Following JLPS 2002:

Motivation: X super R and Y a B.sp. s.t. $X \overset{L}{\twoheadrightarrow} Y$

Then Y^* crudely f.r. in X^*. (M)

- is true even when X ⟨AUS⟩, $X = C(K)$, K a countable compact
- Question: is (M) true for any Banach space X?
- Answer: no
- Theorem 1: Let X be separable B.sp., $\ell_1 \underset{\sim}{\subseteq} X$, $\varepsilon > 0$.

Then $\forall$ Y sep. :

$X \xrightarrow{(1+\varepsilon) L} Y$

In particular: $\ell_1 \subseteq C[0,1]$ so $\exists$ $C[0,1] \overset{L}{\twoheadrightarrow} \ell_1$

but $\ell_1^* = \ell_\infty$ is not crudely f.r. in $C[0,1]^*$

it is because cotype $\ell_\infty = \infty$ and cotype $C[0,1]^* = 2$

Remark: • $\forall$ Y sep. Banach $\ell_1 \twoheadrightarrow Y$

Proposition 2: $\left.\begin{array}{l} \ell_1 \underset{\sim}{\subseteq} X \\ X \text{ separable} \\ \varepsilon > 0 \end{array}\right] \Rightarrow X \xrightarrow{1+\varepsilon} C(\Delta)$, Δ the Cantor set

Proof: James: $\ell_1 \underset{\sim}{\subseteq} X \Rightarrow \ell_1 \overset{1+\varepsilon}{\underset{\sim}{\subseteq}} X$ (if $\{x_i\}$ is equiv to ℓ_1 urb in the 1st ℓ_1, take its block basis that starts sufficiently far)

(James's ℓ_1-distortion theorem, see Albiac-Kalton)

$$\begin{array}{ccc} T : Z \overset{1+\varepsilon}{\sim} \ell_1 & \twoheadrightarrow & C(\Delta) \\ \cap & & \cap \\ S : X & \longrightarrow & \ell_\infty(\Delta) \end{array}$$

let ... extend linearly with the same $\|\cdot\|$

in fact $\overline{S(X)} =: Y \subset \ell_\infty(\Delta)$ is separable

Pelczynski $\exists$ 1-complemented copy of $C(\Delta)$ "$= E$" in $C(\Delta) \subset Y$ (0 dim, sep.)

denote $P : Y \to E$ a projection $\|P\| = 1$

$P \circ S$ is a quotient (onto), $\|P \circ S\| \leq 1+\varepsilon$ $\square$

~~Defin~~ Definition: a metric space X is <u>metrically convex</u> if

$$\forall x,y \in X \ \forall \lambda \in [0,1] \ \exists z \text{ s.t. } d(x,z) = \lambda d(x,y) \ \& \ d(y,z) = 1-\lambda\, d(x,y)$$

Proposition 3: $\forall Y$ sep., complete, metr. convex m.s. : $C(\Delta) \xrightarrow{1-L} Y$

<u>Def.</u>: • Let $(X,d_X), (Y,d_Y)$ be m.s. s.t. $X \cap Y = \{p\}$ for some p

we put <u>ℓ_1-union</u> $X \cup_1 Y = (X \cup Y, d)$ where

$$d(x,y) = \begin{cases} d_X(x,y), & x,y \in X \\ d_Y(x,y), & x,y \in Y \\ d_X(x,p) + d_Y(p,y), & x \in X, y \in Y \end{cases}$$

• a m.s. (T,d) is called SMT if $\exists \{I_n\}_{n=1}^{\infty} \subset 2^T$ s.t.

