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G. Lancien

Coarse embeddings into Banach spaces

1. Linear embeddings

- Let $1 \leq p, q < \infty$. Pitt's theorem says that $l_p \subseteq l_q \Leftrightarrow p = q$
- Also $l_p \subseteq L_q \Leftrightarrow L_p \subseteq L_q \Leftrightarrow (2 \leq q$ and $p \in \{2, q\})$ or $(q < 2$ and $p \in [q, 2])$

  To prove this, use type, cotype, p-stable laws, Kadets-Pełczynski

- Also $L_p \subseteq l_q \Leftrightarrow p = q = 2$

2. Lipschitz embeddings

Def: $f : (M, d) \to (N, \delta)$ is a Lipschitz embedding if

$\exists A, B > 0 \ \forall x, y \in M \quad A d(x,y) \leq \delta(f(x), f(y)) \leq B d(x,y)$

In symbols: "$M \underset{L}{\hookrightarrow} N$"

Tools for B-spaces:
- Gâteaux-diff^{ty} when X is separable and Y has RNP,
- w^*-G-diff^{ty} when X is separable and Y is a dual (or in Y^{**})
- local reflexivity

- l_q $(1 \leq p < \infty)$ and L_q $(1 < q < \infty)$ have RNP
- L_1 has cotype 2
- If $p > 2$, l_p has cotype p.

This enables to get the same table of results in 2. as in 1.

One delicate point: $X \underset{L}{\hookrightarrow} L_1$ gives $X \subseteq L_1^{**}$ ...

Universal space for lip. embeddings: Aharoni: M separable metric space, $M \underset{L}{\hookrightarrow} c_0$.

Open question: does $c_0 \underset{L}{\hookrightarrow} X$ B-space imply $c_0 \subseteq X$?

3. Coarse Lip. embeddings

Def: $f : M \to N$ is a coarse Lip embedding if

$\exists A, B, \vartheta > 0 \ \forall x, y \in M \quad d(x,y) \geq \vartheta \Rightarrow A\, d(x,y) \leq \delta(f(x), f(y)) \leq B\, d(x,y)$

In symbols: $M \underset{CL}{\hookrightarrow} N$

Tools: $f : \underset{sep}{X} \underset{CL}{\hookrightarrow} Y$ defines $\Phi : X \underset{L}{\hookrightarrow} (Y)_{\mathcal{U}} \subseteq (Y_{\mathcal{U}})^{**}$, $\mathcal{U}$ ultrafilter

$x \longmapsto \left(\frac{f(nx)}{n}\right)_n$

w^*- G-diff + $(Y_{\mathcal{U}})^{**}$ finitely representable in $(Y)_{\mathcal{U}}$ and so in Y.

$F \subseteq X$, $\dim F < \infty$ implies $\exists G \subseteq Y \quad F \underset{1+\varepsilon}{\simeq} G$.

$\to$ local properties are preserved and (1) type$(X) \geq$ type(Y)

cotype$(X) \leq$ cotype(Y)

(2) kind of stability of asymptotic uniform smoothness (Kalton)

convexity

this yields: $p \neq q \quad \ell_p \underset{CL}{\not\hookrightarrow} \ell_q$

(3) $1 \leq q < \infty$, X sep., $X \underset{CL}{\hookrightarrow} L_p(\mu)$ yields $X \underset{L}{\hookrightarrow} (L_p(\mu))_{\mathcal{U}} = L_p(\tilde{\mathcal{Z}})$

A separable subspace of an abstract L_p-space is isometric to a subspace of $L_p[0,1]$.

There is a coarse counterpart to the previous question:

$¿\ c_0 \underset{CL}{\hookrightarrow} X \Rightarrow c_0 \subseteq X$?

