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Nonlinear quotients

Seminal paper: Bate-Johnson-Lindenstrauss-Preiss-Schechtman 1999

Def: • (M,d) (N,δ) metric space. A f^n $f: M \to N$ is <u>co-uniformly continuous</u>

if $\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in M \quad f(B(x,\varepsilon)) \supset B(f(x),\delta)$

If δ may be chosen a linear f^n of ε, f is <u>co-Lipschitz</u>

• N is a uniform (vs. Lipschitz) quotient of M if there is an onto f^n $f: M \to N$ that uniformly and co-uniformly continuous (vs Lipschitz and co-Lipschitz). We write $M \overset{u}{\twoheadrightarrow} N$ (vs $M \overset{L}{\twoheadrightarrow} N$).

These def^{ns} are motivated by the open mapping theorem.

The General Question is: X, Y B-space (1) Do $X \overset{u\,or\,L}{\twoheadrightarrow} Y$ imply $X \twoheadrightarrow Y$

(2) Which properties are preserved by nonlinear quotients

Remarks: 1) $f: X \underset{L}{\hookrightarrow} Y \not\Rightarrow X \subseteq Y$. Assume f is G-diff^{able} at $x_0 \in X$

Then $Df(x_0)$ is a linear embedding from X into Y.

and there is a Th: X separable and Y RNP $\Rightarrow$ existence of points of G-diff^{ability}

2) $1 \le p < \infty$. There is $f: L_p \overset{L}{\twoheadrightarrow} \ell_p$ G-diff at 0, but $Df(0) = 0$.

3) If $f: X \twoheadrightarrow Y$ and f Fréchet-diff^{able} at x_0, then $Df(x_0)$ is onto

However, points of F-diff^{ability} are very rare!

$\to$ th (Preiss 90): If X is Asplund (every sep subspace has sep dual) and $f: X \to \mathbb{R}$ Lipschitz, then $\{x \in X : f$ F-diff at $x\}$ is dense

And there is an open question: X Asplund, $f: X \overset{Lip}{\twoheadrightarrow} \mathbb{R}^2 \not\Rightarrow$ existence of points of F-dif.

We therefore study weaker notions: ① $f: X \to Y$, $\varepsilon > 0$: f is ε-F-d at x_0 if

$\exists T \in B(X,Y)\ \exists r > 0\ \|h\| \le r \Rightarrow \|f(x_0+h) - f(x_0) - Th\| \le \varepsilon \|h\|$

N.B: there are Theorems about the existence of points of ε-F-d for arbitrarily small ε's.

Lemma: $f: X \xrightarrow{L} Y$, $\mathrm{Lip} f = 1$, $\forall x \forall r$ $f(B(x,r)) \supset B(f(x), cr)$ $(c<1)$

Assume f is ε-F-d at x_0. Then $\exists T \in B(X,Y)$ $T: X \twoheadrightarrow Y$

Proof: We may suppose $x_0 = 0$ and $f(x_0) = 0$

If ① $\forall \rho > 0$ $f(B(0,\rho)) \supset B(0, c\rho)$

② $\exists r > 0$ $\|f(x) - Tx\| \leq \varepsilon \|x\|$ for $x \in B(0,r)$

Then $T(B(0, \cancel{(1+\varepsilon)} r)) \supset B(0, (c-\varepsilon) r)$

Let $\|y\| \leq (c-\varepsilon) r$ and x_1 with $\|x_1\| \leq \frac{c-\varepsilon}{c} r$ s.t. $y = f(x_1)$

$\|f(x_1) - Tx_1\| \leq \varepsilon \|x_1\| \leq \varepsilon \frac{c-\varepsilon}{c} r$.

Then $f(x_1) = T(x_1) + y_2$, $\|y_2\| \leq \varepsilon \frac{c-\varepsilon}{c} r$.