- $\forall n$ (I_n, d) is isometric to a closed interval or a closed ray
- if we call $T_n = \bigcup_{i=1}^{n} I_n$, then $T_n \cap I_{n+1} = \{p_n\}$

  where p_n is an end point of I_{n+1}

  and $T_{n+1} = T_n \cup_1 I_{n+1}$
- T is the completion of $\bigcup_{n=1}^{\infty} T_n$

• an SMT T is called an ℓ_1-tree if $\exists \{I_n\}$ as above

+ all I_n are rays
+ $\{p_n\}_{n=1}^{\infty}$ is dense in T

- any SMT T is isometric to a subset of ℓ_1:
- $f(p_0) = 0$, $f(I_1) \subset \mathbb{R}^+ e_1$ isometrically
- if $f(x)$ has been defined for $x \in T_n$
- map I_{n+1} isometrically into $f(p_n) + \mathbb{R}^+ e_{n+1}$
- make a completion

Recall: $f : X \to Y$ is 1-Lip. quotient if $\mathrm{Lip}\, f \cdot \mathrm{coLip}\, f = 1$

$$\mathrm{Lip}\, f = \sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)}$$

$$\mathrm{coLip}\, f = \inf \{c : f(B(x,r)) \supset B(f(x), \tfrac{r}{c}) \ \forall x \in X, \forall r > 0\}$$

my question: is the inf in fact min? no let $X^* \ni f$ for a non-reflexive X s.t. $\max f$ is not attained

Lemma 4: Let $f : X \to Y$ be 1-Lipschitz, $X_0 \subset X$ dense

s.t. $\forall x \in X_0$ $\forall r \in \mathbb{Q}^+$

$$B_Y^0(f(x), r) \subset \overline{f(B_X(x,r))}$$

then $\forall x \in X, \forall r > 0$: $B_Y^0(f(x), r) \subset f(B_X^0(x,r))$

Proof: Let $d(f(x), y) < r - \varepsilon$ for some $\varepsilon > 0$.

say $< r - \varepsilon - \eta$ $\quad$ $(x_0, d(x_0, x) < \frac{\eta}{2} \Rightarrow d(f(x_0), f(x)) < \frac{\eta}{2}$ $\Rightarrow d(y, f(x_0)) < r - \varepsilon - \frac{\eta}{2}$

$\exists x_1 \in X$

$d(x_1, x) \leq r - \varepsilon$ $\quad$ (as $d(x_1, x_0) \leq r - \varepsilon - \frac{\eta}{2}$ & $d(x_0, x) < \frac{\eta}{2}$)

$d(f(x_1), y) < \frac{\varepsilon}{2}$

$\exists x_2 \in X$ $\quad d(x_2, x_1) < \frac{\varepsilon}{2}$ & $d(f(x_2), y) < \frac{\varepsilon}{2^2}$

$\vdots$

$\exists x_{n+1} \in X$ $\quad d(x_{n+1}, x_n) < \frac{\varepsilon}{2^n}$ & $d(f(x_{n+1}), y_0) < \frac{\varepsilon}{2^{n+1}}$

$\Rightarrow x_n \to \tilde{x}$ $\quad$: $\|x - \tilde{x}\| < \underbrace{r - \varepsilon}_{d(x_1, x)} + \underbrace{\varepsilon}_{\sum_{i=1}^{\infty} d(x_i, x_{i+1})}$

$f(x_n) \to y = f(\tilde{x})$ $\quad \square$

Proposition 5: Let T be an ℓ_1-tree, Y separable, complete, metrically convex metric space $\Rightarrow \exists\, f : T \xrightarrow{1\text{-L}} Y$.

Proof.: Let $Y_0 = \{y_n\}_{n=1}^{\infty}$ be dense in Y.

Since $\{p_n\}_{n=0}^{\infty}$ is dense & T is connected

$\forall m\ \exists\, \varphi_m : \mathbb{N} \to \mathbb{N}$ s.t. $\{\varphi_n(k)\}_k \cap \{\varphi_m(k)\}_k = \emptyset$, $n \neq m$ and $\bigcup_m \{\varphi_m(k)\}_k = \mathbb{N}$

$(n,k) \mapsto \varphi_n(k)$ is bijection $\mathbb{N}^2 \to \mathbb{N}$

and $p_{\varphi_n(k)} \to p_n$ as $k \to \infty$ $\forall n \in \mathbb{N}$

$\boxed{\begin{array}{l}\text{let } (\varphi_1(k))_{k=1}^{\infty} \subset \mathbb{N} \text{ s.t. } p_{\varphi_1(k)} \to p_1,\ k \to \infty \\ \text{if } (\varphi_1, \dots, \varphi_n) \text{ known disjoint } \varphi_{n+1} \subset \mathbb{N} \setminus \bigcup_{m \leq n} \{\varphi_m\} \\ \text{s.t. } p_{\varphi_{n+1}(k)} \to p_{n+1}, \text{ if afterall there are still some points } (z_n) \text{ put } \varphi_n(0) = z_n\end{array}}$

