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More on coarse embeddings of ℓ_2.

<u>Th</u> (Nowak 2006) If $1 \leq p < \infty$, then $\ell_2 \underset{c}{\hookrightarrow} \ell_p$.

Oshovskii went further using the following:

<u>Th</u> (Odell - Schlumprecht 2004) Let X be a Bspace with an unconditional basis and nontrivial cotype (ℓ_∞^n don't unif^{ly} embed into X "$\ell_\infty^n \not\subseteq X$")

Then $S_X \underset{UH}{\sim} S_{\ell_1}$.

<u>Corollary</u>: (Oshovskii) Let X satisfy the same conditions. Then $\ell_2 \underset{c}{\hookrightarrow} X$.

<u>Th</u> (O-S) Let X have 1-u.b. and be uniformly convex and uniformly smooth. Then $S_X \underset{UH}{\sim} S_{\ell_1}$.

<u>p-convexity and q-concavity</u>: For X with a 1-u.b. (e_i), we will write $(x_i) \in X$ if $\sum x_i e_i \in X$. For $t > 0$ and $x = (x_i) \in X$, $|x|^t = (|x_i|^t)$.

For $1 \leq p, q \leq \infty$, say X is <u>p-convex</u> if $\forall\ (x_j)_{j=1}^n \subseteq X$ $\|(\sum |x_j|^p)^{1/p}\| \leq M(\sum \|x_j\|^p)^{1/p}$ for some $M > 0$. The smallest M is denoted <u>$M^p(X)$</u>.

Every X is 1-convex. Say X is <u>q-concave</u> if the reverse inequality holds. and $M_q(X)$ is defined similarly.

By duality, X is p-convex iff X^* is p'-concave.

<u>p-convexification of X</u>: this is $X^{(p)} = \{(x_i) : \|(x_i)\|_{(p)} = \|\sum |x_i|^p e_i\|^{1/p} < \infty\}$

If X has a 1-unconditional basis (e_i), $X^{(p)}$ has also a 1-unconditional basis again denoted by (e_i).

If X is r-convex, then for $x, y \in X^{(p)}$
$$\|(|x|^{pr} + |y|^{pr})^{\frac{1}{pr}}\|_{(p)}^{pr} = \|(|x|^{pr} + |y|^{pr})^{\frac{1}{r}}\|^r \leq M^r(X)^r(\|x\|^{pr} + \|y\|^{pr})$$

so that $M^{pr}(X^{(p)}) \leq M^r(X)^{\frac{1}{p}}$.

More generally, if X is r-convex and s-concave, $1 \leq r \leq s < \infty$, then $X^{(p)}$ is pr-convex and ps-concave with $M^{pr}(X^{(p)}) \leq M^r(X)^{1/p}$
$M_{ps}(X^{(p)}) \leq M_s(X)^{1/p}$.

Proposition 1: Let X have a 1-ub and $M_q(X)=1$. Then $M^p(X^{(p)})=1=M_{pq}(X^{(p)})$.

Proposition 2: Let $1<q<\infty$ and X q-convex and with 1-ub (e_i). Then X admits an equivalent norm for which $M_q(X)=1$ and (e_i) is 1-ub.

Proposition 3: Let $1<p\le 2\le q<\infty$ and X with 1-ub and $M^p(X)=1=M_q(X)$. Then X is uniformly smooth and uniformly convex.

Proof: by duality, it suffices to show that X is u.c.

Lemma: Let $q\geq 2$. Then for any $1<p<\infty$ there exists $C=C(p,q)$ st

$$\left|\frac{s-t}{C}\right|^q+\left|\frac{s+t}{2}\right|^q\le\left(\frac{|s|^p+|t|^p}{2}\right)^{1/p}$$

Proof: We may assume that $s=1>t\geq-1$. Consider $\varphi(t)=\left(\frac{1+|t|^p}{2}\right)^{\frac{q}{p}}-\left(\frac{1+t}{2}\right)^q$

φ is ≥ 0 on $[-1,1[$ and $\varphi''(1)>0$, so $\frac{\varphi(t)}{(1-t)^2}$ is bounded from below, hence so is $\frac{\varphi(t)}{(1-t)^q}$.

Fix $\varepsilon>0$ and let $x,y\in S_X$ be st $\|x-y\|=\varepsilon$. By the lemma, there is $C=C(p,q)$ s.t. $\left\|\left(\left(\frac{|x-y|}{C}\right)^q+\left(\frac{|x+y|}{2}\right)^q\right)^{1/q}\right\|\le\left\|\left(\frac{|x|^p+|y|^p}{2}\right)^{\frac{1}{p}}\right\|$

Apply $M^p(X)=1=M_q(X)$ to obtain $\left(\left\|\frac{x-y}{C}\right\|^q+\left\|\frac{x+y}{2}\right\|^q\right)^{\frac{1}{q}}\le\frac{(\|x\|^p+\|y\|^p)^{1/p}}{2^{1/p}}$

Thus $\frac{\varepsilon^q}{C^q}\le 1-\left\|\frac{x+y}{2}\right\|^q\le q\left(1-\frac{\|x+y\|}{2}\right)$. Hence $\delta_X(\varepsilon)\geq\frac{\varepsilon^q}{C^q q}$.