Let x_2 s.t. $y_2 = f(x_2)$ and $\|x_2\| \leq \frac{\varepsilon}{c} \frac{c-\varepsilon}{c} r$.

then $\|f(x_2) - Tx_2\| \leq \varepsilon \|x_2\| \leq \varepsilon \frac{\varepsilon}{c} \frac{c-\varepsilon}{c} r$

$y = T(x_1 + x_2) + y_3$, $\|y_3\| \leq \dots$ in the end, one obtains

$y = T\left(\sum_1^\infty x_i\right)$, $\|x_i\| \leq \left(\frac{\varepsilon}{c}\right)^{i-1} \frac{c-\varepsilon}{c} r$ $\|x\| \leq (c-\varepsilon) r$.

$\|x\| \leq \frac{1}{1-\frac{\varepsilon}{c}} \frac{c-\varepsilon}{c} r = r$.

Claim: Assume only (1) and instead of (2) $\exists r > 0$ $\|f(x) - T(x)\| \leq \varepsilon r$ for $x \in B(0,r)$.
Then the same conclusion holds. (Th 11.14 in Benyamini–Lindenstrauss)
Gilles doesn't know how to work this out.

Def: • X, Y B-spaces. $\mathrm{Lip}(X,Y)$ has the Approximation by Affine Functions Property (AAP) if $\forall B$ ball in X $\forall f: B \xrightarrow{\mathrm{Lip}} Y$ $\forall \varepsilon > 0$ $\exists B_1 \subset B$ ball $\exists T: B_1 \xrightarrow{\text{affine}} Y$ $\forall x \in B_1$ $\|f(x) - Tx\| \leq \varepsilon r$, where r is the radius of B_1

• $\mathrm{Lip}(X,Y)$ has UAAP if moreover $\exists c(\varepsilon) > 0$ $r \geq c(\varepsilon) R$, where R is the radius of B.

Main results of BJLPS: Th 1: $\mathrm{Lip}(X,Y)$ has UAAP iff $\begin{cases} X \text{ or } Y \text{ is f.d, and} \\ X \text{ and } Y \text{ are superreflexive} \end{cases}$

Li–Naor obtained a quantitative version

Lindenstrauss–Preiss '96: X sr, $\dim Y < \infty$, $f: X \xrightarrow{\mathrm{Lip}} Y$. Then $\forall \varepsilon > 0$ f admits points of ε-F-d

Th 2: X sr (i.e., X admits an equivalent uc norm).

then $X \xrightarrow{u} Y \Rightarrow Y \simeq \frac{X_u}{Z}$

Cor: If $1 < p < \infty$ and $L_p \xrightarrow{u} Y$, then $L_p \twoheadrightarrow Y$

(here the authors need: $f: X$ s.r. $\xrightarrow{\mathrm{Lip}} E$ f.d are AAP)

Th 3. Lip$(X, \mathbb{R})$ has AAP iff X is Asplund

Th 4: Lip$(\mathbb{R}, X)$ has AAP iff there is no bounded generalised δ-separated martingale in X [which is implied by, and does not imply, RNP]

Th 5: $X \xrightarrow{L} Y$, X Asplund, implies Y Asplund

Proof of Th 5: ① Assume X separable Asplund (i.e. X^* sep). Then Y is sep.

We need to show Y^* sep: If not, $\exists\, (y^*_\gamma)_{\gamma\in\Gamma}$, Γ uncountable, $\|y^*_\gamma\| = 1$, $\|y^*_\gamma - y^*_{\gamma'}\| \geq \frac{1}{2}$ for $\gamma \neq \gamma'$

Then $f_\gamma : X \xrightarrow{f \text{ Lip}} Y \xrightarrow{y^*_\gamma} \mathbb{R}$. As X is Asplund, there are $x_\gamma \in B_X$ where f_γ is F-diff (Preiss' thm). Let $\varepsilon > 0$.