- $f_0(p_0)$ arbitrary $\in Y$

suppose f_n has been defined, 1-Lipschitz

$f_n : T_n \to Y \Rightarrow \exists!\,(m,k)$ s.t. $n = \varphi_m(k)$

let $f_{n+1} : T_{n+1} \to Y$ be the extension of f_n s.t.

- the set $\{x \in I_{n+1} : d(p_n, x) \leq d(f(p_n), y_k)\}$

  is $\underbrace{\text{mapped onto}}_{\text{isometrically}}$ a geodesic arc between $f(p_n)$ and y_k

- the set $\{x \in I_{n+1} : d(p_n, x) > d(f(p_n), y_k)\}$

  is mapped onto y_k

then f_{n+1} is 1-Lip.

let f be the unique 1-Lip. $: T \to Y$ which extends $f_n\ \forall n$

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we need to show that for every $p_N \in \overline{\{p_n\}_{n=1}^{\infty}}$, $r \in \mathbb{Q}^+$ and

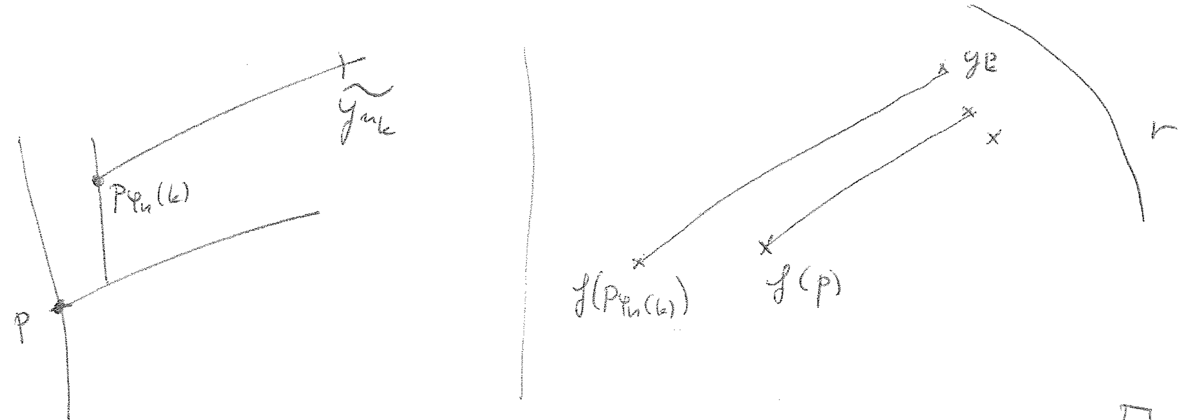
$x \in B^o(f(p_N), r)$ $\cancel{\exists}$ $x \in f(B(p_N, r))$

let $\varepsilon > 0$ be arbitrary s.t. $d(f(p), x) < r - \varepsilon$

$\exists k$ s.t. $d(p_{\varphi_N(k)}, p) < \frac{\varepsilon}{4}$ & $d(x, y_k) < \frac{\varepsilon}{2}$

$\Rightarrow d(f(p), y_k) < r - \frac{\varepsilon}{2}$ & $d(f(p_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}$

$\Rightarrow d(p, \tilde{y}_k) \leq \underbrace{d(p, p_{\varphi_N(k)})}_{< \varepsilon/4} + \underbrace{d(p_{\varphi_N(k)}, \tilde{y}_k)}_{= d(f(p_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}} < r$



$\square$

Proposition 6: For every SMT T: $C(\Delta) \xrightarrow{1-L} T$.

