y)$;

then $D^{k+1}f(x) = D(D^kf)(x) \in \mathcal{L}(X;\mathcal{L}({}^kX;Y)) \overset{\text{isometric}}{\cong} \mathcal{L}({}^{k+1}X;Y)$

Proposition: If $D^kf(x)$ exists, then $D^kf(x)\in\mathcal{L}^s({}^kX;Y)$

Let us define: $d^kf(x) = \widehat{D^kf(x)}$ is the k^{th} derivative in $\mathcal{P}({}^kX;Y)$

Taylor formula: Let $f:\mathcal{U}\to Y$, $x\in\mathcal{U}$. If f has $d^kf(x)$, then

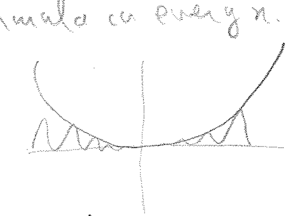
$$\Big\|f(x+h) - \sum_{j=0}^{k} \frac{1}{j!} \overbrace{d^jf(x)[h]}^{\mathcal{P}^k(X;Y)}\Big\| \leq o(\|h\|^k)$$

Forward remark: A smooth f^n behaves like a polynomial: prop^t of smooth f^{ns} will be det'd by the linear structure of X because the polynomials depend on it.

Note: in the complex setting, a differentiable function is automatically analytic, C^∞-smooth, with a converging Taylor series!

Q: what about the converse? Suppose that f has a Taylor formula in every x.

The existence of good polynomial approximations does not imply that the function is $\mathcal{C}^2$ even in 1 dimension!



Definition: Let $f:\mathcal{U}\to Y$ ~~be locally uniformly continuous~~ and $x\in\mathcal{U}$.

Suppose that $\exists\, P_k^x \in\mathcal{P}^k(X;Y)$ $\|f(x+h)-P_k^x(h)\| \leq \omega_x(h)\|h\|^k$ with $\omega_x(h)\xrightarrow[h\to 0]{} 0$

We say that f is T^k-smooth at x.

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Note that P^x_k is unique if it exists.

Also, if $d^k f(x)$ exists, then $P^x_k = \sum_j \frac{1}{j!} d^j f(x)$

Theorem (Johanis) (converse Taylor theorem) Let $f: U \to Y$ be locally uniformly cont.

TFAE: ① $f \in C^k$.

② f is T^k-smooth in U: $\forall x \, \exists P^x_k$ $\lim_{\substack{(y,h)\to(x,0)\\ h\neq 0}} \frac{\|R_k(y,h)\|}{\|h\|^k} = 0$

where $R_k(x,h) = f(x+h) - P^x_k(h)$

Example: $P \in \mathcal{P}(^nX;Y)$ is C^∞-smooth (i.e., C^k-smooth for every $k \in \mathbb{N}$)

$$D^k P(x)[h_1,\dots,h_k] = n(n-1)\cdots(n-k+1)\,\check{P}(^{n-k}x, h_1,\dots,h_k)$$

$$d^k P(x)[h] = n(n-1)\cdots(n-k+1)\,\check{P}(^{n-k}x, {}^k h)$$

and $d^n P(x)[h]$ is "constant" in x, $= n!\,P(h)$

and $d^{n+1} P \equiv 0$.

Other approach: (Fréchet, through local behaviour)

Theorem (Fréchet, Michael, Mazur-Orlicz): X, Y real B-spaces, $f: X \xrightarrow{\text{continuous}} Y$.

wouldn't work with linear.

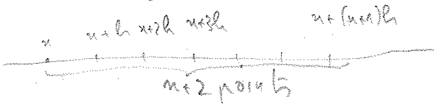
TFAE: ① $f \in \mathcal{P}^n(X;Y)$

1 ⇒ 2 ⇒ 4 ② $f|_E \in \mathcal{P}^n(E;Y)$ for every 1-dim affine subspace of X

③ f is C^{n+1}-smooth and $D^{n+1} f = 0$

④ $\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} P(x+kh) = 0$ for all $x, h \in X$

② → ①: one needs some kind of uniformity: it is provided by the fixed degree n!



fact: $f: X \to Y$ is in $\mathcal{P}(^nX;Y)$ iff
$\varphi \circ f: X \to \mathbb{R}$ is in $\mathcal{P}(^nX)$ for every $\varphi \in Y^*$

We consider $f \xrightarrow{\text{evaluation}} f(x)$ as a linear functional, so ④ says something about $n+2$ functionals on $\mathcal{P}(^nX)$ that has $n+1$ dimensions: they must be linearly dependent and ④ gives the correct coefficients.

Petr Hájek

2^e cours

Definition : Let U be a convex subset of X with nonempty interior, X, Y B-spaces

- $C(U;Y)$ is the space of continuous functions and
  $C_u(U;Y)$ —— uniformly —— on closed bounded convex $V \subset U$ (CCB)
  $C_u^m(U;Y)$ —— m times continuously differentiable f^{ns} with all derivatives are in C_u.
  $C_w(U;Y)$ —— weakly continuous on CCB sets
  $C_{wu}(U;Y)$ —— uniformly ——

$T \in C_{wsc}(U;Y)$ if $\forall V$ CCB $\subset U$ $\forall \{x_n\}_{n=1}^{\infty} \subset V$ weakly convergent to x
$\{T(x_n)\}$ is $\|\cdot\|$-convergent to $T(x)$

$T \in C_K(U;Y)$ if $\forall V$ CCB $\subset U$ $\overline{T(V)}$ is $\|\cdot\|$-compact
$T \in C_{wK}(U;Y)$ —— w-compact.

