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Theorem (Albiac '08): If $1 < p < q < \infty$, ℓ_p strongly embeds into ℓ_q [a strong embedding is an embedding that is coarse and uniform]

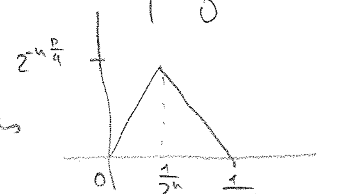
Proof: We will find an $f: \ell_p \to \ell_q$ and $A, B > 0$ s.t.

$$A \|x-y\|_p^{p/q} \le \| f(x) - f(y) \|_q \le B \|x-y\|_p$$

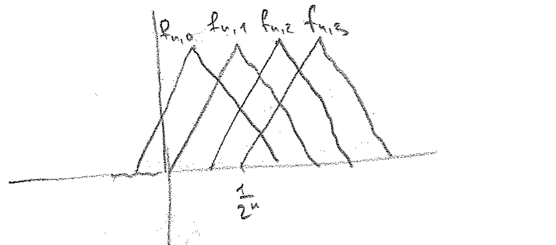
- We will work with $\ell_r(\mathbb{Z})$ instead of $\ell_r(\mathbb{N})$
- We will find $T: \ell_p \to \ell_q(\ell_q(\ell_q)) \underset{\text{isom}}{\cong} \ell_q$

<u>1st step</u>: to find $f: \mathbb{R} \to \ell_q(\ell_q)$, $A, B > 0$, s.t. $A^q |s-t|^p \le \|f(s) - f(t)\|_q^q \le B |s-t|^p$.

Let $n \in \mathbb{Z}$ and set
$$f_n(t) = \begin{cases} 2^{n(1-\frac{p}{q})} t & \text{if } t \in [0, \frac{1}{2^n}] \\ -2^{n(1-\frac{p}{q})}(t - \frac{1}{2^{n-1}}) & \text{if } t \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}] \\ 0 & \text{otherwise.} \end{cases}$$

Then the graph of f is 

and $\operatorname{Lip} f_n = 2^{n(1-\frac{p}{q})}$

Let $k \in \mathbb{Z}$ and define $f_{n,k} = f_n(t - \frac{k-1}{2^{n+1}})$, so that 

Define
$$f: \mathbb{R} \to \ell_q(\ell_q)$$
$$t \longmapsto (f_{n,k}(t))_{n,k=-\infty}^{+\infty}$$

<u>Upper estimate</u>: let $s, t \in \mathbb{R}$, $N \in \mathbb{Z}$, $\frac{1}{2^{N+1}} < |s-t| \le \frac{1}{2^N}$.

For $n \le N$, $k \in \mathbb{Z}$, we have
$$|f_{n,k}(s) - f_{n,k}(t)|^q \le 2^{n(q-p)} |s-t|^q = 2^{n(q-p)} |s-t|^p |s-t|^{q-p}$$
$$\le 2^{n(q-p)} |s-t|^p \frac{1}{2^{N(q-p)}} = 2^{(n-N)(q-p)} |s-t|^p$$
because $\operatorname{Lip} f_{n,k} = 2^{n(1-\frac{p}{q})}$

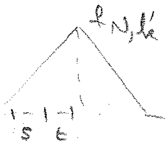
Suppose $n > N$ and $k \in \mathbb{Z}$. $|f_{n,k}(s) - f_{n,k}(t)|^q \le 2^{-np} = 2^p 2^{(N-n)} 2^{-(N+1)p} < 2^p 2^{(N-n)p} |s-t|^p$
$0 \le |f_{n,k}| \le 2^{-n\frac{p}{q}}$

Note that the estimates don't depend on k. For every $n \in \mathbb{Z}$, there are $k_1^n, \dots, k_8^n$ such that neither s nor t do belong to $\operatorname{supp} f_{n,k}$ for other values of k.

