Coarse embeddings of ℓ_2

Recall that • $f: M \to N$ is a <u>coarse</u> embedding if $\exists \rho_1, \rho_2 : \mathbb{R}_+ \circlearrowleft$ with $\rho_1 \xrightarrow[\infty]{} \infty$ such that $\rho_1 \circ d(x,y) \le \delta(f(x), f(y)) \le \rho_2 \circ d(x,y)$

• we proved last time (Nowak '06) $\ell_2 \hookrightarrow \ell_p$ $(1 \le p \le \infty)$ and (Mendel-Naor) $L_p \hookrightarrow L_q$ $(0 < p \le q < \infty)$

Let us address today: ¿ X B-space of ∞ dim $\Rightarrow \ell_2 \underset{c}{\hookrightarrow} X$?

<u>Th</u> (Odell-Schlumprecht '94): If X is a B-space with unconditional basis and nontrivial cotype $[\ell_\infty^n \not\subseteq X]$, then $S_X \underset{UH}{\sim} S_{\ell_1}$ (and therefore $\underset{UH}{\sim} S_{\ell_2}$)

<u>Cor</u>: (Ostrovskii 2007) X with u.b. and $\ell_\infty^n \not\subseteq X$ implies $\ell_2 \underset{c}{\hookrightarrow} X$

. Here « $\ell_\infty^n \not\subseteq X$ » means « $\forall C \, \exists n \, \forall T: \ell_\infty^n \simeq G_n \subseteq X$ $\|T\| \|T^{-1}\| > C$. »

. We proved last time: $\forall R, \varepsilon \, \exists \varphi : \ell_2 \to S_{\ell_2}$ $\begin{cases} \|x-y\|_2 \le R \Rightarrow \|\varphi(x) - \varphi(y)\|_2 \le \varepsilon \\ \lim\limits_{S \to \infty} \inf\limits_{\|x-y\|_2 \ge S} \|\varphi(x) - \varphi(y)\|_2 = \sqrt{2}. \end{cases}$
[Baudier-Guentner]

<u>Proof of Cor</u>: Let (e_n) be a 1-unconditional basis of X, $\mathbb{N} = \bigcup\limits_{i=1}^{\infty} N_i$, $N_i \cap N_j = \emptyset$ for $i \ne j$, $\# N_i = \infty$, $X_i = \overline{\mathrm{span}}\{e_n : n \in N_i\}$

O-S shows that $\exists \varphi_i : S_{X_i} \underset{UH}{\sim} S_{\ell_2}$. Then take $\varphi = \varphi_{R,\varepsilon}$ given by [B-G] and let $f_i = \varphi_i \circ \varphi$. Then $\|x-y\|_2 \le R \Rightarrow \|f_i(x) - f_i(y)\| \le \omega_{\varphi_i}(\varepsilon)$

$\lim\limits_{S \to \infty} \inf\limits_{\|x-y\|_2 \ge S} \|f_i(x) - f_i(y)\| \ge \delta_i$ if $\omega_{\varphi^{-1}}(\delta_i) < \sqrt{2}$, that is, if δ_i is small.

Let us apply this for $R = i$ and $\varepsilon > 0$ s.t. $\omega_{\varphi_i}(\varepsilon) \le \frac{\delta_i}{i 2^i}$: this yields

(i) $\|x-y\|_2 \le i \Rightarrow \|f_i(x) - f_i(y)\|_{X_i} \le \frac{\delta_i}{i 2^i}$

(ii) $\lim\limits_{S \to \infty} \inf\limits_{\|x-y\|_2 \ge S} \|f_i(x) - f_i(y)\|_{X_i} \ge \delta_i$

Fix $x_0 \in \ell_2$ and set $f : \ell_2 \to X = \bigoplus\limits_{i=1}^{\infty} X_i$

$$x \longmapsto \left(\frac{i}{\delta_i} (f_i(x) - f_i(x_0)) \right)_{i=1}^{\infty}$$

• $\sum\limits_i \frac{i}{\delta_i} \|f_i(x) - f_i(x_0)\| = \sum\limits_{i < \|x - x_0\|} + \sum\limits_{i \ge \|x - x_0\|} \frac{1}{2^i}$: therefore f is well-def

• $\|f(x) - f(y)\| \le \sum\limits_{i < \|x-y\|} \frac{i}{\delta_i} \times 2 + \sum\limits_{i \ge \|x-y\|} \frac{1}{2^i}$

$$\le \sum_{i < \|x-y\|} \frac{2i}{\delta_i} + 2 = \rho_2(\|x-y\|).$$

Lower bound: it is enough to prove that $\forall h \in \mathbb{R}^+ \, \exists S \; \|x-y\| \ge S \Rightarrow \|f(x) - f(y)\| \ge h$.

or to prove that $\forall h \in \mathbb{R}^+ \, \exists S \, \exists i \; \|x-y\| \ge S \Rightarrow \frac{i}{\delta_i} \|f_i(x) - f_i(y)\| \ge h$ take $i \ge h$ use (ii).