Same notation $\mathcal{P}_K, \mathcal{P}_{wu}, \ldots$ for space of polynomials

$\to \mathcal{P}_\phi = C_\phi \cap \mathcal{P}$ for $\phi \in \{w, wu, wsc, \ldots\}$

There are many relationships between these notions.

Prop : Let $U \subset X$ and Y. Let $T \in C_K(U,Y)$. Then $T \in C_\phi(U;Y) \iff f\circ T \in C_\phi(U;\mathbb{R})\ \forall f \in Y^*$
(will not be proved)

Prop : $C_{wu}(U;Y) \subset C_K(U;Y)$.

Remark : . T is w-uniformly continuous if $\forall \varepsilon > 0\ \exists \delta > 0\ \exists f_1, \ldots, f_n$ $|f_i(x) - f_i(y)| < \delta \Rightarrow \|T(x) - T(y)\| < \varepsilon$
. T is w —— if T maps w-cv nets to $\|\cdot\|$-cv nets
. T is wu —— w-Cauchy
. $C_w \subset C_{wsc}$ and $C_{wu} \subset C_{wsc}$

Proof : Consider $M : V \longrightarrow \mathbb{R}^n$, $x \longmapsto (f_1(x), \ldots, f_n(x))$. Then $\overline{M(V)}$ is relatively compact.
Thus $M(V)$ contains a ρ-net : choose $\{x_j\}_{j=1}^{N}$ s.t. $\{M(x_j) : j = 1..n\}$ is ρ-dense in $\overline{M(V)}$


$M(V)$

Then, for each $x \in V$, $T(x)$ will be close to some $T(x_j)$. Combine
Thus weakly uniformly continuous is always compact.

<u>Theorem</u> [Aron–Prolla] Let X, Y. $C_{wu}(U;Y) = \overline{\mathcal{P}_f^n(X;Y)}^{\tau_b}$

<u>Proof</u>: $\mathcal{P}_f(X;U)$: A first simple example: $f \otimes y \in X^* \otimes Y : x \mapsto f(x)\,y$

Then: $x \mapsto f^n(x)\,y$ is a polynomial of n-th degree.
Then take linear combinations.

- τ_b is the topology of uniform convergence on CCB subsets of U.

- Def: X has the Dunford–Pettis property (DPP) if $\mathcal{L}_{wK}(X;Y) \subset \mathcal{L}_{wsc}(X;Y)$ or $\mathcal{L}_{wsC}(X;Y)$

  i.e., $T(x_n) \to T(x)$ if $x_n \xrightarrow{w} x$

- th [Aron, Hervés, Valdivia]: $\mathcal{L}_{wsc}(^nX;Y) = \mathcal{L}_{wsC}(^nX;Y)$

  $\mathcal{L}_{w} \qquad \mathcal{L}_{wu}$

  and the same holds for space of polynomials.

- th [Ryan] If X has DPP, then $\mathcal{L}_{wK}(^nX;Y) \subset \mathcal{L}_{wsc}(^nX;Y)$

  $\mathcal{P} \qquad \mathcal{P}$

  ↳ This is by no means obvious: polys disturb the linear structure!

  ex: linear maps send w-Cauchy seq to w-Cauchy seq, but this is false i.g. for polynomials

  but: weakly conditionally Cauchy seq are also preserved for polys.

Can these statements be transferred from poly to smooth functions?
This is by no means obvious. Taylor series approximate only locally.

<u>Theorem</u>: Let $\ell_1 \not\hookrightarrow X$ and Y be a B-space. Then $C_{wsc}(U;Y) = C_w(U;Y)$

$C_{wsC}(U;Y) = C_{wu}(U;Y)$

→ continuity props can be checked on seq and not nets!

Proof of the 2nd statement. $C_{wu} = C_{wsC}$: $\Rightarrow$ is clear. $\Leftarrow$: proof by contradiction

. Let T be wsC on V $CCB \subset U$, but not wu on V.

Then $\exists \varepsilon > 0\ \forall \varphi_1, \dots, \varphi_n \in X^*\ \forall \delta > 0\ \exists a_1, \dots, a_n \in \mathbb{R}\ \exists x, y$ $\left[\begin{array}{l} |\varphi_i(x) - a_i| < \delta \\ |\varphi_i(y) - a_i| < \delta \\ \|T(x) - T(y)\| > \varepsilon \end{array}\right]$ weak shifted neighborhood

Let $\Gamma = \{(\varphi_1, \dots, \varphi_n; a_1, \dots, a_n; \delta)\}$; x_Φ, y_Φ

We will order Γ in the following way: $\Psi > \Phi$ if Ψ may be written as an extension of Φ: $(\varphi_1, \dots, \varphi_n, \varphi_{n+1}, \dots, \varphi_N; a_1, \dots, a_n, a_{n+1}, \dots, a_N; \tilde{\delta})$ and $\tilde{\delta} < \delta$.

Then $0 \in \overline{\{x_\Phi - y_\Phi : \Phi \in \Gamma\}}^{w}$ (Take a look at $\lim_{(\Gamma, <)}$)

Let us use Kaplansky's theorem: w-topology is countably tight.