$\forall \gamma\ \exists \delta > 0$ $\|h\| \leq \delta_\gamma$ $\|f(x_\gamma + h) - f(x_\gamma) - Df(x_\gamma)h\| \leq \varepsilon\|h\|$

[here one could use ε-F-d and use Lindenstrauss' Preiss thm]

By passing to an uncountable subset of Γ, we may assume $\delta = \inf_{\gamma\in\Gamma} \delta_\gamma > 0$

~~Then~~ and $\forall \gamma \neq \gamma'$ $\|x_\gamma - x_{\gamma'}\| < \varepsilon\delta$ (X is separable!)

$\|Df_\gamma(x_\gamma) - Df_{\gamma'}(x_{\gamma'})\| < \varepsilon$ (X^* is separable)

$|f_\gamma(x_\gamma) - f_{\gamma'}(x_{\gamma'})| < \varepsilon\delta$ ($\mathbb{R}$ is sep.)

Let $\|h\| \leq \delta$
$$\begin{aligned}|f_\gamma(x_\gamma + h) - f_{\gamma'}(x_\gamma + h)| &\leq |f_\gamma(x_\gamma+h) - f_\gamma(x_\gamma) + Df_\gamma(x_\gamma)h| + |f_\gamma(x_\gamma) - f_{\gamma'}(x_{\gamma'})| \\ &\quad + |Df_\gamma(x_\gamma)h - Df_{\gamma'}(x_{\gamma'})h| + |Df_{\gamma'}(x_{\gamma'})h - f_{\gamma'}(x_{\gamma'}) \\ &\quad - f_{\gamma'}(x_{\gamma'}+h)| + |f_{\gamma'}(x_{\gamma'}+h) - f_{\gamma'}(x_\gamma + h)| \\ &\leq \varepsilon\delta + \varepsilon\delta + \varepsilon\delta + \varepsilon\delta + \varepsilon\delta = 5\varepsilon\delta < \frac{c\delta}{4} \text{ for } \varepsilon \text{ small enough.}\end{aligned}$$

But $f: X \to Y$ is a Lipschitz quotient: $\exists c > 0\ \forall x\ \forall r\ f(B(x,r)) \supset B(f(x), cr)$

$$\sup_{\|h\|<\delta} |f_\gamma(x_\gamma + h) - f_{\gamma'}(x_{\gamma'} + h)| = \sup_{\|h\|\leq\delta} |\langle y^*_\gamma - y^*_{\gamma'}, f(x_\gamma + h)\rangle|$$
$$\geq \sup_{\|y\|\leq c\delta} |\langle y^*_\gamma - y^*_{\gamma'}, f(x_\gamma) + y\rangle| \geq \|y^*_\gamma - y^*_{\gamma'}\| c\delta \geq \frac{c\delta}{2}$$

$\to$ pick y norming $y^*_\gamma - y^*_{\gamma'}$ s.t. $\operatorname{sgn}\langle y^*_\gamma - y^*_{\gamma'}, y\rangle = \operatorname{sgn}\langle y^*_\gamma - y^*_{\gamma'}, f(x_\gamma)\rangle$

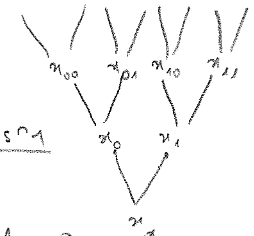
This is a contradiction!

② General case: by separable exhaustion: $X \underset{L}{\sim} Y$, X Asplund, pick $Y_0 \subset Y$, Y_0 separable. We want to show Y_0^* sep. $X_0 = \overline{\operatorname{span}}\, f^{-1}(Y_0)$ is sep. Let $Y_1 = \overline{\operatorname{span}}(f(X_0))$ and $X_1 = \overline{\operatorname{span}}(f^{-1}(Y_1))$. Then let $\tilde{X} = \overline{\bigcup X_n}$ and $\tilde{Y} = \overline{\bigcup Y_n}$. These are separable. $f|_{\tilde{X}} : \tilde{X} \to \tilde{Y}$ is a bijection; $\tilde{X}$ sep Asplund $\Rightarrow$ $\tilde{Y}$ sep Asplund $\Rightarrow$ Y_0 Asplund.

