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2e cours

Definition: Let U be a convex subset of X with nonempty interior, X, Y B-spaces

- $C(U;Y)$ is the space of continuous function and
  $C_u(U;Y)$ ——— uniformly ——— on closed bounded convex $V \subset U$ (CCB)
  $C_u^m(U;Y)$ ——— m times continuously differentiable f^{ns} with all derivatives are in C_u.
  $C_w(U;Y)$ ——— weakly continuous on CCB sets
  $C_{wu}(U;Y)$ ——— uniformly ———

$T \in C_{wsc}(U;Y)$ if $\forall V$ CCB $\subset U$ $\forall \{x_n\}_{n=1}^{\infty} \subset V$ weakly convergent to x
$\{T(x_n)\}$ is $\|\cdot\|$-convergent to $T(x)$

$T \in C_K(U;Y)$ if $\forall V$ CCB $\subset U$ $\overline{T(V)}$ is $\|\cdot\|$-compact

$T \in C_{wK}(U;Y)$ ——— w-compact.

Same notation $\mathcal{P}_K, \mathcal{P}_{wu}, \dots$ for space of polynomials

$\rightarrow \mathcal{P}_\phi = C_\phi \cap \mathcal{P}$ for $\phi \in \{w, wu, wsc, \dots\}$

There are many relationships between these notions.

Prop: Let $U \subset X$ and Y. Let $T \in C_K(U;Y)$. Then $T \in C_\phi(U;Y) \iff f \circ T \in C_\phi(U;\mathbb{R})$ for $f \in Y^*$
(will not be proved)

Prop: $C_{wu}(U;Y) \subset C_K(U;Y)$.

Remark:
- T is w-uniformly continuous if $\forall \varepsilon > 0\ \exists \delta > 0\ \exists f_1, \dots, f_n$ $|f_i(x) - f_i(y)| < \delta \Rightarrow \|T(x) - T(y)\| < \varepsilon$
- T is w ——— if T maps w-cv nets to $\|\cdot\|$-cv nets
- T is wu ——— w-Cauchy
- $C_w \subset C_{wsc}$ and $C_{wu} \subset C_{wsc}$

Proof: Consider $M: V \longrightarrow \mathbb{R}^n$, $x \longmapsto (f_1(x), \dots, f_n(x))$. Then $\overline{M(V)}$ is relatively compact.

Thus $M(V)$ contains a ρ-net: choose $\{x_j\}_{j=1}^{N}$ s.t. $\{M(x_j) : j = 1..n\}$ is ρ-dense in $\overline{M(V)}$

$M(V)$

Then, for each $x \in V$, $T(x)$ will be close to some $T(x_j)$. Combine

Thus weakly uniformly continuous is always compact.

Theorem [Aron–Prolla] Let X, Y. $C_{wu}(U;Y) = \overline{\mathcal{P}_f^n(X;Y)}^{\tau_b}$

Proof: $\cdot\ \mathcal{P}_f(X;U)$: A first simple example: $f \otimes y \in X^* \otimes Y: x \mapsto f(x)y$

Then: $x \mapsto f^n(x)y$ is a polynomial of n^{th} degree.

Then take linear combinations

- τ_b is the topology of uniform convergence on CCB subsets of U.

- Def: X has the Dunford–Pettis property (DPP) if $\mathcal{L}_{wK}(X;Y) \subset \mathcal{L}_{wsc}(X;Y)$ or $\mathcal{L}_{wsC}(X;Y)$

i.e., $T(x_n) \to T(x)$ if $x_n \xrightarrow{w} x$

- Th [Aron, Hervelin, Valdivia]: $\mathcal{L}_{wsc}(^nX;Y) = \mathcal{L}_{wsC}(^nX;Y)$

$\mathcal{L}_{w} \qquad \mathcal{L}_{wu}$

and the same holds for space of polynomials.

- Th [Ryan] If X has DPP, then $\mathcal{L}_{wK}(^nX;Y) \subset \mathcal{L}_{wsc}(^nX;Y)$

$\mathcal{P} \qquad \mathcal{P}$

↳ This is by no means obvious: polys disturb the linear structure!

ex: linear maps send w-Cauchy seq to w-Cauchy seq, but this is false e.g. for polynomials

but: weakly conditionally Cauchy seq are also preserved for polys.

Can these statements be transferred from poly to smooth functions?
This is by no means obvious! Taylor series approximate only locally.

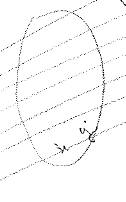
Theorem: Let $\ell_1 \not\hookrightarrow X$ and Y be a B-space. Then $C_{wsc}(U;Y) = C_w(U;Y)$

$C_{wsC}(U;Y) = C_{wu}(U;Y)$

→ continuity props can be checked on seq and not nets!

