Gilles : quotients non linéaires 7/3/2012

(1) Reminder:

X, Y B-space

Lemma 1 $f : X \to Y$, lip $f = 1$, $f(0)=0$. Assume $\exists\, 0<c<1$ $\forall \rho>0$ $f(\rho B_X) \supset c\rho B_X$ ~~$\forall x\in X$ $f(B(x,r)) \supset B(f(x),cr)$~~

and that (2) if $\varepsilon \in]0,c[$, $\exists r>0$ $\exists T\in B(X,Y)$ $\forall x \in rB_X$ $\|Tx - f(x)\| \le \varepsilon \|x\|$.

Then $T(rB_X) \supset (c-\varepsilon) r B_Y$.

The Dubious lemma of last time consists in replacing (2) by

(2') $\exists \varepsilon < c$ $\exists T$ $\exists r>0$ $\forall x\in rB_X$ $\|Tx - f(x)\| \le \varepsilon r$.

Then $T(rB_X) \supset (c-\varepsilon) rB_Y$ (same conclusion as before)

A weaker version of this holds true. (A. Procházka 2012)

(2'') $\exists \varepsilon < c^4$ $\exists T$ $\exists r>0$ $\forall x \in rB_X$ $\|Tx - f(x)\| \le \varepsilon r$

Then $T(rB_X) \subset c(1-c)\, rB_Y$.

Proof: $\forall \rho>0$ $\rho B_Y \subset f\left(\frac{\rho}{c} B_X\right)$ $\quad f(c\rho B_X) \subset c\rho B_Y$

so that $\rho B_Y \setminus c\rho B_Y \subset f\left(\frac{\rho}{c}B_X\right) \setminus f(c\rho B_X)$

$\subset f\left(\frac{\rho}{c} B_X \setminus c\rho B_X\right)$.

Set $r_n = c^n r$, $n \ge 0$: $r_1 B_Y \setminus r_2 B_Y \subset f(r_0 B_X \setminus r_2 B_X)$

If $x \in D_n = r_n B_X \setminus r_{n+2} B_X$, define $f_n(x) = c^n f\left(\frac{x}{c^n}\right)$

Then $x \in D_n \iff \frac{x}{c^n} \in D_0$ and $r_{n+1} B_Y \setminus r_{n+2} B_Y \subset f_n(r_n B_X \setminus r_{n+2} B_X)$

* Let $y_0 \in r_1 B_Y = crB_Y$. There is $n_1 \ge 0$ s.t. $r_{n_1+2} < \|y_0\| \le r_{n_1+1} \Rightarrow \exists x_1 \in D_{n_1}$ $y_0 = f_{n_1}(x_1)$

Note that if $x_0 \in D_0$, $\|f(x) - T(x)\| \le \varepsilon r \le \frac{\varepsilon \|x\|}{c^2}$ $\quad (\|x\| \ge c^2 r)$ — This is the key point; note that $\varepsilon < c^4$!

$\|f(x) - T(x)\| \le c^2 \|x\|$

$x \in D_n$ $\|f_n(x) - T(x)\| \le c^2 \|x\|$.

and $\|f_{n_1}(x_1) - T(x_1)\| \le c^2 \|x_1\| \le r_2$

Let $y_1 = y_0 - T(x_1) = f_{n_1}(x_1) - T(x_1)$: $\|y_1\| \le r_2$

Then there is $n_2 \ge 1$ s.t. $y_1 \in r_{n_2+1} B_Y \setminus r_{n_2+2} B_Y$ and $y_1 = f_{n_2}(x_2)$ and $x_2 \in r_{n_2} B_X \setminus r_{n_2+2} B_X$

We have $\|x_2\| \le r_{n_2} \le r_1$. $\|\underbrace{f_{n_2}(x_2) - T(x_2)}_{y_2}\| \le c^2 \|x_2\| \le r_3$.