Therefore there is a <u>sequence</u> $(\Phi_n) \subset \Gamma$ such that $0 \in \overline{\{x_{\Phi_n} - y_{\Phi_n}\}_n}^{w}$

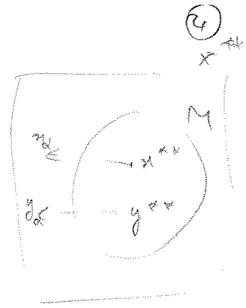
There is a th by Bourgain and Fremlin-Talagrand BFT: If $\ell_1 \not\hookrightarrow X$, consider $Z = [x_{\Phi_n}, y_{\Phi_n}]$.
Then $\ell_1 \not\hookrightarrow Z$; there is a subsequence $\{\Phi_k\}_{k=1}^{\infty}$ such that $0 = w\text{-}\lim x_{\Phi_k} - y_{\Phi_k}$.
Denote the sequences as $\{x_k\}$ and $\{y_k\}$: $0 = w\text{-}\lim x_k - y_k$
By Rosenthal's ℓ_1-Theorem, wlog, $\{x_k\}, \{y_k\}$ is w-Cauchy. Let $v_k = x_k - y_k$.
Consider $y_1, \overset{x_1}{\overset{\shortparallel}{y_1 + v_1}}, y_2, \overset{x_2}{\overset{\shortparallel}{y_2 + v_2}}, \ldots$: this is a w-Cauchy seq perturbed by a w-null seq.
Thus it is itself w-Cauchy: How do $T(x_k), T(y_k)$ behave?
We have $\|T(x_k) - T(y_k)\| > \varepsilon$ ↯

This should even give a characterisation of "$\ell_1 \not\hookrightarrow$"

<u>Theorem</u>: [Local reflexivity version by Johnson, Lindenstrauss, Rosenthal]
If $M \subset X^{**}$, $N \subset X^*$ have finite dimension and $\varepsilon > 0$, there is $T: M \to X$ s.t. $\|T\| < 1+\varepsilon$, $\|T^{-1}\| < 1+\varepsilon$,
and $\langle f, T(x) \rangle = \langle f, x \rangle$ for $f \in N$ and $x \in M$. $\quad T|_{M \cap X} = \mathrm{Id}_{M \cap X}$

Th: idea: given $T: B_X \to Y$, we want to construct $\tilde{T}: B_{X^{**}} \to Y^{**}$.
If T is linear, it suffices to consider the bitranspose. If T isn't ...?
Constructing T^{**}, you set for $x^{**} = \lim_{\mathcal{U}} x_\alpha$ $T^{**}(x^{**}) = \lim T(x_\alpha)$
If T is nonlinear, one has to be very careful about the ultrafilter $\mathcal{U}$:
always use the same and thus have a uniform way of approximating.
Let $\Gamma = \{\underbrace{(M, N, \varepsilon)}_{\alpha} : M \subset X^{**}, N \subset X^*, \varepsilon > 0\}$
Consider the corresponding T_α of the local reflexivity theorem: $\langle f, T_\alpha(x) \rangle = \langle f, x \rangle$ for $f \in N$, $x \in M$.
and consider $(X)_{\mathcal{U}} = \prod_{\alpha \in \Gamma} X$.
Given x^{**}, computing $T_\alpha(x^{**})$ will be possible if $x^{**} \in M$; otherwise we don't mind.
Consider a nontrivial ultrafilter on Γ: Γ has a natural order:
$\alpha > \beta$ if $M_\alpha \supset M_\beta$, $N_\alpha \supset N_\beta$, $\varepsilon_\alpha < \varepsilon_\beta$
For every x^{**}, $\{\alpha : x^{**} \in M\} \subset \mathcal{U}$ because x^{**} can be added to every M.
X^{**} is embedded in $(X)_{\mathcal{U}}$, isometrically. For each x^{**}, there is a net
$x_\alpha = T_\alpha(x^{**})$ such that $x^{**} = w^*\text{-}\lim_{\mathcal{U}} x_\alpha$.

<u>Theorem</u> [Choi Hájek Lee 2012]: Let $T: B_X \to Y$ be uniformly continuous and $\mathcal{U}$ the ultrafilter as above.
Then we let $\tilde{T}(x^{**}) = w^*\text{-}\lim_{\mathcal{U}} T(x_\alpha)$ if $x^{**} = w^*\text{-}\lim_{\mathcal{U}} x_\alpha$ [exists by compactness of w^* closed bounded sets of the bidual]. Then, if T is linear, $\tilde{T} = T^{**}$; also $\widetilde{T+S} = \tilde{T} + \tilde{S}$.



- If T has modulus of continuity $\omega(\delta)$, then also $\tilde T$.
- If $T\in \mathcal{P}^n(X;Y)$, then $\tilde T\in \mathcal{P}^n(X^{**},Y^{**})$
- If T is UC^n, then $\tilde T$ is UC^n: moduli are preserved.
- If T is in C_K or in C_{wK}, then $\tilde T(B_{X^{**}})\subset Y$.

Proof: If $x^{**}=\lim x_\alpha$, $y^{**}=\lim y_\alpha$, then $\underset{\|u\|\le\varepsilon}{\|x^{**}-y^{**}\|}$: for large α, $x^{**},y^{**}\in M_\alpha$
$\|x_\alpha-y_\alpha\|$

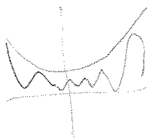
If $\|x_\alpha-y_\alpha\|<\delta$, then $\|T(x_\alpha)-T(y_\alpha)\|<\omega(\delta)$. So this passes to x^{**},y^{**}.