Proof of the 2^{nd} statement. $C_{wu} = C_{wsC}$: $\Rightarrow$ is clear. $\Leftarrow$: proof by contradiction

. Let T be wsC on V CCB $\subset U$, but not wu on V.



φ_i hyperplane

$T(y)$, $T(x)$ sep'd by ε

Then $\exists \varepsilon > 0\ \forall \varphi_1, \dots, \varphi_n \in X^*\ \forall \delta > 0\ \exists a_1, \dots, a_n \in \mathbb{R}\ \exists x, y$ $\left[\begin{array}{l} |\varphi_i(x) - a_i| < \delta \\ |\varphi_i(y) - a_i| < \delta \\ \|T(x) - T(y)\| > \varepsilon \end{array}\right]$ weakly shifted neighborhood

Let $\Gamma = \{(\varphi_1, \dots, \varphi_n; a_1, \dots, a_n; \delta)\}$; x_Φ, y_Φ

We will order Γ in the following way: $\Psi > \Phi$ if Ψ may be written as an extension of Φ: $(\varphi_1, \dots, \varphi_n, \varphi_{n+1}, \dots, \varphi_N; a_1, \dots, a_n, a_{n+1}, \dots, a_N; \tilde{\delta})$ and $\tilde{\delta} < \delta$.

Then $0 \in \overline{\{x_\Phi - y_\Phi : \Phi \in \Gamma\}}^w$ (Take a look at $\lim_{(\Gamma, <)}$)

Let us use Kaplansky's theorem: w-topology is countably tight.

Therefore there is a sequence $(\Phi_n) \subset \Gamma$ such that $0 \in \overline{\{x_{\Phi_n} - y_{\Phi_n}\}_n}^w$

There is a th by Bourgain and Fremlin-Talagrand BFT: If $\ell_1 \not\hookrightarrow X$, consider $Z = [x_{\Phi_n}, y_{\Phi_n}]$.

then $\ell_1 \not\hookrightarrow Z$; there is a subsequence $\{\Phi_k\}_{k=1}^{\infty}$ such that $0 = w\text{-}\lim x_{\Phi_k} - y_{\Phi_k}$.

Denote the sequences as $\{x_k\}$ and $\{y_k\}$: $0 = w\text{-}\lim x_k - y_k$

By Rosenthal's ℓ_1-theorem, wlog, $\{x_k\}, \{y_k\}$ is w-Cauchy. Let $v_k = x_k - y_k$.

Consider $y_1, \underbrace{y_1 + v_1}_{x_1}, y_2, \underbrace{y_2 + v_2}_{x_2}, \ldots$: this is a w-Cauchy seq perturbed by a w-null seq

thus it is itself w-Cauchy: How do $T(x_k)$, $T(y_k)$ behave?

We have $\|T(x_k) - T(y_k)\| > \varepsilon$ ↯

This should even give a characterisation of "$\ell_1 \not\hookrightarrow$"

<u>Theorem</u>: [Local reflexivity version by Johnson, Lindenstrauss, Rosenthal]
If $M \subset X^{**}$ have finite dimension and $\varepsilon > 0$, there is $T: M \to X$ s.t. $\|T\| < 1+\varepsilon$
$N \subset X^*$ $\|T^{-1}\| < 1+\varepsilon$,
and $\langle f, T(x)\rangle = \langle f, x\rangle$ for $f \in N$ and $x \in M$. $T|_{M \cap X} = \mathrm{Id}_{M \cap X}$

Th: idea: given $T: B_X \to Y$, we want to construct $\tilde{T}: B_{X^{**}} \to Y^{**}$.
If T is linear, it suffices to consider the bitranspose. If T isn't ... ?

Constructing T^{**}, you set for $x^{**} = \lim_{\mathcal{U}} x_\alpha$ $T^{**}(x^{**}) = \lim T(x_\alpha)$

If T is nonlinear, one has to be very careful about the ultrafilter $\mathcal{U}$:
always use the same and thus have a uniform way of approximating.

Let $\Gamma = \{\underbrace{(M, N, \varepsilon)}_{\alpha} : M \subset X^{**}, N \subset X^*, \varepsilon > 0\}$

Consider the corresponding T_α of the local reflexivity theorem: $\langle f, T_\alpha(x)\rangle = \langle f, x\rangle$
and consider $(X)_{\mathcal{U}} = \prod_{\alpha \in \Gamma} X$. for $f \in N$, $x \in M$.

Given x^{**}, computing $T_\alpha(x^{**})$ will be possible if $x^{**} \in M$; otherwise we don't mind.