4. Coarse embeddings

Def. $f : M \to N$ is a coarse embedding if $\exists \rho_1, \rho_2 : \mathbb{R}^+ \circlearrowleft \ \lim_{+\infty} \rho_1 = +\infty$

$\forall x, y \quad \rho_1 \circ d(x,y) \leq \delta(f(x), f(y)) \leq \rho_2 \circ d(x,y)$

"$M \underset{C}{\hookrightarrow} N$"

Motivation: Coarsely embed special metric spaces like Cayley graphs of finitely presented groups into Hilbert or superreflexive space enables to prove for them Baum Connes or Novikov conjecture

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Vague questions: does $\underset{C}{\hookrightarrow}$ superreflexive space imply $\underset{C}{\hookrightarrow} \ell_2$?

does $\underset{C}{\hookrightarrow} L_p$ imply $\underset{C}{\hookrightarrow} \ell_2$?

universal space for $\underset{C}{\hookrightarrow}$?

for separable, compact, locally finite, bounded geometry metric sp.

there is one positive result: $\boxed{\text{Th}}$ (Baudier) $\ell_\infty^n \subseteq X$, M proper metric sp. (i.e., balls are rel. compact), then $M \underset{C}{\hookrightarrow} X$ with $X = (\sum \ell_\infty^n)_{\ell_2}$

Open question: does $\dim X = \infty$ imply $\ell_2 \underset{C}{\hookrightarrow} X$?

$\boxed{\text{Th}}$ Ostrovskii: $M \subset \ell_2$, M locally finite (balls are finite), then $M \underset{L}{\hookrightarrow} X$ ($\dim X = \infty$) [uses Dvoretzky's theorem]

Let us try to put up a table as in 1., 2. and 3.

A few positive results: $\boxed{\text{Th 1}}$ (Nowak '06) $1 \leq p < \infty$, $\ell_2 \underset{C}{\hookrightarrow} \ell_p$

$\boxed{\text{Th 1'}}$ (Ostrovskii building on Odell-Schlumprecht): If X has an unconditional basis and $\dim X = \infty$, and X has nontrivial cotype, then $\ell_2 \underset{C}{\hookrightarrow} X$

Application: X superreflexive space with an unc. basis

$\boxed{\text{Th 2}}$ (Mendel-Naor '04) $0 < p \leq q$ $L_p \underset{C}{\hookrightarrow} L_q$, and $\ell_p \underset{C}{\hookrightarrow} \ell_q$

one has to check that the same proof works.

Obstructions to coarse embeddings

th: Mendel-Naor ("metric cotype" to appear in Ann. Math.) X, Y B. spaces with $\ell_1^n \not\subseteq Y$, then $X \underset{C}{\hookrightarrow} Y$ implies cotype $X \leq$ cotype Y

Th: Kalton '07: . $c_0 \underset{C}{\hookrightarrow} Y$ implies $\exists n$ $Y^{(n)}$ nonseparable, so Y is not reflexive

. $X \underset{C}{\hookrightarrow} Y$ reflexive implies X has property (Q).

. $\exists Y$ Schur space $c_0 \underset{C}{\hookrightarrow} Y$ (n.b.: $c_0 \not\subseteq Y$ as Y contains ℓ_1, here $\ell_1^{\ell_2}$)

Table: (I) $\ell_p \underset{C}{\hookrightarrow} \ell_q \Leftarrow \ell_p \underset{C}{\hookrightarrow} L_q \Leftrightarrow L_p \underset{C}{\hookrightarrow} L_q \Leftarrow p \leq q$ or $q < p \leq 2$

(II) $p \leq 2 \Rightarrow L_p \underset{C}{\hookrightarrow} \ell_q$; $2 < q < p \Rightarrow L_p \underset{C}{\not\hookrightarrow} \ell_q$.

open question: does $2 < p \leq q$ imply $L_p \underset{C}{\hookrightarrow} \ell_q$?

in particular, ¿ $2 < p \Rightarrow L_p \underset{C}{\hookrightarrow} \ell_p$?

proof: I) $p \le q$, $\ell_p \hookrightarrow \ell_q$, $\ell_p \subseteq L_p \hookrightarrow L_q$ (Mendel-Naor)

$q < p < 2$ $\ell_p \subseteq L_p \hookrightarrow L_2$ (MN) $= \ell_2 \hookrightarrow \ell_q \subseteq L_q$

$2 < p$, $q < p$: use cotype!

