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"Smoothness and weak continuity"

(we will stick to the real case)

Introduce the calculus in Banach space theory

Let X, Y be Banach spaces and $T: X \longrightarrow Y$.

We will try to define that T is C^k-smooth. This is parallel to the theory of uniform and Lipschitz maps, but different.

Def: $X_1, \dots, X_n, Y$ Banach spaces. $M: X_1 \times \dots \times X_n$ is n-linear if

$M(\lambda_1 x_1 + \lambda_2 x_2, x_2', \dots, x_n') = \lambda_1 M(x_1, x_2', \dots, x_n') + \lambda_2 M(x_2, x_2', \dots, x_n')$

and the same for all entries. We write $M \in L(X_1, \dots, X_n; Y)$ and set

$\|M\| = \sup_{\|x_i\| \leq 1} \|M(x_1, \dots, x_n)\|$ and this makes a Banach space $\mathcal{L}(X_1, \dots, X_n; Y)$.

In the special case $X_1, \dots, X_n = X$, we write $\mathcal{L}({}^nX; Y)$

Def: Let X, Y B-spaces (Banach?). Then $P: X \longrightarrow Y$ is an n-homogeneous polynomial if $\exists M \in L({}^nX; Y)$ $P(x) = M(\underbrace{x, \dots, x}_{n \text{ times}})$. We say P is bounded if $\|P\| = \sup_{\|x\| \leq 1} \|P(x)\|_Y < \infty$

$\rightarrow \mathcal{P}({}^nX; Y) \ni P = \hat{M}$ and n is the degree of the polynomial

Th: If $M \in L({}^nX; Y)$, TFAE: (1) M is bdd (2) M is continuous (3) M is Borel-measurable (4) M is separately continuous (5) M is bounded on some open set (6) M is Lipschitz on every bounded set.

Fact: P is bounded iff M is bounded

Def: We say that $M \in \mathcal{L}({}^nX; Y)$ is symmetric if $M(x_1, \dots, x_n) = M(x_{\pi(1)}, \dots, x_{\pi(n)})$ for every $\pi \in \mathfrak{S}_n$.

Note: There is a natural mapping $M \longrightarrow M^s$, $M^s(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\pi \in S_n} M(x_{\pi(1)}, \dots, x_{\pi(n)})$

In fact, this mapping is a bounded projection with image $\mathcal{L}^s({}^nX; Y)$

Fact: $P(x) = M(x, \dots, x) = M^s(x, \dots, x)$

Polarisation formula: For every $P \in \mathcal{P}({}^nX; Y)$, there is a unique n-linear and symmetric mapping $M = \check{P}$ such that $P = \hat{M}$: $M(x_1, \dots, x_n) = \frac{1}{n! 2^n} \sum_{\varepsilon \in \{-1,1\}^n} P(\sum \varepsilon_i x_i) \varepsilon_1 \cdots \varepsilon_n$

Moreover, $\|P\| \leq \|\check{P}\| \leq \frac{n^n}{n!} \|P\|$

This means that $P \longrightarrow \check{P}$ is an isomorphism $\mathcal{P}({}^nX; Y) \longrightarrow \mathcal{L}^s({}^nX; Y)$.

Remark: • If $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$, then $\mathcal{P}(^kX,Y)$ is indeed the "usual polynomials".

• $\mathcal{L}(X_1;\mathcal{L}(X_2,\dots,X_n;Y)) = \mathcal{L}(X_1,\dots,X_n;Y)$

This will enable us to define higher derivatives!

Def: P is a polynomial of degree $\le n$ if $P = \underset{\text{cst } f: X\to Y}{P_0} + \dots + P_n$ with $P_j \in \mathcal{P}(^jX;Y)$.

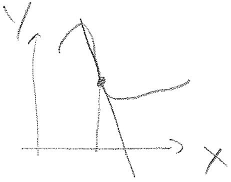
Fact: The P_j's are uniquely determined by

Moreover, $\mathcal{P}^n(X;Y) = \{P : \partial^\circ P \le n\}$ endowed with $\|P\| = \sup_{x\in B_X}\|P(x)\|_Y$

and $\mathcal{P}^n(X;Y) \cong \mathcal{P}(^0X;Y)\oplus\dots\oplus\mathcal{P}(^nX;Y)$ is a B-space isomorphism

Def: $U\subset X$ open, Y B-space, $f: U\to Y$, $x\in U$. We say that $L\in\mathcal{L}(X;Y)$ is a Fréchet derivative at x if $\lim_{\|h\|\to 0}\frac{\|f(x+h)-f(x)-L(h)\|}{\|h\|}=0$

