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Th: If X B-space, $\ell_\infty^n \subseteq X$ (uniformly /n), then $S_X \underset{UH}{\approx} S_{\ell_2}$

The proof is based on: ① If $S_X \underset{UH}{\approx} S_Y$, then $B_X \underset{UH}{\approx} B_Y$ (Lemma 1)

(we are not saying that $X \underset{UH}{\approx} Y$; this could contradict metric dy representation)

② $B_{c_0} \underset{unif}{\not\hookrightarrow} B_{\ell_2}$ (Theorem 1, Raynaud 1983)

③ If $B_{\ell_\infty^n} \underset{equi\text{-}unif.}{\hookrightarrow} \ell_2$, then $B_{c_0} \underset{unif.}{\hookrightarrow} \ell_2$ (Theorem 2)

Proof of Lemma 1: put $\psi(x) = \|x\|\, \varphi\left(\frac{x}{\|x\|}\right)$: then $\psi^{-1}(z) = \|z\|\, \varphi^{-1}\left(\frac{z}{\|z\|}\right)$

Also $\|x\| = \|\psi(x)\|$, so that $\psi(B_X) = B_Y$

Let $\varepsilon > 0$ be fixed, let $x, y \in B_X$: • If $\|x\| \vee \|y\| < \frac{\varepsilon}{2}$, then $\|\psi(x) - \psi(y)\| < \varepsilon$

• Otherwise $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \le \frac{2\|x-y\|}{\|x\| \vee \|y\|} \le \frac{4}{\varepsilon}\|x-y\|$; choose $\delta < \frac{\varepsilon}{2}$ s.t.

$\omega_\varphi\left(\frac{4}{\varepsilon}\delta\right) < \frac{\varepsilon}{2}$; then $\underbrace{\left\|\varphi\left(\frac{x}{\|x\|}\right) - \varphi\left(\frac{y}{\|y\|}\right)\right\|}_{q} < \frac{\varepsilon}{2}$

and $\left\|\|x\|\varphi\left(\frac{x}{\|x\|}\right) - \|y\|\varphi\left(\frac{y}{\|y\|}\right)\right\| \le \big|\|x\| - \|y\|\big| + q < \varepsilon$

We will now prove Theorem 1. First define:

Def: Let (M, d) be a metric space. It is stable if for all free ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$ and all $(x_n), (y_n) \subset M$, bounded, we have

$$\lim_{\mathcal{U},n} \lim_{\mathcal{V},m} d(x_n, y_m) = \lim_{\mathcal{V},m} \lim_{\mathcal{U},n} d(x_n, y_m).$$

Examples: • ℓ_2 is stable because $\|x_n - y_m\|^2 = \|x_n\|^2 - 2\langle x_n, y_m\rangle + \|y_m\|^2$

and B_{ℓ_2} is w-cpct: $w\text{-}\lim_{\mathcal{U},n} x_n = x \in B_{\ell_2}$ $w\text{-}\lim_{\mathcal{V},m} y_m = y$: then

$$\lim_{\mathcal{U}} \lim_{\mathcal{V}} \|x_n - y_m\|^2 = \lim_{\mathcal{U}} \left(\|x_n\|^2 - \langle x_n, y\rangle + \lim_{\mathcal{V}} \|y_m\|^2\right)$$

$$= \lim_{\mathcal{U}} \|x_n\|^2 - \langle x, y\rangle + \lim_{\mathcal{V}} \|y_m\|^2$$

• c_0 is not stable: consider $x_n = (\overbrace{1, \dots, 1}^{n}, 0, \dots)$

$y_n = (\underbrace{0, \dots, 0, -1}_{n}, 0, \dots)$

Def: A basis (x_n) of X is <u>K-spreading</u> if $\forall N \geq 1$ $\forall a_1, \dots, a_N \in \mathbb{R}$ $\forall n_1 < n_2 < \dots < n_N$

$$\frac{1}{K} \left\| \sum_1^N a_n x_n \right\| \leq \left\| \sum_1^N a_k x_{n_k} \right\| \leq K \left\| \sum_1^N a_n x_n \right\|$$

examples: canonical bases of ℓ_p or c_0, summing basis of c_0

Def: A basis (x_n) of X is <u>bimonotone</u> if the basis projections satisfy $\|P_n\| = \|I - P_n\| = 1$.

Lemma: If (x_n) is K-spreading, X can be renormed so that (x_n) is 1-spreading and bimonotone.