Proposition 4: Let $1<p<\infty$ and X has 1-ub (e_i). Then the 'Mazur' map

$$G_{p,X}: S_{X^{(p)}}\longrightarrow S_X$$
$$x\longmapsto\left((\operatorname{sgn} x_i)\,|x_i|^p\right)$$

is a uniform homeomorphism

Proof: Let $x=(x_i)$, $y=(y_i)\in S_{X^{(p)}}$. Let $I_+=\{i:\operatorname{sgn} x_i=\operatorname{sgn} y_i\}$, $I_-=\{i:\operatorname{sgn} x_i\neq\operatorname{sgn} y_i\}$

It suffices to show that for $\delta=\|x-y\|_{(p)}$, $2^{1-p}\delta^p\le\|G_{p,X}(x)-G_{p,X}(y)\|\le\delta+\delta^{\frac{p}{2}}+(1-(1-\delta)^p)$

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$$\|G_{p,X}(x)-G_{p,X}(y)\| = \left\|\sum{}'\big((\operatorname{sgn} x_i)|x_i|^p-(\operatorname{sgn} y_i)|y_i|^p\big)e_i\right\|$$

$$=\Big\|\underbrace{\sum_{i\in I_+}'(|x_i|^p-|y_i|^p)e_i}_{=d_+}+\underbrace{\sum_{i\in I_-}'(|x_i|^p+|y_i|^p)e_i}_{=d_-}\Big\|$$

By 1-u. of (e_i) and $a^p-b^p\geq(a-b)^p$
and $a^p+b^p\geq 2^{1-p}(a+b)^p$ for $a\geq b\geq 0$

so that $\|d_++d_-\|\geq\left\|\sum_{i\in I_+}'\big||x_i|-|y_i|\big|^pe_i+2^{1-p}\sum_{i\in I_-}'\big|(|x_i|+|y_i|)\big|^pe_i\right\|$

$$\geq 2^{1-p}\left\|\sum|x_i-y_i|^p e_i\right\|=2^{1-p}\|x-y\|_{(p)}^p.$$

For the upper estimate, first note that $\|d_-\|\leq\|\sum_{i\in I_-}'|x_i-y_i|^pe_i\|\leq\|x-y\|_{(p)}^p\leq\delta^p$

Set $q=1-\sqrt{\delta}$ and $c=(1-q)^{-p}=\delta^{-p/2}$

For $a,b\geq 0$ with $0\leq b\leq qa$, we have $c(a-b)^p-(a^p-b^p)\geq c(1-q)^pa^p-a^p$
$$=a^p(c(1-q)^p-1)$$
$$=0.$$

Let $I_+'=\{i\in I_+ : |y_i|<q|x_i| \text{ or } |x_i|<q|y_i|\}$

$I_+''=I_+\setminus I_+'$. Then $d_+=d_+'+d_+''$, where $d_+'=\sum_{i\in I_+'}'(|x_i|^p-|y_i|^p)e_i$.

By $(*)$, $\|d_+'\|\leq c\|\sum_{i\in I_+'}\big||x_i|-|y_i|\big|^pe_i\|\leq\delta^{-p/2}\|x-y\|_{(p)}^p=\delta^{p/2}$

<u>Proof</u> of Odell-Schlumprecht: Let X have q-concave u.b. (e_i) and $\ell_\infty^n\not\hookrightarrow X$

We need the following:

<u>Th</u> (Maurey-Pisier) Let X be a B-space with same hypotheses:
then X is q-concave for every $q>q'$.

By an equivalent renorming, we may assume (e_i) is monotone.

By Maurey-Pisier X is q-concave for some $q\in[2,\infty[$.

By Prop'n 2, we may assume $M_q(X)=1$, hence $M^2(X^{(2)})=1=M_{2q}(X^{(2)})$

by Prop'n 1, note also that $X^{(2)}$ has a 1-ub. Moreover by Prop'n 3, $X^{(2)}$ is UC and US and hence $G_{2,X}: S_{X^{(2)}}\longrightarrow S_X$ is a uniform homeomorphism (Prop'n 4).

By Odell-Schlumprecht theorem from last week, there is a uniform homeomorphism $F : S_{X(2)} \longrightarrow S_{\ell_1}$. Then $F \circ G_{2,X}^{-1} : S_X \longrightarrow S_{\ell_1}$ is a uniform homeomorphism.

There is a converse: X with 1-uub: If $S_X \underset{UH}{\sim} S_{\ell_2}$, then $\ell_\infty^n \not\subset X$.
(Enflo's result)