By Fréchet's theorem: $\sum_{k=0}^{n+1}(-1)^{n+1-k}\binom{n+1}{k}T(x+kh)=0$ for $x,h\in X$.

Claim: then the same holds for the $\tilde T(x^{**}+kh^{**})$! Again, they are the limit of the $T(x_\alpha+kh_\alpha)$ along $\mathcal U$, for which individually the identity holds.

Q: If T is UC^n-smooth, what about $\tilde T$?

Let us use the converse Taylor theorem: for each α $\|T(x)-P^a(x-a)\|=\|T(x)-\sum_{j=0}^{n}\frac{d^jT(a)[x-a]}{j!}\|$
$\le\omega(\|x-a\|)\|x-a\|^k$



where ω is an absolute modulus of continuity depending only on T.

Let us set $P^{a^{**}}(x^{**})=w^*\text{-}\lim_{\mathcal U}P^{a_\alpha}(x_\alpha)$. Miracle! this works, though one lets simultaneously $a_\alpha\to a^{**}$, $x_\alpha\to x^{**}$! Another method of proof goes by induction.

For a fixed a^{**}, is this a polynomial in X^{**}? For every x^{**},h^{**}, $\sum_{k=0}^{n+1}P^{a^{**}}(x^{**}+kh^{**})=0$ because $\sum P^{a_\alpha}(x_\alpha+kh_\alpha)=0$. Thus it is a polynomial!

Then it is easy to prove that $P^{a^{**}}$ is a Taylor approximation.

From the next lecture on, the style will change!

Note: Pb: If $T:X\to Y$ is C^n-smooth, is $\tilde T:X^{**}\to Y^{**}$ C^n-smooth.
Not clear at all!

In some cases, the above results can be strengthened.
If X is an $\mathcal{L}_\infty$ (a $C(K)$-space), then for any Z s.t. $X\hookrightarrow Z$, for any $T:B_X\xrightarrow{UC^n}Y$, there is a $\tilde T:B_Z\xrightarrow{UC^n}Y^{**}$

Add: Spaces with DPP: $C(K)$, $L_1(\mu)$, $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces : For certain spaces of analytic f's, $A(\mathbb{D})$, $H^\infty(\mathbb{D})$, it might not even be known.

It is natural to introduce quantitative versions of DPP, like wc, wuC, and then $(p,1)$-summing, weakly summing, etc...

3^e cours

Definition: Let $\lambda \in]0,1]$. We say that X is a W_λ-space if
$$f \in C^1_u(B_X) \Rightarrow f \in C_{wsc}(\lambda B_X)$$

Fact: If X is a W_1-space and $\ell_1 \not\hookrightarrow X$. Then, for all B-space Y and ultrafilter $\mathcal{V}$,
$$T \in C^1_u(B_X; Y),\ x^{**} \in \lambda B_{X^{**}},\ x_\alpha \xrightarrow[\alpha \to \mathcal{V}]{} x^{**} \Rightarrow T(x_\alpha) \xrightarrow[\alpha \to \mathcal{V}]{w^*} x_\alpha.$$

Proof: (Recall that $\tilde{T}$ was defined through a special ultrafilter $\mathcal{U}$.)
In fact, $\tilde{T}$ is uniformly w^*-to-w^*-continuous.
Fix any $f \in Y^*$: $f \circ T \in C^1_u(B_X)$, so that it is C_{wsc}-continuous on λB_X.
We proved last time that $\ell_1 \not\hookrightarrow X \Rightarrow f \circ T \in C_{wu}(\lambda B_X)$.
Therefore there are $\varphi_1, \dots, \varphi_n$ functionals so that $|\varphi_i(x) - \varphi_i(y)| < \delta$ implies $|f \circ T(x) - f \circ T(y)| < \varepsilon$. So these objects witness that $\tilde{T}$ satisfies the same

Example: every Schur space X is a W_1-space. E.g., ℓ_1 is W_1.
$\hookrightarrow$ every wsC seq is norm convergent.

Th: Let $\ell_1 \not\hookrightarrow X$ and $T \in C^k_u(B_X; Y)$, $k \geq 1$. Then TFAE:

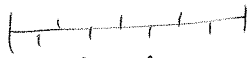
1. $T \in C_{wsC}$
2. $T \in C_{wu}$
3. $dT \in C_{wu}(B_X; \mathcal{L}(X;Y))$ and $dT(x) \in \mathcal{L}_K(X;Y)$.
4. $dT \in C_K(B_X; \mathcal{L}(X;Y))$ and $dT \in \mathcal{L}_K$.
5. $T \in C_K$
6. $d^jT \in C_{wu}(X; \mathcal{P}(^jX;Y))$ and $d^jT(x) \in \mathcal{P}_K(\quad)$ for $j < k$.
7. $d^jT \in C_K(X; \mathcal{P}(^jX;Y))$ ....

Fact: Suppose $T \in C^1_u(B_X; Y)$ with a modulus $\omega(\delta)$ of continuity of dT, i.e. $\|x-y\| < \delta \Rightarrow \|dT(x) - dT(y)\| \leq \omega(\delta)$. Then $T(x) - T(y) = \langle dT(y), x-y\rangle + \eta$ as soon as $|\eta| \leq \underbrace{\omega(\|x-y\|)\cdot\|x-y\|}_{\text{a quantitative } o(\|x-y\|)}$.

Theorem: If K is scattered, then $C(K)$ is a W_1-space.