Lemma 7: T is a 1-absolute Lipschitz retract

Proof:
Observation: X, Y 1-abs Lip. retracts $\Rightarrow X \cup_1 Y$ is 1-a.L.r.
$|X \cap Y| = 1$

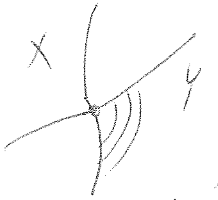
Recall: X is 1-a.L.r. if $\forall Y$ m.s. s.t. $X \subset Y$
$\exists R : Y \to X$, 1-Lip. s.t. $\forall x \in X : Rx = x$.

* classical: X is 1-a.L.r. $\Leftrightarrow$ X is metrically convex and every collection of mutually intersecting balls has a common point

let $\{B(x_\alpha, r_\alpha)\}_{\alpha\in\Gamma_1} \cup \{B(y_\beta, \rho_\beta)\}_{\beta\in\Gamma_2}$ be mutually intersecting balls in $X \cup_1 Y$

$\Rightarrow \forall \alpha, \beta \quad r_\alpha + \rho_\beta \geq d(x_\alpha, p) + d(y_\beta, p)$

- if $r_\alpha \geq d(x_\alpha, p)$ & $\rho_\beta \geq d(y_\beta, p)$ $\forall \alpha, \beta \Rightarrow p$ is common point
- if not



$\exists \alpha_0 \in \Gamma_1 : r_{\alpha_0} < d(x_{\alpha_0}, p)$

$\Rightarrow \{B(x_\alpha, r_\alpha) : \alpha \in \Gamma_1\} \cup \{B(p, \rho_\beta - d(y_\beta, p)) : \beta \in \Gamma_2\}$ are mutually intersecting in $X \Rightarrow \exists$ point in common

$\square$

Let $T \subset S$. Each T_n is 1-a.L.r. $P_n : S \longrightarrow T_n$

$\forall n \in \mathbb{N}$ $\quad P_n \in \prod_{s \in S} \left(B_{\ell_1}(0, d(s,p_0)); w^*\right)$ $\quad$ (better ~~$P_n \in \prod (T \cap B_{\ell_1} \dots$~~)

as if 1-Lip map $f : S \to T$ satisfies $f(p_0) = p_0 \Rightarrow d(f(s), p_0) \leq d(s, p_0)$ $\forall s$

$\Rightarrow \exists$ a cluster point P, necessarily 1-Lip, retraction

(if $\|Ps - Pt\|_{\ell_1} > d(s,t) \Rightarrow \exists x \in c_0 : P \in \underbrace{\{Q : |(Qs)x - (Qt)x| > d(t,s)\}}_{\tau \text{ open}} \not\ni P_n$

(retract : - $\forall t \in \bigcup_{n=1}^{\infty} T_n$ $\exists$ only finitely many n s.t. $P_n(t) \neq t$

- $\forall t \in \overline{\bigcup^{\infty} T_n}$ ... lipschitzness of P

- $\forall s \in S$ ~~...~~ $Ps \in T$ as T is w^*-closed

because $T = \bigcap_{m \geq 1} \left(P_m(T_n) \times \overline{\text{span}}\{e_k : k > m\}\right)$

1) $\forall m \geq n$ $\quad T_m \subset P_n(T_n) \times \overline{\text{span}}\{e_k : k > n\}$

$\Rightarrow \bigcup_{m \geq n} T_m \subset$ ――――――

$\Rightarrow \underbrace{\bigcap_{n \geq 1} \bigcup_{m \geq n} T_m}_{\bigcup_{m=1}^{\infty} T_m} \subset \underbrace{\bigcap_{n \geq 1} P_n(T_n) \times \overline{\text{span}}\{e_k : k > n\}}_{w^*\text{-closed}} = F$

~~...~~ $\forall x \in F \Rightarrow x \in T$

indeed as $\forall n$: $P_n x \in P_n(T_n) \Rightarrow \underbrace{P_n x \times (0, 0, \dots)}_{y_n} \in T_n$

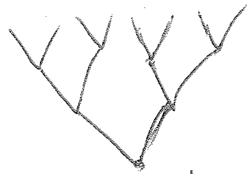
$\|x - y_n\| = \sum_{m > n}^{\infty} x_m \longrightarrow 0$

$\square \leftarrow$ end of proof of L7.

