<u>Recall</u> [Odell-Schlumprecht]: If X has an u.b. and $\ell_\infty^n \not\subseteq X$, then $S_X \underset{UH}{\sim} S_{\ell_1} \underset{UH}{\sim} S_{\ell_2}$

This theorem has a <u>converse</u> due to Enflo: If the ℓ_∞^n embed uniformly into X, then $S_X \underset{UH}{\not\sim} S_{\ell_2}$.

We will construct a sequence of finite metric spaces M_n that are uniformly bounded so that every sequence (T_n) of embeddings $M_n \to \ell_2$ is not equi-bi-uniform.

<u>Recall Def:</u> $(T_n : A_n \to B_n)$ is <u>equi-bi-uniform</u> if $\forall \varepsilon>0\ \exists \delta>0\ \forall n$ ~~$\forall x,y\in A_n$~~

$$\begin{cases} \forall x,y\in A_n \quad d(x,y)<\delta \Rightarrow d(T_n x, T_n y) < \varepsilon \\ \forall T_n x, T_n y \in B_n \quad d(T_n x, T_n y)<\delta \Rightarrow d(x,y) < \varepsilon \end{cases}$$

is this true?! [<u>Note</u> that

- $S_X \underset{UH}{\overset{\varphi}{\sim}} S_{\ell_2} \Rightarrow B_X \underset{UH}{\overset{\varphi}{\sim}} B_{\ell_2}$, putting $\varphi(x) = \|x\|\,\varphi\left(\frac{x}{\|x\|}\right)$ (radial extension)
- $\ell_\infty^n \subseteq X \iff \exists \tilde{T}_n : \ell_\infty^n \xrightarrow{\text{linear}} X \quad \frac{\|x-y\|}{C} \le \|\tilde{T}_n x - \tilde{T}_n y\| \le \|x-y\|$
- for every finite metric space M, there is an isometry $I : M \to \ell_\infty^{|M|}$ called Frechet map: $I(m) = (d(m,n))_{n\in M}$

$$\max_n |d(m,n) - d(m',n)| = d(m,m').$$

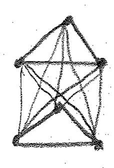
We will consider $T_n = \varphi \circ \tilde{T}_n \circ I_n$, where $I_n : M_n \to \ell_\infty^{|M_n|}$

Let us consider the metric in the metric space, denoted by d.

Let M be a metric space. A <u>segment</u> s in M is $s \subset M$, $\# s = 2$ (If $\# s = 1$, it is a trivial segment). The <u>length of</u> s is $\ell(\{a,b\}) = d(a,b)$.

A <u>double n-simplex</u> in M is a couple $\{A,B\}$, where $A\subset M$, $B \subset M$, $\# A\cup B = 2n$. $A = \{a_1,\dots,a_n\}$, $B = \{b_1,\dots,b_n\}$.
Their set is denoted by <u>$\Delta_n(M)$</u>.

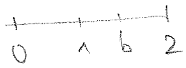
Let $D \in \Delta_n(M)$ ($\{A,B\}$): then any segment $\{a_i,a_j\}$ or $\{b_i,b_j\}$, $i\neq j$, is called an <u>edge of D</u>. Their set is <u>$E(D)$</u>. Any segment $\{a_i,b_j\}$ is called a connecting line of D. Their set is $C(D)$


A
B

<u>Déf</u>: M has generalised roundness $p\in\mathbb{R}$ [$\Theta(M)=p$] if p is the supremum of those q with the property $\sum_{c\in C(D)} \ell(c)^q > \sum_{e\in E(D)} \ell(e)^q$ for $n\in\mathbb{N}$ and $D\in\Delta_n(M)$

1)

- For every m. sp. M we have $\Theta(M) \geq 0$ because $\# E(\Delta) = n(n-1)$, $\# C(\Delta) = n^2$.
- If the segment $[0,2]$ sits isometrically in M, then $\Theta(M) \leq 2$:



Let $\Delta_b = \{\{0,2\},\{1,b\}\} \subset [0,2]$ and define $f_p(b) = \sum'_{c \in C(\Delta)} \ell(c)^p - \sum'_{e \in E(\Delta)} \ell(e)^p$.

