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Stochastic metric decompositions D'après Lee-Naor.

Déf. 3.1: Let (Y, d) be a metric space and X a subspace of Y. Then $(\Omega, \mu, \{\Pi^i(\cdot), \gamma^i(\cdot)\}_{i\in I})$ is a stochastic decomposition of Y wrt X if I is some index set, (Ω, μ) is a prob. space, and $\{\Pi^i(\omega)\}_{i\in I}$ is a partition of Y into Borel sets for every $\omega \in \Omega$, and $\{i\in I : \exists \omega \in \Omega \; x \in \Pi^i(\omega)\}$ is countable for every $x \in Y$, and $\gamma^i : \Omega \to X$ is a Borel measurable function for every $i \in I$ st. $d(\gamma^i(\omega), \Pi^i(\omega)) \le 2\, d(X, \Gamma^i(\omega))$ for every $\omega \in \Omega$, and $\{\omega \in \Omega : x \in \Gamma^i(\omega)\}$ is measurable.

Def 3.2: (bounded S.D.): Let $\Delta > 0$. We say that the SD is bounded by Δ if $\forall \omega \in \Omega \, \forall i \in I$ $\operatorname{diam}(\Pi^i(\omega)) < \Delta$.

Def 3.3: (padded decomposition) We say a SD is (ε, δ)-padded if $\omega \mapsto d(x, Y\setminus\Gamma^i(\omega))$ is measurable for every $x \in Y$ and $i \in I$ and, if $d(x, X) \le \varepsilon\Delta$, then $\mu\left(\bigcup_{i\in I}\{\omega : d(x, Y\setminus\Gamma^i(\omega)) \ge \varepsilon\Delta\}\right) \ge \delta$.

$\to$ fix ω and consider the partition $\{\Pi^i(\omega)\}_{i\in I}$ of Y: $\cancel{\forall x}\; \exists!\, i_0 \in I$ $x \in \Gamma^{i_0}(\omega)$, so that $d(x, Y\setminus\Gamma^i(\omega)) = d(x, \Gamma^{i_0}(\omega)^c)$ and we want that $B(x, \varepsilon\Delta) \subset \Pi^{i_0}(\omega)$

Def 3.4: (thick decomposition): a SD is (ε, δ)-thick if $\forall x \in X \, \forall i \in I$ $\omega \mapsto d(x, Y\setminus\Gamma^i(\omega))$ is measurable and if $d(x, X) \le \varepsilon\delta \Rightarrow \int_\Omega \sum_{i\in I} \min\{d(x, Y\setminus\Gamma^i(\omega)), \Delta\}\, d\mu(\omega) \ge \delta\Delta$

¿ Why take the minimum here ?

N.B.: (ε, δ)-padded $\Rightarrow (\varepsilon, \varepsilon\delta)$-thick: $\int_\Omega \frac{\sum_{i\in I} d(X, Y\setminus\Gamma_i(\omega))}{\varepsilon\Delta}\, d\mu(\omega) \ge \mu\left(\bigcup_{i\in I}\{\omega : d(x, Y\setminus\Gamma^i(\omega)) \ge \varepsilon\delta\}\right) \ge \delta$

Def 3.5: (separating decomposition) A SD is (ε, δ)-separated if $\forall x, y \in Y$ with $d(\{x,y\}, X) \le \varepsilon\Delta$ one has $\int_\Omega \sum_{i\in I} |1_{\Gamma^i(\omega)}(x) - 1_{\Gamma^i(\omega)}(y)|\, d\mu(\omega) \le \frac{2d(x,y)}{\delta\Delta}$

§3.2: Construction

Let (X, d) be a metric space and $R > r > 0$. Let $C(X, R, r)$ be the largest cardinality of a set $N \subseteq X$ satisfying $\forall x \ne y \in N$ $r \le d(x,y) \le R$

Lemma 3.11: Let $\Delta > 0$. Every (X, d) admits an $(\varepsilon, \frac{1}{2})$-padded Δ-bounded finitely supported decomposition of X wrt itself, with $\varepsilon = \frac{1}{256 \log C(X, 2\Delta, \frac{\Delta}{4})}$

Proof: Consider a 4Δ-padded decomposition; assume N is a Δ-net for X.

