Kalton's graphs.

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Last time: (Raynaud 1983) $B_{c_0} \underset{unif}{\not\hookrightarrow} \ell_2$. In particular, $c_0 \underset{unif}{\not\hookrightarrow} \ell_2$.

Theorem (Kalton 2007): Let X be a B-space so that $X^{(n)}$ is separable for all n. Then c_0 does not coarsely or uniformly embed into X.

Cor: c_0 does not coarsely nor unif^{ly} embed into a reflexive space.

Def. $M \subseteq \mathbb{N}$ infinite, $k \in \mathbb{N}$, $G_k(M) = \{\sigma \subset M : |\sigma| = k\}$
$G_0(M) = \{\emptyset\}$

$\sigma, \tau \in G_k(M)$, $\sigma = (m_1, \dots, m_k)$ in increasing order;
$\tau = (n_1, \dots, n_k)$

σ, τ define an edge if $m_1 \le n_1 \le m_2 \le n_2 \le \dots \le m_k \le n_k$
or $n_1 \le m_1 \le n_2 \le m_2 \le \dots \le n_k \le m_k$.

Then $G_k(M)$ is a connected graph. We put a metric d on $G_k(M)$:
- any edge has length 1
- if $\sigma, \tau \in G_k(M)$, $d(\sigma, \tau)$ is the geodesic distance.

Then $\operatorname{diam} G_k(M) = k$ (For example, $d(\sigma, \tau) = k$ if $m_k < n_1$ or $n_k < m_1$.

Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let X be a B-space and $f : G_k(\mathbb{N}) \xrightarrow{bdd} X$

Define $\partial f : G_{k-1}(\mathbb{N}) \longrightarrow X^{**}$

$(m_1, \dots, m_{k-1}) \longmapsto w^*\lim\limits_{m_k \in \mathcal{U}} f(m_1, \dots, m_k)$ by w^*-cpacity of bdd sets in X^{**}.

By iterating, we obtain $\partial^i f : G_{k-i}(\mathbb{N}) \longrightarrow X^{(2i)}$ for $1 \le i \le k$.

$\partial^k f$ is a constant map: $\partial^k f \in X^{(2k)}$

If X is reflexive, then $\partial^i f : G_k(\mathbb{N}) \longrightarrow X$ and $\partial^k f \in X$.

Lemma 1: Let $h : G_k(\mathbb{N}) \longrightarrow \mathbb{R}$ be a bounded f^n and $\varepsilon > 0$.
Then there is an infinite $M \subset \mathbb{N}$ such that $\forall \sigma \in G_k(M)$ $|h(\sigma) - \partial^k h| < \varepsilon$.

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Proof: by induction: Let $M = \{m_1, m_2, \dots\}$, $m_1 < m_2 < \dots$ so that

$\forall i \in \mathbb{N}$ m_i is picked so that • if $\sigma \subset \{m_1, \dots, m_i\}$, $1 \le |\sigma| \le k$, then $|\partial^{k-|\sigma|} h(\sigma) - \partial^k h| < \varepsilon$

It is possible: pick $\underline{m_1}$ so that $|\partial^{k-1} h(m_1) - \partial^k h| < \varepsilon$. (def of ∂^k)

~~if~~ $m_1, \dots, m_i$ picked, pick m_{i+1} so that ..., but we have to check the condition only for $\sigma \ni m_{i+1}$; in order to do so, fix $\sigma \subset \{m_1, \dots, m_i\}$, $1 \le |\sigma| \le k-1$, ~~and~~ then $|\partial^{k-|\sigma|-1} h(\sigma \cup \{m\}) - \partial^k h| < \varepsilon$

holds for m in an element $A_\sigma \in \mathcal{U}$, ~~choose $m_{i+1} \in A_\sigma$ with $m_{i+1} > m_i$~~

Then $A = \bigcap_{\substack{\sigma \subset \{m_1, \dots, m_i\} \\ 1 \le |\sigma| \le k-1}}$ satisfies $A \neq \emptyset$ and $A \in \mathcal{U}$.

Choose $m_{i+1} \in A$: then (1) holds for $\{m_1, \dots, m_{i+1}\}$.

Def: X B-space, $f : G_k(\mathbb{N}) \longrightarrow X$, $g : G_k(\mathbb{N}) \longrightarrow X^*$ bounded maps.

Define $f \otimes g : G_{2k}(\mathbb{N}) \longrightarrow \mathbb{R}$
$(m_1, \dots, m_{2k}) = \langle f(m_2, \dots, m_{2k}), g(m_1, \dots, m_{2k-1}) \rangle$

Lemma 2: $\partial^{2i}(f \otimes g) = \partial^i f \otimes \partial^i g$ for $1 \le i \le k$.