Proof: exercise: consider $|||\sum_1 a_n x_n||| = \sup \|\sum' a_k x_{n_k}\|$

then $|||\sum a_n x_n||| \geq |||\sum a_k x_{n_k}|||$ but $\leq$ is not so clear!

Theorem: Let X be a B-space with a K-spreading basis (x_n), s.t. $B_X \underset{UH}{\sim}$ to some stable metric space. Then (x_n) is unconditional.

(If $X \overset{\varphi}{\sim} (M, d)$, let us write $d(\varphi(x), \varphi(y)) = d(x,y)$.)

Proof We may suppose that (x_n) is 1-spreading and bimonotone.

Fix $m \in \mathbb{N}$, $(a_i)_{i=1}^m \subset \mathbb{R}$, $(\vartheta_i)_{i=1}^m \subset \{-1, +1\}$; let $\alpha > 0$, $r > 0$ be such that

$d(x,y) < \alpha$ implies $\|x - y\| < \frac{1}{2}$ (U.C. of φ^{-1})

$\|x - y\| < r$ implies $d(x,y) < \frac{\alpha}{2}$ (U.C. of φ.)

It is enough to prove that (*) $1 = \|\sum' a_i x_i\| \geq \|\sum' \vartheta_i a_i x_i\| \Rightarrow B \geq r$.

We shall prove (**): $\|\sum_{i=1}^m x_i\| \geq \|\sum \vartheta_i x_i\|$ implies $\|\sum_1 \vartheta_i x_i\| \geq r \|\sum' x_i\|$

We have $1 = \left\| \sum_{i=1}^m x_i \right\| \underset{\text{1-spreading}}{=} \left\| \sum_{k=1}^m x_{n_k} \right\| \leq \Big\| \underbrace{\sum_{i \leq l} x_{n_i}}_{x} \Big\| + \Big\| \underbrace{\sum_{i > l} x_{n_i}}_{y} \Big\|$

$\leq 2 \Big\| \overbrace{\sum_{i \leq l} x_{n_i} - \sum_{i > l} x_{n_i}}^{z} \Big\|$

because $\|x\| = \|P_l z\| \leq \|z\|$, $\|-y\| = \|(I - P_l) z\| \leq \|z\|$ with $l = \#\{i : \vartheta_i = +1\}$

Then $\|z\| \geq \frac{1}{2}$ and $d(\sum_{i \leq l} x_{n_i}, \sum_{i > l} x_{n_i}) \geq \alpha$ for every sequence $n_1 < \dots < n_m$

Then (A) $= \lim_{\mathcal{U}_1, n_1} \lim_{\mathcal{U}_2, n_2} \dots \lim_{\mathcal{U}_m, n_m} d(\sum_{i \leq l} x_{n_i}, \sum_{i > l} x_{n_i}) \geq \alpha$

(B) $=$ ________ $d(\sum_{\vartheta_i = +1} x_{n_i}, \sum_{\vartheta_i = -1} x_{n_i}) \leq \sup \{ d(x,y) : \|x - y\| \leq \|\sum_{i=1}^m \vartheta_i x_i\| \}$

$\|\sum_{\vartheta_i = +1} x_{n_i} - \sum_{\vartheta_i = -1} x_{n_i}\|$

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stability implies $A=B$, so that $\alpha \leq \sup\{ d(x,y) : \cdots \}$

If $\| \sum_i \varepsilon_i x_i \| < r$, then $\sup\{ \quad \} \leq \frac{\alpha}{2}$ by definition of r

Let us prove this stability: changing the order in which the limits are taken corresponds to mixing the x_i's affected with $+1$ and the x_i's affected with -1. (see Benyamini-Lindenstrauss for details!)

Theorem: If finite subsets of a metric space (M,d) embed equi-unif$^{\ell y}$ into a Hilbert space, then (M,d) itself does.

Cor: $B_{\ell_\infty^m} \underset{\text{equiunif}}{\not\hookrightarrow}^{\ell y} \ell_2$

$\boxed{\text{Th 1: } B_{c_0} \underset{\text{unif}}{\not\hookrightarrow}^{\ell y} \ell_2}$

↳ proof: If we had $B_{c_0} \underset{\text{unif.}}{\hookrightarrow} \ell_2$, we would get a contradiction as the summing basis is 1-spreading and not unconditional.

2 options → get to studying <u>kernels</u>

→ give Kalton's proof (interlaced graphs as a generalisation of these sequences)