We write $L = Df(x)$ and $L(h) = Df(x)[h]$



Def: Suppose $U\subset X$ is open, Y is a Banach space and $f: U\to Y$ has k^{th} derivative in U $D^kf: U\to\mathcal{L}(^kX;Y)$; then $D^{k+1}f(x) = D(D^kf)(x)\in\mathcal{L}(X;\mathcal{L}(^kX;Y)) \overset{\text{isometric}}{\cong} \mathcal{L}(^{k+1}X;Y)$

Proposition: If $D^kf(x)$ exists, then $D^kf(x)\in\mathcal{L}^s(^kX;Y)$

Let us define: $d^kf(x) = \widehat{D^kf(x)}$ is the k^{th} derivative in $\mathcal{P}(^kX;Y)$

Taylor formula: Let $f: U\to Y$, $x\in U$. If f has $d^kf(x)$, then

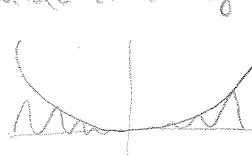
$$\Big\|f(x+h) - \overbrace{\sum_{j=0}^{k}\frac{1}{j!}d^jf(x)[h]}^{\mathcal{P}^k(X;Y)}\Big\| \le o(\|h\|^k)$$

Forward remark: A smooth f^n behaves like a polynomial: prop^s of smooth f^{ns} will be det'd by the linear structure of X because the polynomials dependent.

Note: in the complex setting, a differentiable function is automatically analytic, C^∞-smooth, with a converging Taylor series!

Q: what about the converse? Suppose that f has a Taylor formula on every x.

The existence of good polynomial approximations does not imply that the function is $\mathcal{C}^2$ even in 1 dimension!



Definition: Let $f: U\to Y$ ~~be locally uniformly continuous~~ and $x\in U$.

Suppose that $\exists\, P_k^x\in\mathcal{P}^k(X;Y)$ $\|f(x+h)-P_k^x(h)\|\le\omega_x(h)\|h\|^k$ with $\omega_x(h)\xrightarrow[h\to 0]{}0$

We say that f is T^k-smooth at x.

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Note that P_k^x is unique if it exists.

Also, if $d^k f(x)$ exists, then $P_k^x = \sum_j \frac{1}{j!} d^j f(x)$

__Theorem__ (Johanis) (converse Taylor theorem) Let $f: U \to Y$ be locally uniformly cont.

TFAE: 1. $f \in C^k$.

2. f is T^k-smooth in U: $\forall x \, \exists P_k^x \quad \lim_{\substack{(y,h)\to(x,0)\\ h\neq 0}} \frac{\|R_k(y,h)\|}{\|h\|^k} = 0$

where $R_k(x,h) = f(x+h) - P_k^x(h)$

__Example:__ $P \in \mathcal{P}(^n X; Y)$ is C^∞-smooth (i.e., C^k-smooth for every $k \in \mathbb{N}$)

$$D^k P(x)[h_1, \dots, h_k] = n(n-1)\cdots(n-k+1)\check{P}(^{n-k}x, h_1, \dots, h_k)$$

$$d^k P(x)[h] = n(n-1)\cdots(n-k+1)\check{P}(^{n-k}x, {}^k h)$$

and $d^n P(x)[h]$ is "constant" in x, $= n!\, P(h)$

and $d^{n+1} P \equiv 0$.

__Other approach:__ (Fréchet, through local behaviour)

Theorem (Fréchet, Michael, Mazur-Orlicz): X, Y real B-spaces, $f: X \xrightarrow{\text{continuous}} Y$.

wouldn't work with linear.

$1 \Rightarrow 2 \Rightarrow 4$

TFAE: 1. $f \in \mathcal{P}^n(X; Y)$

2. $f|_E \in \mathcal{P}^n(E; Y)$ for every 1-dim affine subspace of X

3. f is C^{n+1}-smooth and $D^{n+1} f \equiv 0$

4. $\sum_0^{n+1} (-1)^{n+1-k} \binom{n+1}{k} P(x+kh) = 0$ for all $x, h \in X$

2 $\to$ 1: one needs some kind of uniformity: it is provided by the fixed degree n!

$x \quad x+h \quad x+2h \quad x+3h \quad \dots \quad x+(n+1)h$

$\underbrace{\qquad\qquad\qquad\qquad}_{n+2 \text{ points}}$

fact: $f: X \to Y$ is in $\mathcal{P}(^n X; Y)$ iff

$\varphi \circ f: X \to \mathbb{R}$ is in $\mathcal{P}(^n X)$ for every $\varphi \in Y^*$

We consider $f \xrightarrow{\text{evaluation}} f(x)$ as a linear functional, so 4 says something about $n+2$ functionals on $\mathcal{P}(^n X)$ that has $n+1$ dimensions: they must be linearly dependent and 4 gives the correct coefficients.