$$\|f(s) - f(t)\|_q^q = \sum_{n,k=-\infty}^{\infty} |f_{n,k}(s) - f_{n,k}(t)|^q = \Big(\sum_{n \le N} + \sum_{n > N}\Big) \sum_{k \in \mathbb{Z}} |f_{n,k}(s) - f_{n,k}(t)|^q$$
$$= \sum_{n \le N} \sum_{i=1}^{8} |f_{n,k_i^n}(s) - f_{n,k_i^n}(t)|^q + \sum_{n > N} \sum_{i=1}^{8} |f_{n,k_i^n}(s) - f_{n,k_i^n}(t)|^q$$
$$\le 8 \sum_{n \le N} 2^{(n-N)(q-p)} |s-t|^p + 8 \sum_{n > N} 2^p 2^{(N-n)p} |s-t|^p$$
$$= 8 \Big(\underbrace{\sum_{n \le N} 2^{(n-N)(q-p)} + \sum_{n > N} 2^{(N-n)p}}_{\frac{2^{q-p}}{2^{q-p}-1} + \frac{2^p}{2^p - 1} < \infty} \Big) |s-t|^p.$$

<u>Lower estimate</u>: let $N \in \mathbb{Z}$ st $\frac{1}{2^{N+2}} < |s-t| \leq \frac{1}{2^{N+1}}$ and wlog $s<t$.

Let $k' \in \mathbb{Z}$ maximal st $s \in \operatorname{supp} f_{N,k'}$: then



$s \in \left[\frac{k'-1}{2^{N+1}}, \frac{k'-1}{2^{N+1}} + \frac{1}{2^{N+1}}\right[$

$t \in \left[\frac{k'-1}{2^{N+1}}, \frac{k'-1}{2^{N+1}} + \frac{1}{2^{N}}\right[$

Therefore $|f_{N,k'}(s) - f_{N,k'}(t)|^q = 2^{N(q-p)}|s-t|^q$

$$= 2^{N(q-p)}|s-t|^p|s-t|^{q-p} \geq 2^{N(q-p)}|s-t|^p \frac{1}{2^{(N+2)(q-p)}}$$

and $\|f(s)-f(t)\|_q^q = \sum_{n,k=-\infty}^{+\infty} |f_{n,k}(s) - f_{n,k}(t)|^q$ $\qquad = \frac{1}{2^{2(q-p)}}|s-t|^p$

$$\geq |f_{N,k'}(s) - f_{N,k'}(t)|^q \geq \frac{1}{2^{2(q-p)}}|s-t|^p$$

So that $\quad A^q|s-t|^p \leq \|f(s)-f(t)\|_q^q \leq B^q|s-t|^p$.

<u>2^{nd} step</u>: $T : \ell_p \to \ell_q(\ell_q(\ell_q))$ $\quad$ If $x=(x_i), y=(y_i) \in \ell_p$, then
$(x_i) \mapsto (f(x_i))$. $\quad \|T(x)-T(y)\|_q^q = \sum_{i\in\mathbb{Z}} \|f(x_i)-f(y_i)\|_q^q \begin{cases} \leq B^q \underbrace{\sum |x_i-y_i|^p}_{=\|x-y\|_p^p} \\ \geq A^q \sum |x_i-y_i|^p \end{cases}$

Thus $\quad A\|x-y\|_p^{\frac{p}{q}} \leq \|T(x)-T(y)\|_q \leq B\|x-y\|_p^{\frac{p}{q}}$

<u>Theorem</u>: $\ell_p \underset{c}{\hookrightarrow} \ell_q \iff \max(2,p) \leq \max(2,q)$

<u>Proof</u>: If $1 \leq p \leq 2$, $1 \leq q < \infty$, then $\ell_p \underset{c}{\hookrightarrow} \ell_2 \underset{c}{\hookrightarrow} \ell_q$
(just proved, Nowak) (Nowak)

- $2 < p \leq q < \infty$ implies $\ell_p \underset{c}{\hookrightarrow} \ell_q$ (just proved)
- $2 < p$, $p > q$ implies $\ell_p \underset{c}{\not\hookrightarrow} \ell_q$ (Mendel, Naor) [this uses metric cotype]
[If $X \underset{c}{\hookrightarrow} Y$ and Y has nontrivial type, then $\operatorname{cotype}(X) \leq \operatorname{cotype}(Y)$]
cotype $\ell_p = \max(2,p)$