By induction, we get $y_0 = T(x_1 + \dots + x_n) + y_n$ $\quad \|x_n\| \le r_{n-1}$ $\quad \|y_n\| \le r_{n+1}$

Let $x = \sum x_n$ and $y = Tx$: Then $\|x\| \le r \sum_0^\infty c^n = \frac{r}{1-c}$

$$c\,r\,B_Y \subset T\left(\frac{r}{1-c} B_X\right)$$
$$c(1-c)B_Y \subset T(B_X)$$

② Elementary facts on uniform quotients

Lemma 3 : let $f : X \to Y$ uniformly continuous. Then $\forall d>0$ $\|x-x'\| > d \Rightarrow \|f(x)-f(x')\| \le \frac{2\omega_f(d)}{d}\|x-x'\|$

Proof: x ——— x' split $[x,x']$ into $\left\lfloor \frac{\|x-x'\|}{d} \right\rfloor + 1$ segments of length $\le d$

then $\|f(x)-f(x')\| \le \left(\left\lfloor \frac{\|x-x'\|}{d} \right\rfloor + 1\right)\omega_f(d)$

$$\le \frac{\omega_f(d)}{d}\|x-x'\| + d\,\frac{\omega_f(d)}{d} \le \frac{2\,\omega_f(d)}{d}\|x-x'\| \qquad (\|x-x'\|>d)$$

Lemma 4 $f : X \to Y$ co-uniformly continuous.

then $\forall d>0\ \exists c>0$ $r \ge d \Rightarrow \forall x\in X$ $f(B(x,r)) \supset B(f(x), cr)$

Proof: Let $\delta = \delta(d)$ given by the def of co-uniform continuity

$(\forall x\in X,\ f(B(x,d)) \supset B(f(x),\delta))$

Let $r \ge d$. Then $r = nd + d'$, $n \ge 1$, $d' < d$.

Fix $x\in X$, $y \in X$, $\|y - f(x)\| \le n\delta$, split $\underset{f(x)}{y_0}\ y_1 \cdots \underset{y}{y_n}$ $\|y_{i+1}-y_i\| \le \delta$

An easy induction yields x_n s.t. $y = f(x_n)$ $\|x_n - x\| \le nd' \le r$

$$B(f(x), n\delta) \subset f(B(x,r))$$

$\underset{\text{so that } \frac{r}{2d} \le n}{r \le 2nd}$ $\quad B\left(f(x), \frac{\delta}{2d} r\right) \subset f(B(x,r))$ ◻

Ultrapowers : $\mathcal{U}$ free ultrafilter on I and $\ell_\infty^I(X) = \{x = (x_i)_{i\in I},\ (x_i) \text{ bounded in } X\}$

$\mathcal{N} = \{x \in \ell_\infty^I(X) : \lim_{\mathcal{U}} \|x_i\|_X = 0\}$.

Then $X_{\mathcal{U}} = \ell_\infty^I(X)/\mathcal{N}$ equipped with $\|\tilde{x}\| = \lim_{\mathcal{U}} \|x_i\|$ if $x = (x_i)_{i\in I} \in \tilde{x}$.

Prop 5 : If $X \xrightarrow{u} Y$, then $X_{\mathcal{U}} \xrightarrow{L} Y_{\mathcal{U}}$

Proof: • $\exists f : X \to Y$ $\exists C \ge 1$ $\forall r \ge 1$ $\forall x\in X$ $B\left(f(x), \frac{r}{C}\right) \subset f(B(x,r)) \subset B(f(x), Cr)$

• $f_n(x) = \frac{f(nx)}{n}$. $\forall r \ge \frac{1}{n}$ $\forall x$ $B\left(f_n(x), \frac{r}{C}\right) \subset f_n(B(x,r)) \subset B(f_n(x), Cr)$

$$\tilde{f} : X_{\mathcal{U}} \to Y_{\mathcal{U}} \qquad \mathcal{U} \text{ free ultrafilter on } \mathbb{N}.$$
$$\tilde{x} = (x_n) \longmapsto \left(\frac{f(x_n)}{n}\right)_n$$

Then $\forall r>0\ \forall \tilde{x}$ $B\left(\tilde{f}(\tilde{x}), \frac{r}{C}\right) \subset \tilde{f}(B(\tilde{x},r)) \subset B(\tilde{f}(\tilde{x}), Cr)$

This means that $\tilde{f}$ is a Lipschitz quotient. ◻

(3) UAAP (Uniform affine AP), superreflexivity and applications.