Consider a nontrivial ultrafilter on Γ: Γ has a natural order:

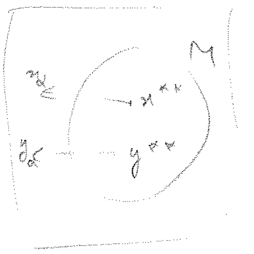
$\alpha > \beta$ if $M_\alpha \supset M_\beta$, $N_\alpha \supset N_\beta$, $\varepsilon_\alpha < \varepsilon_\beta$

For every x^{**}, $\{\alpha : x^{**} \in M\} \in \mathcal{U}$ because x^{**} can be added to every M.

X^{**} is embedded in $(X)_{\mathcal{U}}$, isometrically. For each x^{**}, there is a net

$x_\alpha = T_\alpha(x^{**})$ such that $x^{**} = w^*\text{-}\lim_{\mathcal{U}} x_\alpha$

<u>Theorem</u> [Choi Hájek Lee 2012]: Let $T: B_X \to Y$ be uniformly continuous and $\mathcal{U}$ the ultrafilter as above.
Then we let $\tilde{T}(x^{**}) = w^*\text{-}\lim_{\mathcal{U}} T(x_\alpha)$ if $x^{**} = w^*\text{-}\lim_{\mathcal{U}} x_\alpha$ [exists by compactness of w^* closed bounded sets of the bidual]. Then, if T is linear, $\tilde{T} = T^{**}$; also $\widetilde{T+S} = \tilde{T} + \tilde{S}$.



- If T has modulus of continuity $\omega(\delta)$, then also $\tilde{T}$.
- If $T \in \mathcal{P}^n(X;Y)$, then $\tilde{T} \in \mathcal{P}^n(X^{**}, Y^{**})$
- If T is UC^n, then $\tilde{T}$ is UC^n: moduli are preserved.
- If T is in Γ_K or in Γ_{wK}, then $\tilde{T}(B_{X^{**}}) \subset Y$.

Proof: If $x^{**} = \lim x_\alpha$, $y^{**} = \lim y_\alpha$, then $\underbrace{\|x^{**} - y^{**}\|}_{\|x_\alpha - y_\alpha\|}$: for large α, $x^{**}, y^{**} \in M_\alpha$ $\|u_p \to \varepsilon$

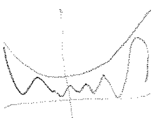
If $\|x_\alpha - y_\alpha\| < \delta$, then $\|T(x_\alpha) - T(y_\alpha)\| < \omega(\delta)$. So this passes to x^{**}, y^{**}.

By Fréchet's theorem: $\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} T(x + kh) = 0$ for $x, h \in X$.

Claim: then the same holds for the $\tilde{T}(x^{**} + kh^{**})$! Again, they are the limit of the $T(x_\alpha + kh_\alpha)$ along $\mathcal{U}$, for which individually the identity holds

<u>Q</u>: If T is UC^n-smooth, what about $\tilde{T}$?

Let us use the converse Taylor theorem: for each α $\|T(x) - P^a(x-a)\| = \|T(x) - \sum_{j=0}^{n} \frac{d^jT(a)[x-a]}{j!}\|$
$\leq \omega(\|x-a\|)\|x-a\|^k$



where ω is an absolute modulus of continuity depending only on T.

Let us set $P^{a^{**}}(x^{**}) = w^*\text{-}\lim_{\mathcal{U}} P^{a_\alpha}(x_\alpha)$. Miracle! this works, though one lets simultaneously $a_\alpha \to a^{**}$, $x_\alpha \to x^{**}$! Another method of proof goes by induction.

For a fixed a^{**}, is this a polynomial in X^{**}? For every x^{**}, h^{**}, $\sum_{k=0}^{n+1} \cdots P^{a^{**}}(x^{**} + kh^{**}) = 0$ because $\sum \cdots P^{a_\alpha}(x_\alpha + kh_\alpha) = 0$. This it is a polynomial!

Then it is easy to prove that $P^{a^{**}}$ is a Taylor approximation.

From the next lecture on, the style will change!

Note: Pb: <u>If</u> $T: X \to Y$ is C^n-smooth, is $\tilde{T}: X^{**} \to Y^{**}$ C^n-smooth
Not clear at all!

In some cases, the above results can be strengthened.

If X is an $\mathcal{L}_\infty$ (a $C(K)$-space), then for any Z s.t. $X \hookrightarrow Z$, for any $T: B_X \xrightarrow{UC^n} Y$, there is a $\tilde{T}: B_Z \xrightarrow{UC^n} Y^{**}$

Add: Spaces with DPP: $C(K)$, $L_1(\mu)$, $\mathcal{L}_\infty$-space, $\mathcal{L}_1$-space : For certain spaces of analytic f., $A(\mathbb{D})$, $H^\infty(\mathbb{D})$, it might not even be known.

It is natural to introduce quantitative versions of DPP, like wc, wuC, and then $(p,1)$-summing, -weakly summing, etc...