Lemma 3: $f : G_k(\mathbb{N}) \longrightarrow X$ bounded maps and $\varepsilon > 0$: there is an infinite $M \subset \mathbb{N}$ so that $\|f(\sigma)\| < \|\partial^k f\| + \omega_f(1) + \varepsilon$ for $\sigma \in G_k(M)$

$\omega_f(1) = \sup\{\|f(\sigma) - f(\tau)\| : d(\sigma, \tau) \le 1\}$

Proof: Define $g : G_k(\mathbb{N}) \to X^*$ so that $\|g(\sigma)\| = 1$ and $\langle f(\sigma), g(\sigma) \rangle = \|f(\sigma)\|$

By Lemma 2, we have $|\partial^{2k}(f \otimes g)| = |\partial^k(f) \otimes \partial^k(g)| = |\langle \partial^k f, \partial^k g \rangle| \le \|\partial^k f\| \, \|\partial^k g\| \le \|\partial^k f\|$, $\|\partial^k g\| \le 1$

By Lemma 1, there is an infinite $A \subset \mathbb{N}$ so that ~~$f \otimes g$~~
$|f \otimes g(\sigma) - \partial^{2k}(f \otimes g)| \le \varepsilon$ for $\sigma \in G_k(A)$

Let $A = \{m_1 < n_1 < m_2 < n_2 < \dots\}$ and $M = \{m_1, m_2, \dots\}$

then, if $\sigma \in G_k(M)$, $\sigma = (m_{u_1}, \dots, m_{u_k})$, then $\|f(\sigma)\| = \langle f(\sigma), g(\sigma) \rangle$
$= \langle f(\sigma) - f(n_{u_1}, \dots, n_{u_k}), g(\sigma) \rangle + \langle f(n_{u_1}, \dots, n_{u_k}), g(\sigma) \rangle = f \otimes g(m_{u_1}, n_{u_1}, \dots, m_{u_k}, n_{u_k})$

$\le \omega_f(1)$ since σ is at distance 1 from $(n_{u_1}, \dots, n_{u_k})$.

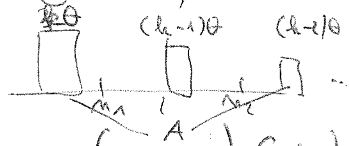
Lemma 4. Let $k\in\mathbb{N}$, $\varepsilon>0$, X B-space so that $X^{(2k)}$ is separable. If $f_i : G_k(\mathbb{N}) \longrightarrow X$ are bounded maps and $i\in I$, I uncountable, then there are $i\neq j\in I$ and an infinite $M\subset\mathbb{N}$ so that

$$\|f_i(\sigma)-f_j(\sigma)\| < \omega_{f_i}(1)+\omega_{f_j}(1)+\varepsilon \text{ for all } \sigma\in G_k(M).$$

Proof: Since $X^{(2k)}$ is separable, there are $i\neq j$ so that $\|\partial^k f_i-\partial^k f_j\|<\frac{\varepsilon}{2}$.

By Lemma 3, $\|f_i(\sigma)-f_j(\sigma)\| = \|(f_i-f_j)(\sigma)\| < \|\partial^k(f_i-f_j)\| + \omega_{f_i-f_j}(1)+\frac{\varepsilon}{2}$

Proof of the Theorem: suppose ~~on the contrary~~ that there is $h: c_0 \longrightarrow X$ bounded on bounded sets. Let $k\in\mathbb{N}$, $0<\theta<\infty$, $A\subset\mathbb{N}$ infinite. Define $s_n(A)=\sum_{i\le n,\, i\in A} e_i \in c_0$.



Let $f_{\theta,A,k}(\sigma)=\theta\sum_{j=1}^{k} s_{n_j}(A)\in c_0$ for $\sigma=(n_1,\dots,n_k)\in G_k(\mathbb{N})$

then $h\circ f_{\theta,A,k} : G_k(\mathbb{N})\longrightarrow X$.

Fix θ, k; let $\varepsilon>0$. By Lemma 4, (there are uncountably many infinite subsets of $\mathbb{N}$), we find $A\neq B$ and an infinite $M\subset\mathbb{N}$ so that

$\forall \sigma\in G_k(M)$ $\|h(f_{\theta,A,k}(\sigma))-h(f_{\theta,B,k}(\sigma))\| < \omega_{h\circ f_{\theta,A,k}}(1)+\omega_{h\circ f_{\theta,B,k}}(1)+\varepsilon$

$< 2\,\omega_h(\theta)+\varepsilon.$

as f_k varies by θ from one vertex to the next $\longrightarrow$ because $\omega_{f_{\theta,A,k}}(1)\le\theta$ and the same for B.

On the other hand, since $A\neq B$, there is $\sigma\in G_k(M)$ so that $\|f_{\theta,A,k}(\sigma)-f_{\theta,B,k}(\sigma)\| = k\theta$.

Therefore h cannot be a coarse embedding (choose θ, $\varepsilon=1$).

h cannot either be a uniform embedding:

If (M_1,d_1), (M_2,d_2) are metric spaces, $g: M_1\longrightarrow M_2$,

$\varphi_g(t) = \inf\{d_2(g(x),g(y)) : d_1(x,y)\ge t\}$, $t>0$,

then $\varphi_g(d_1(x,y)) \le d_2(g(x),g(y)) \le \omega_g(d_1(x,y))$ for $x,y\in M_1$.

g is a uniform embedding if $\omega_g(t)\xrightarrow{t\to 0} 0$ and $\varphi_g(t)>0$ for all t.

We have $\varphi_h(k\theta)\le 2\,\omega_h(\theta)$; if $\omega_h(\theta)\xrightarrow{\theta\to 0} 0$, then $\varphi_h(t)=0$ for all t.