Then $f_p(b) \xrightarrow[b \to 1]{} 2 \cdot 2 \cdot 1^p - (0 + 2^p) = 4 - 2^p$ which is ≥ 0 iff $p \leq 2$.

<u>Lemma</u> $\Theta(L_2(\mu)) = 2$

Proof: $\leq$ is ok. We need $\sum'_{c \in C(\Delta)} \ell(c)^2 \geq \sum'_{e \in E(\Delta)} \ell(e)^2$ for $\Delta \in \Delta_n(L_2(\mu))$.

$$\int \sum'_{1 \leq i,j \leq n} (a_i - b_j)^2 - \sum'_{1 \leq i < j \leq n} (a_i - a_j)^2 - \sum'_{1 \leq i < j \leq n} (b_i - b_j)^2 \, d\mu \geq 0$$

because

$$= \Big(\sum'_i a_i - \sum'_j b_j \Big)^2 \qquad \sum' (a_i - b_i)^2 + 2 \sum_{i<j} (a_i - b_i)(a_j - b_j)$$

We construct (M_n) so that $\forall Y \ \Theta(Y) > 0$, $\forall T_n : M_n \to Y$ (T_n) is not e-b-u.

The initial idea is to take $\Delta \in \Delta_n(M_n)$ with long edges and short connecting lines: if $T_n : M_n \to Y$ and $\Theta(Y) = p > 0$, (T_n) e-b-u, then

$$\sum'_{c \in C(\Delta)} \ell(T_n c)^p \geq \sum'_{e \in E(\Delta)} \ell(T_n e)^p \geq n(n-1) \inf_{e \in E(\Delta)} \ell(T_n e)^p \geq n(n-1)\varepsilon$$

$$\sum'_{c \in C(\Delta)} \ell(T_n c)^p \leq n^2 \sup_{c \in C(\Delta)} \ell(T_n c)^p \xrightarrow[\sup \ell(c) \to 0]{} 0 \qquad \text{because } T_n^{-1} \text{ is u.c.}$$

This is not possible as $\sup_{c \in C(\Delta)} \ell(c) \geq \frac{1}{2} \sup_{e \in E(\Delta)} \ell(e)$.

<u>Possible distances in M_n</u>: the ℓ(segments of M) may be ordered as

$\ell_1^n < \ell_2^n < \dots < \ell_{2^m}^n$ if we have constructed M_n in the way below.

We construct M_n so that $\lim \ell_1^n = 0$.

Imagine that for all $n \in \mathbb{N}$ and $i \in [1, m-1]$ there is

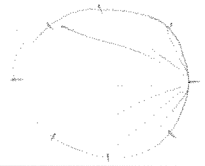
$\Delta \in \Delta_n(M_n)$ so that $\ell(e) = \ell_{2i}^n$ for $e \in E(\Delta)$

$\ell(c) = \ell_{2i-1}^n$ for $c \in C(\Delta)$.

The space M_n:

Let n be even and put $A_n = \{ e^{\frac{2\pi i k}{2^{n+1}}} : k \in [\![0, 2^{n+1}-1]\!] \}$ with the C-distance.

$M_n = \prod_{i=1}^{n^n} A_n$ together with the sup metric.



n=2

If $m \in M_n$, $m = (m_1, \dots, m_{n^n})$

$\tilde{m} \in M_n$, $d(m, \tilde{m}) = \sup_{i \in [\![1, n^n]\!]} d(m_i, \tilde{m}_i)$

A segment $s \in M_n$ is called an i-segment ($s \in S_n^i$) if $\ell(s) = \ell^n_{2^{i-1}}$.

If $s = \{a, b\}$, $a = (a_1, \dots, a_{n^n})$, $b = (b_1, \dots, b_{n^n})$, we require $d(a_j, b_j) = \ell^n_{2^{i-1}}$ for $n^{(n-i)}$ indices j, and $d(a_j, b_j) = 0$ otherwise.