II) $p \le 2$, $L_p \underset{c}{\hookrightarrow} L_2$ (MN) $\hookrightarrow \ell_q$ for all q.

If $2 < q < p$, use cotype

$\boxed{\text{Th}}$ (Nowak) $\ell_2 \underset{c}{\hookrightarrow} \ell_p$ $(1 \le p < \infty)$

$\boxed{\text{Proposition 1}}$ If X is a metric space, $1 \le p < \infty$, s.t. $\exists \delta > 0\ \forall R > 0\ \forall \varepsilon > 0$

$\exists \varphi : X \longrightarrow S_{\ell_p}$ (i) $\sup_{d(x,y) \le R} \|\varphi(x) - \varphi(y)\|_p \le \varepsilon$

(ii) $\liminf_{S \to \infty} \inf_{d(x,y) \ge S} \|\varphi(x) - \varphi(y)\| \ge \delta$

Then $X \underset{c}{\hookrightarrow} \ell_p$

This Prop. gives $\exists \delta > 0\ \forall n\ \exists \varphi_n : X \longrightarrow S_{\ell_p}\ \exists s_n > 0$, $s_n \uparrow +\infty$

$$d(x,y) \le n \implies \|\varphi_n(x) - \varphi_n(y)\|_p \le 2^{-n}$$
$$d(x,y) \ge s_n \implies \|\varphi_n(x) - \varphi_n(y)\|_p \ge \delta.$$

Fix $x_0 \in X$ and let $\Phi : X \longrightarrow \ell_p(\ell_p)$

$$x \longmapsto (\varphi_n(x) - \varphi_n(x_0))_n$$

$$\|\Phi(x)\|_p^p \le \sum_{n < d(x,x_0)} \|\varphi_n(x) - \varphi_n(x_0)\|_p^p + \sum_{n \ge d(x,x_0)} 2^{-np} < \infty$$

so that Φ is well-defined.

Let $x, y \in X$ and $k \in \mathbb{N}$: $(k-1)^{\frac{1}{p}} \le d(x,y) \le k^{\frac{1}{p}}$. Then

$$\|\Phi(x) - \Phi(y)\|_p^p \le \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|_p^p + \sum_{k}^{\infty} 2^{-np}$$
$$\le 2^p (k-1) + 1.$$

$\|\Phi(x) - \Phi(y)\|_p^p \le 2^p (d(x,y))^p + 1$ and there is k s.t. $S_k \le d(x,y) \le S_{k+1}$

and $\|\Phi(x) - \Phi(y)\|_p^p \ge \sum_{n=1}^{k} \|\varphi_n(x) - \varphi_n(y)\|_p^p \ge k \delta^p$.

Let $\rho : t \longmapsto k$ so that $s_k \le t < s_{k-1}$: $\lim_{t \to \infty} \rho(t) = +\infty$.

and $\|\Phi(x) - \Phi(y)\| \ge \delta \rho^{1/p}$. ∎

$\boxed{\text{Proposition 2}}$ $1 \le p, q < \infty$. If (N_p) are the assumptions in Prop. 1, then $(N_p) \iff (N_q)$

Proof: For instance, let $p > q$: consider the Mazur map

$$\Phi : S_{L_p(\mu)} \longrightarrow S_{L_q(\mu)}, \quad f \longmapsto |f|^{p/q} \operatorname{sgn} f . \text{ Then } \Phi(S_{L_p}) = S_{L_q}.$$

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and $C\|f-g\|^{\frac{p}{q}} \le \|\Phi(f)-\Phi(g)\|_q \le \frac{p}{q}\|f-g\|_p$.