Ex: c_0

Lemma: Let $x, y \in \ell_\infty^n$ with $\|x\|_\infty, \|y\|_\infty < \eta$; then there is a partition $A \cup B = \{1, \dots, n\}$, $A \cap B = \emptyset$ such that $\left| \sum_{i \in A} x_i - \sum_{i \in B} x_i \right|, \left| \sum_{i \in A} y_i - \sum_{i \in B} y_i \right| \leq 2\eta$

→ illustration:  for one vector, this is trivial

→ You can prove this for k vectors (2η to be replaced by $k\eta$)

Trick:

Consider $L: \ell_\infty^n \longrightarrow \mathbb{R}^2$. $L(u) = (u \cdot x, u \cdot y)$

This operator has a large kernel. $\exists v \in \ker L \; \forall i$ (but 2) $|v_i| = 1$.

→ Choose $v \in \text{Extremal}(\ker L) \cap B_{\ell_\infty^n}$. If more than two coordinates have $|v_i| \neq 1$. Then $v_{i_1}, v_{i_2}, v_{i_3}$ generate a 3-dim vectorspace in the kernel. But then v is not an extremal point.

$\rightarrow \exists\, x = (x_1, x_2, x_3, 0, \dots)$ $L(x) = (0,0)$

$V \pm \varepsilon x \longrightarrow (0,0)$

Let $A = \{i : v_i = 1\}$, $B = \{i : v_i = -1\}$. Then $\sum v_i x_i = 0$,

so that $\left\| \sum_{i \in A} x_i - \sum_{i \in B} x_i \right\| \leq 2\eta$.

Lemma: Let $\omega(\cdot)$ be a fixed modulus function for the uniform continuity of df

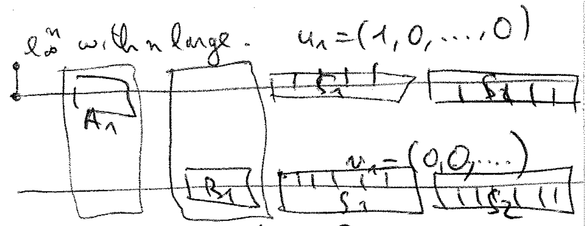
$f: B_{\ell_\infty^n}^+ \longrightarrow \mathbb{R}$, $f(0) = 0$, $df(0) = 0$, f symmetric $[\, f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ for $\pi \in S_n \,]$

then $\forall \varepsilon > 0 \; \exists n \; \forall f \; \forall i \; |f(e_i)| < \varepsilon$

"increasing dimension"

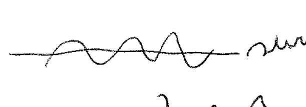
Proof: Idea: $df(u_1)$

$df(v_1) = 0$ because $v = 0$.



df is uniformly continuous and uniformly bounded on $B_{\ell_\infty^n}$

Then $\|df(x)\| < K$ for $x \in B$: K depends on $\omega(\delta)$



derivative normed by ℓ^1-norm.

How many coordinates (K) have $|\;| > \delta$? $A \cdot \delta < K$, ie $A < \frac{K}{\delta}$.

$\ell^n \ni \begin{cases} df(u_1) = ? & A_1 \text{ the set of } i \text{ where } |(df(u_1))_i| \geq \delta \\ df(v_1) = 0 & B_1 \text{ ———— } |(df(v_1))_i| \geq \delta \text{, i.e. } B_1 = \emptyset \end{cases}$

By the lemma, there are two sets S_1, S_2 in $\{1,\dots,n\}\setminus(A_1\cup B_1)$ such that $\left|\sum_{i\in S_1} df(v_1)_i - \sum_{i\in S_2} df(v_1)_i\right| < \delta$
and the same for u.

Now take $w_1 = \begin{cases} (w_1)_i = \rho & \text{if } i\in S_1 \\ (w_1)_i = -\rho & \text{if } i \in S_2 \\ = 0 & \text{otherwise} \end{cases}$

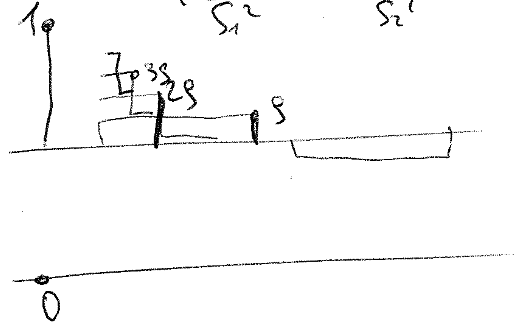
We have $f(v_1+w_1) = f(v_1) + \langle df(v_1), w_1\rangle + \text{error terms} = f(v_1) + \rho\cdot 2\delta + \dots$ (very small)

$f(u_1+w_1) = f(u_1) + \langle df(u_1), w_1\rangle + \text{———}$

Take $u_2 = u_1 + w_1$ and consider the derivative in S_1 and S_2:
$v_2 = v_1 + w_1$

We find S_1^2 and $S_2^2 \subset S_1$, $S_1^2 \cap S_2^2 = \emptyset$ s.t. $\left|\sum_{S_1^2} df(u_2)_i - \sum_{S_2^2} df(u_2)_i\right| < 2\delta$
$\left|\sum_{S_1^2} df(v_2)_i - \sum_{S_2^2} df(v_2)_i\right| < \delta$

Choose w_2: $(w_2)_i = \begin{cases} \rho & \text{if } i\in S_1^2 \\ -\rho & \text{if } i \in S_2^2 \\ 0 & \text{otherwise.}\end{cases}$