Def: $Lip(X,Y)$ has UAAP if $\forall \varepsilon>0\ \exists c(\varepsilon)>0\ \forall$ ball $B\subset X\ \forall f: B \xrightarrow{Lip} Y$
$\exists B_1\subset B$ ball $rad(B_1)\geq c(\varepsilon)\, rad(B)$
$\exists T: B_1 \longrightarrow Y$ affine continuous $\forall x\in B_1\ \|f(x)-Tx\|\leq \varepsilon\, rad(B_1)\, Lip(f)$

Prop 6 If $Lip(X,\mathbb{R})$ has UAAP, then, for any f.d. Y, $Lip(X,Y)$ has UAAP

Proof: We will only prove that $Lip(X,\ell_\infty^2)$ has UAAP. Then it suffices to conduct an induction. If $f=(f_1,f_2): B\longrightarrow(\mathbb{R}^2,\|\cdot\|_\infty)$, $Lip(f)\leq 1 \Leftrightarrow Lip f_i\leq 1$ $(i=1,2)$
$(rad B=1)$

Fix $\varepsilon>0$. There is $B_1\subset B$ with $rad(B_1)\geq c(\varepsilon c(\varepsilon))$

Then there is $g_1: B_1\longrightarrow\mathbb{R}$ st $|f_1-g_1|\leq\varepsilon c(\varepsilon)\, rad(B_1)$ on B_1.

There is $B_2\subset B_1$ with $rad\, B_2\geq c(\varepsilon)\, rad(B_1)$.

There is $g_2: B_2\longrightarrow\mathbb{R}$ st $|f_2-g_2|\leq \varepsilon\, rad(B_2)$ on B_2.

Then on $B_2\subset B_1$, $|f_1-g_1|\leq\varepsilon c(\varepsilon)\, rad(B_1)\leq\varepsilon\, rad(B_2)$

If $g=(g_1,g_2)$, $\|f-g\|_{\ell_\infty^2}\leq\varepsilon\, rad(B_2)$ on B_2

Then $rad\, B_2\geq c(\varepsilon)c(\varepsilon c(\varepsilon))=c_2(\varepsilon)>0$.

Here the U in UAAP is crucial!

Theorem 7 If X is superreflexive ($X_\mathcal{U}$ reflexive), then $Lip(X,\mathbb{R})$ has UAAP.

Theorem 8: If $X \xrightarrow[u]{} Y$, then $\exists\lambda\geq 1\ \forall G\subseteq Y^*$ $\dim G<\infty\ \exists F\subseteq X^*$ $F\underset{\lambda}{\simeq} G$.

It follows that $\exists Z\subseteq X_\mathcal{U}$ $Y\simeq X_\mathcal{U}/Z$

Corollary 9: $1<p<\infty$, $L_p=L_p(\mathbb{R})$. $L_p \xrightarrow[u]{} Y \Leftrightarrow L_p \twoheadrightarrow Y$

Proof of this Cor: Th 8 implies there is $\mathcal{U}$ s.t. $Y\simeq (L_p)_\mathcal{U}/Z$ and there is (Σ,μ) such that $(L_p)_\mathcal{U}\equiv L_p(\Sigma,\mu)$ but this a nonseparable space. and there is a quotient map $Q: L_p(\Sigma,\mu)\to Y$ (because $L_p\twoheadrightarrow Y$ implies Y is separable) and there is $s: Y\longrightarrow L_p(\Sigma,\mu)$ continuous selection s.t. $Q\circ s=Id_Y$ (Bartle-Graves)

Then $E=\overline{span}\ s(Y)$ is separable because s is continuous and Y separable

$E\subseteq L_p(\Sigma,\mu)$.

Let (f_n) be dense in E and $\tau=\sigma$-field generated by the f_n's

Then $F=L_p(\tau,\mu)$ is a separable space! and it contains E.

Then $F\simeq L_p$, ℓ_p or ℓ_p^n. But $Q|_F: F\longrightarrow Y$ is onto and $Y\simeq F/W$.

cf Chapter 1 of Wojtaszczyk, Banach spaces for analysts

Proof of Th 8: Suppose $X \twoheadrightarrow Y$. Then $X_{\mathcal{U}} \underset{L}{\twoheadrightarrow} Y_{\mathcal{U}}$ with $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$.

We want to show that for $G \subseteq Y^*$ there is $F \subseteq X^*$ with $G \underset{\lambda}{\simeq} F$.

It is enough to find $F \subset (X^*)_{\mathcal{U}}$ (finite dimensionality of F).

But $(X^*)_{\mathcal{U}} = (X_{\mathcal{U}})^*$ because X is superreflexive. Similarly, we can work with $G \subseteq Y^* \subset (Y^*)_{\mathcal{U}} \subseteq (Y_{\mathcal{U}})^*$, G f.d.