Compose the φ's in (N_p) with ϕ. ∎

It remains to prove

$\boxed{\text{Proposition 3}}$ ℓ_2 has (N_2).

Proof: Let $H = \mathbb{R} \oplus_2 \ell_2 \oplus_2 \ell_2 \otimes \ell_2 \oplus_2 \dots \oplus(\ell_2\otimes\dots\otimes\ell_2)\oplus\dots$; $H \equiv \ell_2$.

where $\langle x\otimes x', y\otimes y'\rangle = \langle x,y\rangle\langle x',y'\rangle$.

Consider the exponential: $\ell_2 \xrightarrow{\exp} H$

$$y \longmapsto 1 \oplus y \oplus \frac{1}{\sqrt{2!}}(y\otimes y)\oplus\dots\oplus\frac{1}{\sqrt{n!}}(y\otimes\dots\otimes y)$$

We have $\langle e^{y}, e^{y'}\rangle = e^{\langle y,y'\rangle}$. Let $\varphi(x) = e^{-t\|x\|^2}e^{\sqrt{2t}\,x} \oplus \dots$.

then $\langle\varphi(x),\varphi(y)\rangle = e^{-t(\|x\|^2+\|y\|^2)}e^{2t\langle x,y\rangle} = e^{-t\|x-y\|^2}$ and $\|\varphi(x)\|=1$

and $\|\varphi(x)-\varphi(y)\|^2 = 2-2e^{-t\|x-y\|^2} = 2(1-e^{-t\|x-y\|^2})$.

Let $R>0$, $\varepsilon>0$. Let t small enough st we have $\|x-y\|\le R \Rightarrow \|\varphi(x)-\varphi(y)\|\le\varepsilon$.

then $\lim_{\|x-y\|\to+\infty}\|\varphi(x)-\varphi(y)\| = \sqrt{2}$. There is a converse to this!

connected w. cond neg def. kernels

Addendum: Th (Mendel-Naor) $0<p\le q$ $L_p \overset{c}{\hookrightarrow} L_q$

Proof: $0<\alpha<2\beta$. Let $F(x) = \int_{-\infty}^{+\infty}\frac{(1-\cos tx)^\beta}{|t|^{\alpha+1}}\,dt$

$$= 2\int_0^{+\infty}\frac{(1-\cos tx)^\beta}{t^{\alpha+1}}\,dt.$$

If $x>0$, $u=tx$, $F(x) = \left(2\int_0^\infty\frac{1-\cos u}{u^{\alpha+1}}\,du\right)x^\alpha$.

and $|x|^\alpha = F(x)$. Let $1\le p<q<\infty$ and define $T: L_p(\mathbb{R}) \longrightarrow L_q(\mathbb{R}\times\mathbb{R})$ by $Tf(s,t) = \frac{1-e^{itf(s)}}{|t|^{\frac{p+1}{q}}}$.

If $f,g\in L_p(\mathbb{R})$, $\|Tf-Tg\|_q^q = \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{|1-e^{it(f(s)-g(s))}|^q}{|t|^{p+1}}\,dt\,ds$

As $|1-e^{itx}|^2 = 2-2\cos tx$, $\|Tf-Tg\|_q^q = \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{[2-2\cos t(f(s)-g(s))]^{\frac{q}{2}}}{|t|^{p+1}}\,dt\,ds$

and $p<2\frac{q}{2}$, so, for $\begin{cases}\alpha=p\\ \beta=\frac{q}{2}\end{cases}$, $\|Tf-Tg\|_q^q = 2^{\frac{q}{2}}C_{p,q}\int_{\mathbb{R}}|f(s)-g(s)|^p\,ds = \tilde{C}_{p,q}\|f-g\|_p^p$.

Sequel: extend the Mazur map to B-spaces w. an unc. basis.