There are $k = \frac{1}{\rho}$ steps
the final vectors are u_k and v_k.
At each step, $f(u_j) - f(u_{j+1}) \approx 0$
In the end, $|f(u_k) - f(u_1)| < \eta$
$|f(v_k) - f(v_1)| < \eta$ (k times the error term $\hookrightarrow o(\rho)$, by uniformity)

the trick: the symmetry: the only coordinates we care about are those that remain: the "staircase vectors"

u_k: $1 \quad 1-\rho \quad 1-2\rho \quad 1-3\rho \quad \dots$

By symmetry, I can shift this by one. v_k: $1-\rho \quad 1-2\rho \quad 1-3\rho$

$\tilde{v}_k$: $1-\rho \quad 1-2\rho \quad 1-3\rho \quad \dots$

Thus $\|u_k - \tilde{v}_k\| \le \rho$. This means f has the "same" value at u_k and $\tilde{v}_k$:
$|f(u_k) - f(\tilde{v}_k)| \le \omega(\rho)$
$|f(u_k) - f(u_1)| < \eta$ $\qquad |f(v_1) - f(u_1)| < 2\eta + \omega(\rho)$.
$|\underbrace{f(v_k)}_{f(\tilde{v}_k) = f(v_k)} - f(v_1)| < \eta$ But this means that all $|f(e_i)| <$ same

But the only thing that refrains us from continuing our construction would have been that there are not enough coordinates. That is why n needs to be large enough.

<u>Corollary</u>: Let $\omega(\delta)$ be the modulus of continuity of df, $f(0)=0$, $df(0)=0$.
Then $\#\{i : |f(e_i)| > \delta\} < N(\omega(\cdot))$

Proof: Pass from f to $\tilde{f}$ the symmetrisation of f.

If $f(e_i) > \delta$ for $i \in S$, then $\tilde{f} = \frac{1}{|S|!} \sum f \circ T_\pi$ with T_π, π permutation on the index set S

....

Using this, we conclude that if $f \in C^1_u(c_0)$ then f is wsC-continuous

<u>Note</u>: for any weakly null sequence $\{u_n\} \subset B_{c_0}$ $f(u_n) \longrightarrow 0$

– bumps that are describing ℓ^n_∞: $f([u_i]^N_{i=1})$
$\tilde{\ell}^n_\infty$

Corollary: $C^1_u \subset C_{wsc}$. (but we want more: C_{wsC}, but we will lose the quantitative information)

$\rightarrow$ argument by transfinite induction: $C(\alpha)$, α ordinal and for every $C(K)$, K scattered.

and then for every $C(K)$, but this is highly non trivial.

① 20/6/2012

Last course
Smoothness and weak continuity

We saw that

$C^1_u(B_{c_0}) \subset C_{wsc}(B_{c_0})$ and

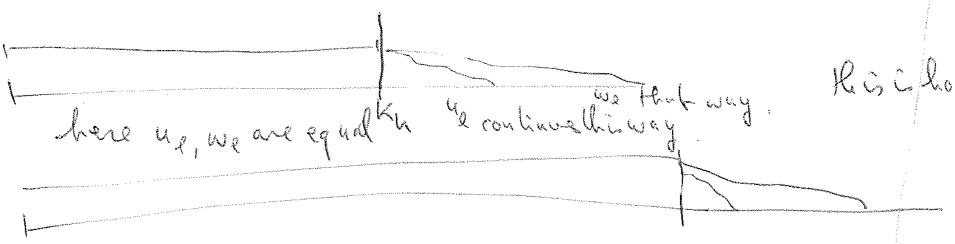
Theorem: If $X = C(K)$, K scattered, then $X \in W_1$, i.e. $C^1_u(B_X) \subset C_{wsc}(B_X)$, i.e. f maps w-Cauchy sequences into norm convergent sequences.

$W_{(2)}$ has been defined because at some points, W_1 is out of reach.

Indications for c_0: by contradiction, construction of $\{u_n\}, \{w_n\} \subset B_X$ that are part of a single w-Cauchy seq., and $f \in C^1_u(B_X)$.

Assume by contradiction that $\exists \beta > 0$ $f(u_n) > 2\beta$ and $f(w_n) = 0$ or viceversa.

Assume wlog that there are intervals $I_n = [0, K_n]$ st $u_\ell|_{I_n} = w_\ell|_{I_n}$ for $\ell \geq n$.

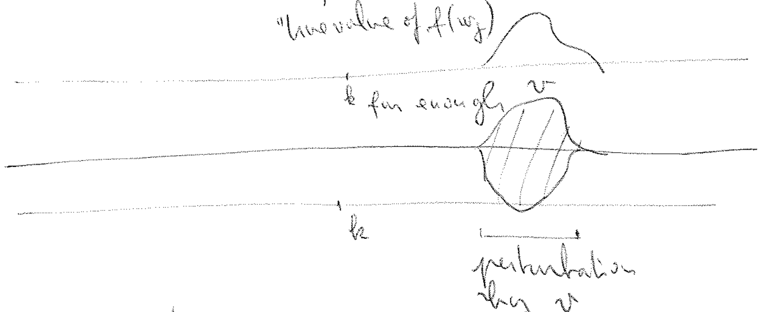


this is how wC sequences look like...

Lemma: Let (w_n) be w-Cauchy in c_0.

~~For all $\beta > 0$~~ There exists an infinite subset $M \subset \mathbb{N}$ and a $k \in \mathbb{N}$ st. for $v \in c_{00}$ supp $v \subset [k, \infty)$ the set $\{j \in M : |f(w_j + v) - 2\beta| > \beta\}$ is finite.