Call $\Delta_n^i(M_n) = \left\{ D \in \Delta_n(M_n) : \begin{array}{ll} \forall c \in C(\Delta) & c \in S_n^i \\ \forall e \in E(\Delta) & e \in S_n^{i+1} \end{array} \right\}$

Assume that $\Delta_n^i(M_n) \neq \emptyset$ for $i \in [\![1, n-1]\!]$ and all n.

Notice that if $n \in \mathbb{N}$ and $i \in [\![1, n-1]\!]$, there is $N_C \in \mathbb{N}$ s.t. for all $s \in S_n^i$ $\#\{D \in \Delta_n^i : s \in C(\Delta)\} = N_C$; whenever $s, t \in S_n^i$ there is an isometry $\varphi: M_n \to M_n$ s.t. $\varphi s = t$. ; if $D_s = \{ j : s_1(j) \neq s_2(j) \}$, then $\# D_s = n^{n-i}$ $\# D_t = n^{n-i}$

Also $\exists\, N_E \in \mathbb{N}$ $\forall s \in S_n^{i+1}$ $\#\{ D \in \Delta_n^i : s \in E(\Delta) \} = N_E$.

If $\Delta \in \Delta_n(M_n)$, $\sum'_{c \in C(\Delta)} \ell(T_n c)^p \geq \sum'_{e \in E(\Delta)} \ell(T_n e)^p$.

Then we sum: $\sum'_{D \in \Delta_n^i} \sum'_{c \in C(\Delta)} \ell(T_n e)^p \geq \sum'_{D \in \Delta_n^i} \sum'_{e \in E(\Delta)} \ell(T_n e)^p$.

$$\underbrace{N_C (\# S_n^i)}_{\# \Delta_n^i \cdot n^2} (\# S_n^i)^{-1} \sum'_{s \in S_n^i} \ell(T_n s)^p \geq \underbrace{N_E (\# S_n^{i+1})}_{\# \Delta_n^i \cdot n(n-1)} (\# S_n^{i+1})^{-1} \sum'_{s \in S_n^{i+1}} \ell(T_n s)^p.$$

By telescoping, $(\# S_n^1)^{-1} \sum'_{s \in S_n^1} \ell(T_n s)^p \geq \left(1 - \frac{1}{n}\right)^{n-1-1} (\# S_n^n)^{-1} \sum' \ell(T_n s)^p \geq \frac{1}{e} \inf_{s \in S_n^n} \ell(T_n s)^p$

$\sup_{s \in S_n^1} \ell(T_n s)^p \geq$

as (T_n) is an e-b-u

$\downarrow$

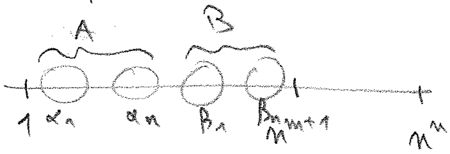
0

Contradiction.

~~10~~ ~~17~~ ~~24~~ 1/11

<u>Last lemma:</u> to prove that $\Delta_n^i(\Pi_n) \neq \emptyset$.

proof:



$2n$ groups of $\frac{n^m}{2}$ coordinate each

Let $D = \{A, B\}$ For $a \in A$, ~~pick~~ $i \in [\![1, n]\!]$ and put $a_j = e^{\frac{\pi i}{2^m}}$ for $j \in \alpha_i$.

$a = (a_1, \dots, a_{n^m})$

$b = (b_1, \dots, b_m)$ For $l \in [\![1, n]\!]$, $l \neq i$, put $a(j) = 1$ for $j \in \bigcup_{l \neq i} \alpha_l$ and $a(j) = e^{\frac{\pi i}{2^{m+1}}}$ for the other j.

For B, to replace α by β.

Q: Does this follow from metric cotype?

But $S_{\ell p} \underset{\mathrm{unif}}{\cong} S_{\ell 2}$

$B_{\ell p} \underset{U}{\not\hookrightarrow} B_{\ell 2}$. (at least for the spaces themselves instead of their unit balls).