Dutrieux: X^* sep., $X \xrightarrow{L} Y$. Then $\exists\, \psi : (0,\omega_1) \to (0,\omega_1)$ $Sz\,Y \le \psi(Sz\,X)$.
and the open question is: $\psi = Id$? This would have appl^{ns} to $\cong$ for $C(K)$ spaces

<u>Th</u>: X Bspace then $\Leftrightarrow$
(i) $Lip(X,\mathbb{R})$ has AAP
(ii) X is Asplund
(iii) $\forall\, \mathcal{U}$ open in X $\forall f : \mathcal{U} \xrightarrow{Lip} \mathbb{R}$ f has a pt of F-d.

Proof: (iii) $\Rightarrow$ (i) is clear. Recall the very definition of Asplund space:
X is <u>Asplund</u> if $\forall\, \mathcal{U}$ open convex $\forall f : \mathcal{U} \xrightarrow[\text{continuous}]{\text{convex}} \mathbb{R}$ $\{x$ pt of F-diff$\}$ is G_δ-dense in $\mathcal{U}$.
(iii) $\Rightarrow$ (ii) $f : \mathcal{U} \xrightarrow[\text{cont.}]{\text{convex}} \mathbb{R}$ is locally Lipschitz; $\{x : f$ F-diff at $x\}$ is G_δ
(ii) $\Rightarrow$ (iii) Preiss
(i) $\Rightarrow$ (ii) Assume X not Asplund. Then there is an equivalent norm on X which is very bad: it is rough: $\|\ \| : X \to \mathbb{R}$ is not approximable by affine functions.



<u>X-valued martingale</u>: $(x_s)_{s \in \{0,1\}^{<\omega}}$. For every s, $x_s = \dfrac{x_{s\frown 0} + x_{s\frown 1}}{2}$

<u>Generalised dyadic martingale</u>: $(\mathcal{F}_n) \uparrow$ σ-fields of $[0,1)$ with $\mathcal{F}_0 = \{\emptyset, [0,1)\}$ with $\mathcal{F}_n$ generated by 2^n atoms (intervals of the form $[a,b)$). The atoms of $\mathcal{F}_{n+1}$ are obtained by splitting the atoms of $\mathcal{F}_n$ into 2 subintervals (of $\neq$ length)
An X-valued martingale (M_n) wrt $\mathcal{F}_n$ is called is <u>generalised dyadic</u>
It is <u>bounded</u> if $\sup \|M_n\|_\infty < \infty$ and <u>δ-separated</u> if $\|M_{n+1} - M_n\| > \delta$
We may see that as $(x_s)_{s \in \{0,1\}^{<\omega}}$. $x_s = \lambda_s x_{s\frown 0} + (1-\lambda_s) x_{s\frown 1}$; $\lambda_s \in (0,1)$.
and $\sup_s \|x_s\| < \infty$.
And δ-separation is expressed by $\|x_s - x_{s\frown 0}\| \geq \delta$ and $\|x_s - x_{s\frown 1}\| \geq \delta$
We have $\lambda_s (x_{s\frown 0} - x_{s\frown 1}) = x_s - x_{s\frown 1}$.
Assume $\|x_s\| \leq 1$ δ-separated. Then $\lambda_s, 1-\lambda_s \geq \frac{\delta}{2}$ (1)
We are ready to prove Th 4!

Proof of Th 4. (⇒) Assume there is such a martingale (M_n). Let $f_n(t) = \int_0^t M_n(s)\,ds$.

Then $\mathrm{Lip}\, f_n \le 1$. If t is an end-point of an atom of $\mathcal{F}_n$, $(f_m(t))_{m \ge n}$ is constant.

Such t's are dense because of (1). Suppose that $f_n \xrightarrow{u} f$, $\mathrm{Lip}\, f \le 1$.