Let's suppose $f(w_j) = 2\beta$
$f(u_j) = 0$.



The statement is more complicated than the proof! By contradiction,

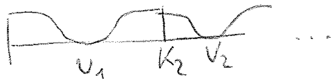
suppose $\forall k$ $\forall \Pi \subseteq \mathbb{N}$ infinite $\exists v$ finitely supported supp $v \subset [k, \infty[$ $\#\{j \;|f(w_j + v) - 2\beta| > \beta\} = \infty$.

Inductive procedure: Choose ~~this will~~ be v_1 for $k = 1$, $M_1 = \mathbb{N}$.

let Π_2 be the subset given by the assumption

let K_2 be such that supp $v_2 \subset [0, K_2]$.

let $\Pi_3 \ldots$



(2)

then for every $j \in M_{n+1}$ $|f(w_j + v_k) - 2\beta| > \beta$ for $k < n$.

[I will also achieve that $M_1 \supset M_2 \supset M_3 \supset \dots$]

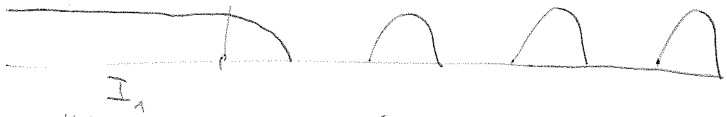
By shifting w_j to the origin, this is equivalent to having $f(0)$, $f(e_k)$, $0 \leftrightarrow w_j$, as e_k is equivalent to v_k.

then $f(0) = 2\beta$ and $f(e_k)$ differs by β.

this can only happen a limited # of times by our quantitative information.

We may now finish the proof:

Assume that the index set $\mathbb{N}$ was replaced by the M of the previous lemma

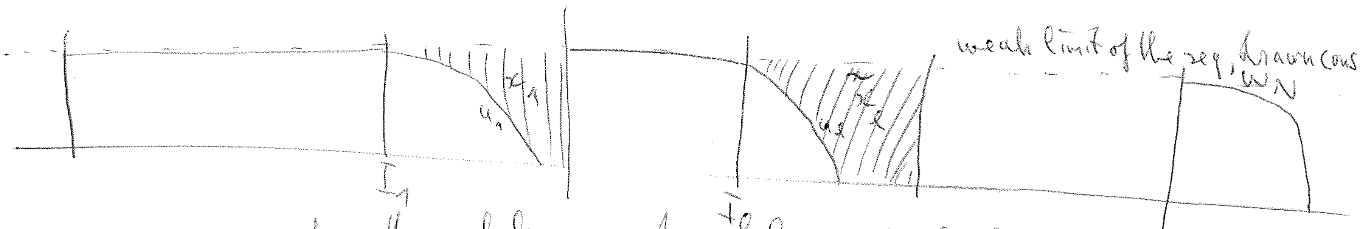


u_k, w_k where $f(u_k) = 0$, $f(w_k) = 2\beta$

By the main lemma, adding these blocks will make us achieve the same value.

$\exists \ell \quad \forall u \quad \operatorname{supp} u \subset [\ell, \infty[\quad f(u_1 + u) = f(u_1)$

weak limit of the seq, drawn constant.



- you can keep the whole w_N and cancel x_1 : $w_N - x_1$
- you can look at u_1 + far-away block :

$f(\qquad\qquad) = f(u_1) = 0$

$\neg \exists M \subset \mathbb{N}$, $\exists h$ $h \in I_\ell$ $|f(w_j + h) - f(w_j)| < \beta$ except for finitely many j.

Therefore $f(w_N - x_1) = f(w_N) \pm \beta$ ↯

this proves that ~~for~~ any wC seq must be convergent.

Thus $c_0 \in W_1$. For any $C(K)$, here is what I do: reduce first to $C([0,\alpha])$ with α a countable ordinal. Then reason similarly.

By known machineries, it extends to any $C(K)$, K scattered.

This was the 1st round.

Next round: for Banach-space valued:

③

Theorem: Let K be scattered, $X = C(K)$, Y arbitrary B-space. Let $T \in C^1_u(B_X; Y)$.

Then TFAE: ① $T \in C_{wsc}(B_X; Y)$ ② $T \in C_{wsC}(B_X; Y)$ ③ $T \in C_K(;)$

④ $T \in C_{wK}(;)$ ⑤ $T \in C_{wu}(;)$

⑥ $T|_{\{x_i\}}$ is convergent for every $\{x_i\} \sim \{e_i\}$ canonical basis of c_0

⑦ $d\tilde{T}(x^{**}) \in \mathcal{L}_K(X^{**}, Y^{**})$ ⑧ $d\tilde{T}(x^{**}) \in \mathcal{L}_{wK}$

Def: Let us say that X has ⊛ if $C^1_u(B_{c_0}; X) \subset C_{wsC}(B_{c_0}; X)$

$\boxed{\text{Theorem:}}$ X has ⊛ if any of the following is true:

(†) ① $c_0 \not\hookrightarrow X^{**}$ ($\Leftrightarrow \ell_1 \not\hookrightarrow X^*$) ③ X has PCP

② X is a dual space and $c_0 \not\hookrightarrow X$ ④ X has (u) and $c_0 \not\hookrightarrow X$

⑤ X is weakly seq. complete ⑥ - X has cotype -

Consider $\mathcal{L}(\ell_\infty; X) = \mathcal{L}_{wK}(\ell_\infty; X)$ provided $c_0 \not\hookrightarrow X$ (Pełczyński)

Thus wK is for free...