Firstly we need to introduce a distribution over partial orders $\prec$ on N st for every ball $B \subset X$ of radius 3Δ, $\prec$ is a uniformly total order on $B \cap N$. ($|B \cap N|$ is finite)

We consider the infinite graph $G = (N, E)$, where $\{x,y\} \in E$ iff $d(x,y) \le 3\Delta$.

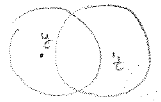
Then $\deg G \le C(X, 6\Delta, \Delta) \le C(X, 8\Delta, \Delta) < \infty$, therefore G admits a proper colouring using a finite number ($M \le C(X,6\Delta,\Delta)+1$) of colouring classes.

These classes $1, 2, \dots, M$. Let $\chi : N \to \{1, \dots, M\}$ be a proper colouring and let σ be a random permutation on $\{1, \dots, M\}$. Define $x \prec y$ by $\sigma(\chi(x)) < \sigma(\chi(y))$

It is easy to see that for every ball of radius 3Δ, every point in $B \cap N$ receives a unique colour and σ induces a uniform r.p on B.

Let us choose $R \in [\Delta, 2\Delta]$ uniformly at random, and for each $y \in N$ define a cluster

$$C_y = \{x \in X : x \in B(y,R) \text{ and if } x \in B(z,R), \text{ then } y \prec z\}$$



$C_y \cap C_z = \emptyset$ if $y \prec z$

Let $P = \{C_y\}_{y \in N}$ be a partition of X and $\mathrm{diam}(C_y) \le 2R \le 4\Delta$.

For $y \in N$ let γ_y be the minimal element of N in C_y. Now fix $t \in [0, \Delta]$ and $x \in X$. Note that $P[B(x,t) \subset \Gamma(w,x)] = 1 - P[B(x,t) \not\subset \Gamma(w,x)]$ (*)

$\Gamma(w,x)$ cuts $B(x,t)$.

There is $z \in N$ s.t. $C_z \cap B(x,t) \neq \emptyset$, let y be an element of it. Then $d(z,x) \le d(x,y) + d(z,y) \le t + f \le t + 2\Delta < 3\Delta$. Then $N \cap B(x, 3\Delta)$ ...

Let $W = B(x, 2\Delta + t) \cap N$ and $m = |W| \le C(X, 6\Delta, \Delta)$. Arrange the points $w_1, \dots, w_m \in W$ in order of increasing distance from x. Let $I_k = [d(x,w_k) - t, d(x,w_k) + t]$

Now let $\mathcal{E}_k$ be the event that w_k is a minimal element in W (for $\prec$) for which C_{w_k} cuts $B(x,t)$: $\forall\, 1 \le k \le m$ $P[\mathcal{E}_k \mid R \in I_{m_0}] \le \frac{1}{k}$?

The event $\mathcal{E}_k$ implies $R \in I_k$

N.B. Consider $1, \dots, m$. w_k appears before $w_1, \dots, w_{k-1}$

By contradiction, if we assume $\exists\, 1 \le i \le k-1$ s.t. $w_i \prec w_k$, then by minimality of w_k C_{w_i} should not cut $B(x,t)$: (i) $C_{w_i} \supset B(x,t)$

(ii) $C_{w_i} \cap B(x,t) = \emptyset$

(*) continues into $P[B(x,t) \text{ cut } \Gamma(\omega,x)] \le \sum_{k=1}^{m} P[\mathcal{E}_k] \le \sum_{k=1}^{m} P[R\in I_k]\, P[\mathcal{E}_k \mid R\in I_k]$

$$\le \sum_{k=1}^{m} \frac{2t}{\Delta}\frac{1}{k} \le \frac{2t}{\Delta}(1+\log m) \le \frac{\Delta t}{8}\log C(X; 6\Delta, \Delta)$$

Another decomposition in the case of compact X.