Let $[a,c[$ atom of $\mathcal{F}_n$ and $[a,b[$, $[b,c[$ atoms of $\mathcal{F}_{n+1}$

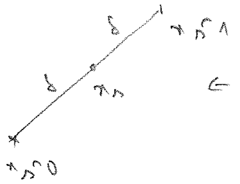
$M_{n+1} = x_{s\frown 0}$ $\quad M_{n+1} = x_{s\frown 1}$

this condition is just a perturbation of the general case

$\lambda_s = \frac{b-a}{c-a} \qquad 1-\lambda_s = \frac{c-b}{c-a}$

Assume that L is affine and $L(a) = f(a)$ and $\|L(x) - f(x)\| \le \frac{\delta^2}{4}(c-a)$

then
$$\|x_{s\frown 0} - x_s\| = \left\| \frac{f(b)-f(a)}{b-a} - \frac{f(c)-f(a)}{c-a} \right\| \le \left\| \frac{L(b)-L(a)}{b-a} - \frac{L(c)-L(a)}{c-a} \right\| + \frac{\delta^2}{4}\left[\frac{c-a}{b-a} + 1\right]$$



← We have $2 \ge \|x_{s\frown 0} - x_{s\frown 1}\| \ge 2\delta$ and $\delta \le 1$

$$= \frac{\delta^2}{4}\left(\frac{1}{\lambda_s} + 1\right) \le \frac{\delta^2}{4}\left(\frac{2}{\delta} + 1\right) = \frac{\delta}{2} + \frac{\delta^2}{4} \le \frac{3}{4}\delta.$$

(⇐) Assume $\mathrm{Lip}(\mathbb{R}, X)$ fails AAP: $\exists f : [0,1[\to X$, $\mathrm{Lip}\, f \le 1$, $\exists \delta > 0$

(*) $\forall [a,c[\subset [0,1[\ \exists b \in [a,c[\quad \Big\| f(b) - \underbrace{\left(\frac{c-b}{c-a} f(a) + \frac{b-a}{c-a} f(c)\right)}_{L(b)} \Big\| > \delta(c-a)$

$L(a) = f(a) \quad L(c) = f(c)$

Let $M_0 = f(1) - f(0)$ and assuming M_n defined, we choose for every atom $[t_n^i, t_n^{i+1}[$ of $\mathcal{F}_n$, $M_{n+1} = \dfrac{f(t_n^{i+1}) - f(t_n^i)}{t_n^{i+1} - t_n^i}$ on $[t_n^i, t_n^{i+1}[$

For each atom $[a,c[$ of $\mathcal{F}_n$ pick $b \in [a,c]$ given by (*)

If $[a,b[$, $[b,c[$ atoms of $\mathcal{F}_{n+1}$, $M_{n+1} = \dfrac{f(b)-f(a)}{b-a}$ on $[a,b[$

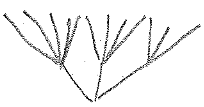
$= \dfrac{f(c)-f(b)}{c-b}$ on $[b,c[$

We have $\|M_n\| \le 1$ because $\mathrm{Lip}\, f \le 1$ and $\left\| \frac{f(b)-f(a)}{b-a} - \frac{f(c)-f(a)}{c-a} \right\| \ge \frac{\delta(c-a)}{b-a} \ge \delta$

$\left\| \frac{f(c)-f(b)}{c-b} - \frac{f(c)-f(a)}{c-a} \right\| \ge \frac{\delta(c-a)}{c-b} \ge \delta$

Remark on AAP ≠ RNP: X fails RNP iff $\exists$ bdd δ-separated martingale in X

iff $\exists$ bdd δ-separated bush in X



$x_s = \sum_{i=1}^{N_s} \lambda_s^i x_{s\frown i}, \ \lambda_s^i \ge 0, \ \sum_i \lambda_s^i = 1, \ \|x_{s\frown i} - x_s\| \ge \delta$

There is an X failing RNP (i.e., a bush) but not containing any generalized dyadic (→ Bourgain–Rosenthal)