We shall now prove a part of [O-S]: the superreflexive case.

Th [partial O-S] Let X be uniformly convex and uniformly smooth with a 1-unconditional basis. Then $S_X \sim S_{\ell_1}$

Preparation: Assume X is uniformly smooth (US): $\forall x \in S_X \, \exists! \, x^* \in S_{X^*}$ $x^*(x)=1$ ($j(x)=x^*$) and j is unif^{ly} cont.

If moreover X is UC (that is, X^* is US) (and X is reflexive), then

$j_X : S_X \longrightarrow S_{X^*}$ and $j_{X^*} : S_{X^*} \longrightarrow S_{X^{**}} = S_X$ is the inverse of j_X.

So $S_X \underset{UH}{\overset{j_X}{\sim}} S_{X^*}$

Lemma: If $\dim X = n$, if X is UC and US and if X has a 1-unc. basis, and if you look at X and X^* as sequence spaces whose canonical bases are 1-unconditional, and also as algebras for the coordinatewise pointwise multiplication, and if you let $F : S_X \longrightarrow S_{\ell_1^n}$, $x \longmapsto x|j(x)|$,

then (i) F is uniformly continuous and ω_F depends only on the modulus of smoothness of X. if X is ℓ_p^n, F is the Mazur map.

(ii) F is 1-1, F^{-1} is uniformly continuous, $\omega_{F^{-1}}$ depends only on the modulus of convexity of X

(iii) F is onto.

Note that $j(|x|) = |j(x)|$. $\sum |x_i| \, |j(x)_i| = \langle |x|, j(|x|)\rangle = 1$.

(i) $\|F(x)-F(y)\|_1 = \|x|j(x)| - y|j(y)|\|_1 \le \|x(|j(x)|-|j(y)|)\|_1 + \|(x-y)|j(y)|\|_1$

$\le \|x(|j(x)|-|j(y)|)\|_1 + \|(x-y)\,|j(y)|\|_1$

$= \qquad + \langle |x-y|, j(|y|)\rangle$

$\le \sum |x_i| \, |j(x)_i - j(y)_i| + \_\_\_\_$

$= \langle |x|, |j(x)-j(y)|\rangle + \_\_\_\_$

$\le \|x\|_X \|j(x)-j(y)\|_{X^*} + \|x-y\|$

$= \|j(x)-j(y)\| + \|x-y\| \le \omega_j(\|x-y\|) + \|x-y\|$

(ii) Let $x, y \in S_X$, $f = F(x) = x|j(x)|$, $h = F(y) = y|j(y)|$.

Assume $\|f-h\|_1 = \varepsilon$. We want to show that $\|x-y\|_X \le \omega(\varepsilon)$.

It is enough to show that $\|\frac{x+y}{2}\| \ge 1-\lambda(\varepsilon)$, or that $\langle y, j(x)\rangle \ge 1-\mu(\varepsilon)$ ($\mu = 2\lambda$)

Let $\Lambda = \{i : \operatorname{sgn} f_i = \operatorname{sgn} h_i\}$: If $i \in \Lambda$, $\operatorname{sgn} x_i = \operatorname{sgn} y_i = \operatorname{sgn} f_i$. Wlog $\forall i \in \Lambda$ $f_i, h_i, x_i, y_i \ge 0$

Define $g : i \longmapsto \begin{cases} 0 & \text{if } i \notin \Lambda \\ \min(f_i, h_i) & \text{if } i \in \Lambda \end{cases}$. Note that $\|f\|_1 = \|h\|_1 = 1$ and $\sum |x_i| \, |j(x_i)| = \langle |x|, j(|x|)\rangle = 1$

We have $\boxed{\|g\|_1 = 1-\frac{\varepsilon}{2}}$: $\sum_{\Lambda^c}|f_i-h_i| = \varepsilon_1 = \underbrace{\sum'_{\Lambda^c}|f_i|}_{\varepsilon_1'} + \underbrace{\sum'_{\Lambda^c}|h_i|}_{\varepsilon_1''}$.

$0 = \sum'_{\Lambda^c}|g_i| = \sum'_{\Lambda^c}|f_i| - \varepsilon_1' = \sum'_{\Lambda^c}|h_i| - \varepsilon_1''$

Let $\Lambda_f = \{i\in\Lambda : f_i \geq h_i\}$ and $\Lambda_h = \Lambda\setminus\Lambda_f$

Let $\varepsilon_2 = \sum'_{\Lambda} |f_i-h_i| = \overbrace{\sum'_{\Lambda_f} f_i - h_i}^{\varepsilon_2'} + \overbrace{\sum'_{\Lambda_h} h_i - f_i}^{\varepsilon_2''}$

then $\sum'_{\Lambda} g_i = \sum'_{\Lambda_f} h_i + \sum'_{\Lambda_h} f_i = \sum'_{\Lambda} f_i - \varepsilon_2' = \sum'_{\Lambda} h_i - \varepsilon_2''$

$\|g\|_1 = \sum'_{\Lambda^c}|g_i| + \sum'_{\Lambda} g_i = 1-(\varepsilon_1'+\varepsilon_2') = 1-(\varepsilon_1''+\varepsilon_2'')$ $\frac{\varepsilon}{2}$

and $\varepsilon_1+\varepsilon_2=\varepsilon$ and $\|g\|_1 = 1-\frac{\varepsilon}{2}$.