Since $X_{\mathcal{U}}$ is superreflexive, we may assume $f : X \underset{L}{\twoheadrightarrow} Y$ replacing X, Y by $X_{\mathcal{U}}, Y_{\mathcal{U}}$. $\operatorname{Lip} f = 1$

we also have $\forall x \in X$ $f(B(x,r)) \supset B(f(x), cr)$, $0 < c < 1$.

If $G \subset Y^*$, $\dim G < \infty$, let $G_\perp = \{y \in Y : \forall g \in G \ \langle y, g\rangle = 0\}$ and $Q : Y \longrightarrow Y/G_\perp$ the canonical quotient. $Q \circ f : X \longrightarrow Y/G_\perp$ f.d. and $\operatorname{Lip}(Q\circ f) \leq 1$

and $\forall x \in X$ $Q\circ f(B(x,r)) \supset B(Q\circ f(x), cr)$

X is superreflexive and $\dim Y/G_\perp < \infty$: thus $(X, Y/G_\perp)$ has UAAP.

For $\varepsilon < c^4$, there is $B_1 = B(x_0, r)$ and $T : B_1 \longrightarrow Y/G_\perp$ affine continuous such that $\forall x \in B_1$ $\|Q\circ f(x) - T(x)\| \leq \varepsilon \operatorname{rad}(B_1)$ $(Q\circ f)(x_0) = T(x_0)$

Let $\tilde{T}$ be the linear part of T: $\|\tilde{T}\| \leq 1+\varepsilon$

Lemma 2 yields $\tilde{T}(B_X) \supset c(1-c) B_{Y/G_\perp}$

Consider $\tilde{T}^* : (Y/G_\perp)^* \equiv G_\perp^\perp = G \longrightarrow X^*$. Then $\tilde{T}^*$ is a $\underbrace{\frac{1+\varepsilon}{c(1-c)}}$-embedding

* Classical results permit to deduce that $Y^* \subseteq (X^*)_{\mathcal{V}}$ [Y^{**} is finitely crudely representable in X^{**}]

As X is superreflexive, X^* is superreflexive, $(X^*)_{\mathcal{V}}$ is reflexive, Y^* is reflexive and thus Y is reflexive. Then $Y \equiv Y^{**} \simeq (X^*)^*_{\mathcal{V}}/Z \simeq \underbrace{(X^{**})_{\mathcal{V}}/Z}_{\text{because } X \text{ is superreflexive}} \simeq X_{\mathcal{V}}/Z$

Th 7 Assume X is superreflexive: $\operatorname{Lip}(X, \mathbb{R})$ has UAAP

Proof We know that X admits an equivalent uniformly smooth norm: we shall assume that $\|\cdot\|_X$ is u.s. Consider $f : B \longrightarrow \mathbb{R}$, B ball of X of radius 1 and $\operatorname{Lip} f = 1$. Fix $\varepsilon > 0$. Let $\operatorname{Lip}_t f = \sup_{\|x-y\| \geq t} \frac{\|f(x)-f(y)\|}{\|x-y\|}$. Case 1. Assume $\operatorname{Lip}_1 f \leq \varepsilon$. Then f has a good approximation by a constant f":

x_0-u x_0 x x_0+u $\|u\|=1$

$\|f(x_0) - f(x_0-u)\| \leq \varepsilon$
$\|f(x_0) - f(x_0-u)\| \leq 2\varepsilon$ and $|f(x) - f(x_0)| \leq 3\varepsilon$

Case 2: $\operatorname{Lip}_1 f \geq \varepsilon$: $\|\ \|_X$ u.s. implies that $\exists \delta(\varepsilon) > 0$ $\forall x \in S_X$ $\forall y \in \delta B_X$ $\|x+y\| = 1 + y_x^*(y) + r(y)$ where y_x^* is the unique norming functional of x in S_{X^*}. Let k be the smallest integer s.t. $(1+\delta\varepsilon)^k \varepsilon > 0$: $k = k(\varepsilon)$. Then $\varepsilon \leq \operatorname{Lip}_1(f) \leq \operatorname{Lip}_{\delta^{-1}}(f) \leq \dots \leq \operatorname{Lip}_{\delta^{-k}}(f) \leq 1$. $|r(y)| \leq \varepsilon\|y\|$