Let $r_n : \Delta \to \{-1,1\}$ be the n-th coordinate projection

$r_n \in C(\Delta)$, $\Delta = \{-1,1\}^{\mathbb{N}}$

Let $E_n \subset C(\Delta)$ be the space of functions that depend on the 1st n coordinates ($\dim E_n = 2^n$)



Observe that $\overline{\bigcup E_n} = C(\Delta)$

By Stone-Weierstrass (xy, $x+y$, λx, $\mathbb{1}$, separates points)

let us observe that for $x \in E_n$, $m > n$, $\forall t \in \mathbb{R}$:

$$\|x + t r_m\| = \|x\| + |t| \quad (*)$$

Let $T \approx \{T_n\}$, $(x_n) \subset \bigcup E_n$ dense s.t. $x_n \in E_n$ $\forall n$

let $\{y_k\} \subset \bigcup T_n$ s.t. $\{y_k\}_k \cap T_n$ is dense in T_n $\forall n$ and $y_k \in T_{k+1}$

$\forall n\ \exists \varphi_n : \mathbb{N} \to \mathbb{N}$ s.t. $(n,k) \mapsto \varphi_n(k)$ is a bijection $\mathbb{N}^2 \to \mathbb{N}$

and $x_{\varphi_n(k)} \to x_n$ as $k \to \infty$ $\forall n \in \mathbb{N}$

- and MOREOVER $k \le \varphi_n(k)$ $\forall n, k$ .. achieved by insisting that $\forall n$ $(\varphi_n(k))_{k=1}^{\infty} \nearrow$

  if some points are left out after all $\Rightarrow$ just forget them

- let $f_0(x_0) = p_0$

  as $x_0 = 0$ and $E_0 := \{0\}$ (and no constants)

Suppose f_n has been defined, 1-Lip.

$$f_n : E_n \to T_n$$

$$\exists!\ (m,k) : \varphi_m(k) = n$$

extend f_n to $\tilde{f}_n : E_n \cup (x_n + \mathbb{R}^+ r_{n+1}) \to T_{n+1}$

by mapping $\{x_n + t r_{n+1} : 0 \le t \le d(f_n(x_n), y_k)\}$ onto (isometrically)

the geodesic arc between $f_n(x_n)$ and y_k

put $\tilde{f}_n(x) = y_k$ if $x = x_n + t r_{n+1}$ with $t > d(f_n(x_n), y_k)$

then $\tilde{f}_n$ is 1-Lip thanks to $(*)$

T_{n+1} is 1-a.L.r.

$$\tilde{f}_n : \underbrace{E_n \cup (x_n + \mathbb{R}^+ r_{n+1})}_{U} \longrightarrow T_{n+1}$$ can be extended

$$f_{n+1} : E_{n+1} \longrightarrow T_{n+1}, \quad 1\text{-Lip.}$$

f the unique 1-Lip extension of all f_n

We need to ~~ve~~ verify that $\forall x_N \in \{x_n\}_{n=1}^{\infty}$, $r \in \mathbb{Q}^+$ and

$$y \in B_T^O(f(x_N), r) \; : \; y \in \overline{f(B_{C(\Delta)}(x_N, r))}$$

let $\varepsilon > 0$ be arbitrary s.t. $d(f(x_N), y) < r - \varepsilon$

$$\exists\, k \text{ s.t. } d(x_{\varphi_N(k)}, x_N) < \frac{\varepsilon}{4} \;\&\; d(y, y_k) < \frac{\varepsilon}{2}$$

$$\Rightarrow d(f(x_N), y_k) < r - \frac{\varepsilon}{2} \;\&\; d(f(x_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}$$

$$\Rightarrow d(x_N, \tilde{y}_k) \le \underbrace{d(x_N, x_{\varphi_N(k)})}_{< \frac{\varepsilon}{4}} + \underbrace{d(x_{\varphi_N(k)}, \tilde{y}_k)}_{= d(f(x_{\varphi_N(k)}), y_k) < r - \frac{\varepsilon}{4}} < r$$

Finish by L4 $\square$