$G = \operatorname{supp} g \subset (\Lambda\cap\operatorname{supp}x\cap\operatorname{supp}y)$. Then $\|g\frac{y}{x}\|_1 \leq \|f\frac{y}{x}\|_1 = \||j(x)|\,|y|\|_1$

Thus $\|g(\frac{y}{x}+\frac{x}{y})\|_1 \leq 2$ $\qquad = \langle |y|, j(|x|)\rangle \leq 1$

Fix $\delta>0$. Let $A = \{i\in G : \frac{y_i}{x_i}\wedge\frac{x_i}{y_i} < 1-\delta\}$: the set where x and y differ somewhat.

Note that $t<1-\delta \Rightarrow t+\frac{1}{t} > \frac{1}{1-\delta}+1-\delta > 2+\delta^2$. Let $C = G\setminus A$.

$2 \geq \|g(\frac{x}{y}+\frac{y}{x})\|_1 = \|\pi_A g(\frac{x}{y}+\frac{y}{x})\|_1 + \|\pi_C g(\frac{x}{y}+\frac{y}{x})\|_1 \geq (2+\delta^2)\|\pi_A g\|_1 + 2\|\pi_C g\|_1$

so that $2\geq 2\|g\|_1 + \delta^2\|\pi_A g\|_1$ and $2\geq 2-\varepsilon+\delta^2\|\pi_A g\|_1$.

So $\|\pi_A g\|_1 \leq \frac{\varepsilon}{\delta^2}$. Similarly, $\|\pi_C g\|_1 = \|g\|_1 - \|\pi_A g\|_1 \geq 1-\frac{\varepsilon}{2}-\frac{\varepsilon}{\delta^2}$

and $\|\pi_C g\frac{y}{x}\|_1 \geq (1-\delta)(1-\frac{\varepsilon}{2}-\frac{\varepsilon}{\delta^2})$ $\quad$ [if $i\in C$, $\frac{y_i}{x_i}\geq 1-\delta$]

Let $\delta = \varepsilon^{1/3}$. Then $\langle \pi_G y, j(x)\rangle = \|\pi_G f\frac{y}{x}\|_1 \geq \|g\frac{y}{x}\|_1 \geq \|\pi_C g\frac{y}{x}\|_1$
$\geq (1-\varepsilon^{1/3})(1-\frac{\varepsilon}{2}-\frac{\varepsilon}{\delta^2}) \geq 1-2\varepsilon^{1/3}$.

But $\langle|y|,|j(x)|\rangle\leq 1$, so $|\langle\pi_{G^c}y, j(x)\rangle| \leq 2\varepsilon^{1/3} \Rightarrow \langle y, j(x)\rangle \geq 1-4\varepsilon^{1/3}$.

(iii) F is a uniform homeomorphism from S_X into $S_{\ell_1^n}$. It is bound to be onto: $S_X \underset{UH}{\sim} S_{\ell_2^n}$, $S_{\ell_1^n}\sim S_{\ell_2^n}$. But by Brouwer's theorem, $S_{\ell_2^n}$ is not contractible whereas every proper subset of $S_{\ell_2^n}$ is.

<u>Corollary</u>: If X is UC, US with a 1-unc. basis, <u>then</u> $S_X \underset{UH}{\sim} S_{\ell_1}$

<u>Proof</u>: Let $\{x_i\}$ be a 1-unc. basis of X and $\{e_i\}$ be the canonical basis of ℓ_1. The lemma shows that $\forall A\subset\mathbb{N}$ finite $\exists F_A: S_{\mathrm{span}\{x_i : i\in A\}} \longrightarrow S_{\mathrm{span}\{e_i : i\in A\}}$ and $\omega_{F_A}, \omega_{F_A^{-1}}$ do not depend on A; the F_A's are compatible.

This yields a uniform homeomorphism from S_X onto S_{ℓ_1}

<u>Announcement</u>: We shall show that X with u.b. and $\ell_\infty^n \not\subset X$, then the 2-convexification $X^{(2)}$ of X is 2-convex and q-concave for some $q<\infty$: apply Th 0 to $X^{(2)}$: $S_X \underset{UH}{\sim} S_{X^{(2)}}$

Note that for X with unc. basis, $S_X \underset{UH}{\sim} S_{\ell_2} \Leftrightarrow \ell_\infty^n \not\subseteq X$

($\Rightarrow$ is due to Enflo.)

Remark: $\ell_2 \underset{L}{\hookrightarrow} c_0$

Open question: X unc. basis, $\ell_\infty^n \subseteq X \overset{?}{\Rightarrow} \ell_2 \underset{c}{\hookrightarrow} X$