Therefore there is $j \leq k$ s.t. $\mathrm{Lip}_{4^{-j}}(f) < (1+\delta\varepsilon)\, \mathrm{Lip}_{4^{-j+1}}(f)$

Denote $\Delta = 2 \cdot 4^{-j}$. $\mathrm{Lip}_{\Delta/2}(f) < (1+\delta\varepsilon)\, \mathrm{Lip}_{2\Delta}(f)$

There is $d \geq \Delta$, $y, z \in B$ with $\|y-z\| = 2d$ s.t. $\mathrm{Lip}_{\frac{d}{2}}(f) \leq \mathrm{Lip}_{\frac{\Delta}{2}}(f) < (1+\delta\varepsilon) \frac{\|f(y)-f(z)\|}{2d}$

We may assume $y = -z$ and $f(z) = -f(-z) > 0$.

(*) Then $\exists\, d \geq \Delta$ $\|z\| = d$ $\mathrm{Lip}_{\frac{d}{2}} f < (1+\delta\varepsilon) \frac{f(z)}{d}$

As $-z, z \in B \Rightarrow 0 \in B$ $\quad \underset{0}{\times} \overset{\frac{\delta d}{2}}{\text{---}} \underset{u}{\times} \text{---} \underset{x_0}{\times}$ $\quad \frac{\delta d}{2} < \frac{1}{4}$. $B(u, \frac{\delta d}{2}) \subset (B \cap \delta d B_X)$

Note that $\frac{\delta d}{2} \geq \psi(\varepsilon) > 0$ for some function ψ

Let z^* be the norming functional of $\frac{z}{d}$.

$\|w+z\|, \|w-z\| \geq \frac{d}{2}$ because $\|z\| = d$, $\delta < \frac{1}{2}$.

Consider
$$
\begin{aligned}
f(w) = f(w) - f(-z) - f(z) &\leq \mathrm{Lip}_{\frac{d}{2}}(f)\, \|w+z\| - f(z) \\
&\leq (1+\delta\varepsilon) f(z) \left\| \tfrac{z}{d} + \tfrac{w}{d} \right\| - f(z) \\
&\leq (1+\delta\varepsilon) f(z) \left[1 + z^*\left(\tfrac{w}{d}\right) + \varepsilon \left\| \tfrac{w}{d} \right\| \right] - f(z) \\
& \qquad \left(\left\| \tfrac{w}{d} \right\| \leq \delta \right) \\
&\leq \frac{f(z)}{d} z^*(w) + [\varepsilon\delta + \varepsilon\delta + \delta^2\varepsilon + \delta^2\varepsilon^2]
\end{aligned}
$$

Thus $f(w) \leq \frac{f(z)}{d} z^*(w) + 3\varepsilon\delta$

* Similarly,
$$
\begin{aligned}
-f(w) = f(z) - f(w) - f(z) &\leq \mathrm{Lip}_{\frac{d}{2}} \|z - w\| - f(z) \\
&\leq \frac{f(z)}{d} z^*(-w) + 3\varepsilon\delta
\end{aligned}
$$

so that $f(w) \geq \frac{f(z)}{d} z^*(w) - 3\varepsilon\delta$.

Thus $f(w) - \frac{f(z)}{d} z^*(w) \leq 3\varepsilon\delta$ on a ball of radius $\geq \frac{\delta d}{2}$. ☒

<u>One last comment</u> on the current paper. It states the following result

<u>Prop</u>: $\mathrm{Lip}(X,Y)$ has UAAP $\Leftrightarrow$ X superreflexive and $\dim Y < \infty$

or

Y ——————— X ——

Another result we could discuss ① If $\ell_1 \subseteq X$, then $X \twoheadrightarrow \ell_1$ (JLPS'02: Lipschitz quotient of metric trees) This yields separable Lip quotients which are not quotients

② If $1 \leq q < p < \infty$, then $\ell_p \not\xrightarrow{L}\!\!\!\rightarrow \ell_q$ (another JLPS'02)

③ If $1 < p < q < \infty$, $\ell_p \not\xrightarrow{u}\!\!\!\rightarrow \ell_q$ (Lima-Randrianarivony '11)

Last remark: If you have a uniform quotient $L_p \xrightarrow{u} Y$, then $Y \simeq L_p/Z$ and $Y \simeq L_p/W$.