Main problem: Is it so that X has ⊛ $\Leftrightarrow$ $c_0 \not\hookrightarrow X$?

$\longrightarrow$ If $T: c_0 \xrightarrow{C^1_u} X$, is T compact?

3rd level of results: Take $X = C(K)$ for any K compact

(4th level: $\mathcal{L}_{\infty,\lambda}$ spaces)

Theorem: Let K be compact, $X = C(K)$, $\{x_n\}$ wC in B_X, $\varepsilon > 0$

then there is $Z \hookrightarrow X^{**}$ s.t. $Z \overset{\text{isometrically}}{\cong} C[0,\alpha[$, α a countable ordinal

and there is $\{y_n\} \subset Z$ wC s.t. $\|x_n - y_n\| < \varepsilon$ for all n.

[This is a variation on Zippin's lemma]

$\longrightarrow$ this transfers structure in X to structure in $C[0,\alpha[$

"Z has another direction in X^{**} than X but contains a seq. point by point close to the original sequence.

Sketch of proof: assume K is metric. Then $X^* = M(K)$ and $X^{**} = ??$ very huge.

But $\{x_n\}$ is wC. then $x = w^*\text{-}\lim x_n$ lives in X^{**}

Consider $x = \tau_p$-limit of the x_n, x Baire 1-f^n (Borel)

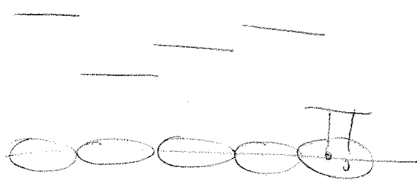
Because K is compact and $\{x_n\}$ is uniformly bounded, ~~Bes~~ the Lebesgue thm yields that if $\mu \in M(K)$ $\int x_n \, d\mu = \int x \, d\mu$

Theorem: $x = \tau_p\text{-}\lim \{x_n\}$: fix $\varepsilon = \frac{1}{n}$; $\Delta = \{\frac{j}{n}; -n \le j \le n\}$

and replace x by a discretely valued-sequence.

Idea: If x is discretely valued: for every $t \in K$, look at $x_n(t)$: it is eventually constant

Then define equivalence classes where the x_n are constant.



$\hookrightarrow$ a workaround to get K to be countable.

K

Theorem: Let $X = C(K)$, K compact, (1) ~~either~~ $C^1_u(X,Y) \cap C_{wK} \subset C_{wsc}(B_X;Y)$

(2) ~~A~~ if Y has (*) then $C^1_u(B_X;Y) \subset C_{wsc}(B_X;Y)$

N.B.: in the 2nd case, $C^1_u(B_X;Y)$ need not be weakly compact.

Sketch of proof: Let $T \in C^1_u \cap C_{wK}$. Assume by contradiction that there is $\{x_n\}$ wC in X st $\{T(x_n)\}$ is ε-not convergent [not ε-close to a convergent sequence]

1st step: extend T to $\tilde{T}: B_{X^{**}} \to Y^{**}$

By a Zippin-style lemma, there is $\{y_n\}$ and $Z \hookrightarrow X^{**}$, $Z = C[0,\alpha]$
$\to = C(K)$, K scattered.

such that for all k $|\tilde{T}(y_n) - \tilde{T}(x_n)| < \frac{\varepsilon}{4}$

i.e., $\tilde{T} \in C^1_u(Z)$ with same modulus for $d\tilde{T}$

Then $\{\tilde{T}(y_n)\}$ is not $\frac{\varepsilon}{2}$-convergent. End of proof.

Note that T wK $\Rightarrow \tilde{T}$ wK.

There are many other results!

Theorem: If is X is $\mathcal{L}_{\infty,\lambda}$, then $C^1_u(B_X;Y) \cap C_{wK} \subset C_{wsc}(\lambda B_X;Y)$

and similarly there is a "(*)"-result

Proof: in fact, by a machinery

X^{***} is complemented in some $C(K)$ space.

$T \rightsquigarrow \tilde{T} \rightsquigarrow \tilde{\tilde{T}} : C(K) \to Y$

Theorem: If X is W_λ and $\ell_1 \not\hookrightarrow X$, assume that Y^* has type.

Then $C_u^1(B_X; Y) \subset C_K(\lambda B_X; Y)$

$\rightsquigarrow$ we have (*) is close to "$\ell_1 \not\hookrightarrow X^*$" and we want to strengthen it to having type.

Corollaries: • Theorem (Bates): $\forall X, Y\ \exists T : X \xrightarrow{\text{onto}} Y$, $T \in C^1$ and Lipschitz.

For higher smoothness, Bates proved (under weak assumptions on X) that

$\forall Y\ \exists T : X \xrightarrow{\text{onto}} Y$ C^∞-smooth.

→ Proof by Hájek: If X^* has type, then $\forall Y$ Banach $\nexists T : X \xrightarrow{\text{onto}} Y$ s.t. T is a polynomial

Ex: $X = \ell_2$, Y separable $\exists P : X \xrightarrow{\text{onto}} Y$ 2-homogeneous.

• There is no C^2-smooth surjection from $W_\lambda \ni X$, $\ell_1 \not\hookrightarrow X$ onto Y s.t. Y^* has type.

Note: (†) resembles Aharoni's Theorem.

$\hookrightarrow \Leftrightarrow$ X has skipped BCS.