Th 3.17: If σ is a nondegenerate Borel measure on X (ie $\sigma(B(x,r)\subset X)>0$), then $\forall \Delta>0$ $\exists$ Δ-decomposition $\int_\Omega \sum_{i\in I} |1_{\Gamma^i(\omega)}(x) - 1_{\Gamma^i(\omega)}(y)|\, d\mu(\omega)$

$$\le \frac{2d(x,y)}{\Delta}\left[1+\log\frac{\sigma(B_X(x,5\Delta))}{\sigma(B_X(x,\Delta))}\right]$$

Proof: Let $\Omega' = \left(\prod_{i=1}^{\infty} X, \mu'\right)$ and $\Omega = \Omega' \times [2\Delta, 4\Delta]$ and $P = \mu' \times \lambda$.

Consider $\Omega' \ni \omega = (x_1, x_2, x_3, \dots)$ and $R\in[2\Delta,4\Delta]$. Let us construct a seq of subsets of Y, $\{\Gamma^i(\omega,k) : i\in\mathbb{N}\}$, $\Gamma^k(\omega,k) = B(x_k, k)\setminus \bigcup_{i=1}^{k-1}\Gamma^i(\omega,k)$.

<u>Claim</u>: $\{\Gamma^i(\omega,k)\cap S\}_{i\in\mathbb{N}}$ is a partition of $S = \{y\in Y : d(X,y) < \frac{\Delta}{2}\}$

For every $x\in X$ $j_R(x) = \inf\{j : d(x_j, x)\le R\}$: $j_R(x)$ is finite for all $x\in B(X, 2\Delta)$ with probability 1: $\mu'\{j_R(x) = \infty\} = 0 = \prod_{i\in\mathbb{N}} \underbrace{\sigma(d(x_i,x) > R)}_{\text{uniformly} < 1} = 0$.

Fix $x\in Y$ st $d(x,X) < \frac{\Delta}{2}$ and denote ν the distribution $\nu([\alpha,\beta]) = \sigma(\{z\in X : \alpha\le d(z,x)\le\beta\})$

Fix $t\in\Delta$ and let D_R be the set of all $z\in X$ st $B(x,t)$ is cut by $B(z,R)$.

Finally, let $\Omega_{i,R} = \{\omega\in\Omega' : \omega_i\in D_R\}\setminus \bigcup_{j=1}^{i-1}\{\omega\in\Omega' : \omega_j\in D_R\}$

Consider $\Omega'\ni\omega\in\Omega_{i,R}$: then $\underline{R\in[d(x,\omega_i)-t, d(x,\omega_i)+t]}$.

[Let $y\in B(\omega_i,R)\cap B(x,t)$: so $d(x,\omega_i)\le d(x,y)+d(y,\omega_i)\le t+R$

Let $y\in B(x,t)\setminus B(\omega_i,R)$: so $d(x,\omega_i)\ge d(y,\omega_i) - d(x,y)\ge R-t$.

We have $\underline{\Delta < d(\omega_i, x)\le 5\Delta}$.

[If $d(\omega_i,x)\le\Delta$, i.e. $\omega_i\in B(x,\Delta)$, then $B(x,t)\subset B(x,\Delta)\subset B(\omega_i,R)$ because $t\le\Delta$ and $R\ge 2\Delta$, which is in contradiction with our def^{n} of « cuts »

then $d(\omega_i,x)\le R+t\le 4\Delta+\Delta = 5\Delta$.

If $\omega\in\Omega_{i,R}$, $d(\omega_j,x) > d(\omega_i,x)$ for all $j<i$.

For all $l>0$ $\mu'(\Omega_{i,R} \mid d(x,\omega_i) = l) \le 1_{\{R\in[l-t,l+t]\}}\cdot 1_{\{\Delta<l\le 5\Delta\}}\ \mu'\{\omega : 1\le j\le i, d(\omega_j,x) \dots$

$$\le \qquad\qquad [1-\nu[0,l]]^{i-1}$$

Hence $\mathbb{P}[\, B(x,t) \text{ is cut}] = \frac{1}{2\Delta}\int_{2\Delta}^{4\Delta}\left(\sum_{i=1}^{\infty}\mu'(\Omega_{i,R})\right)dR$

$$= \frac{1}{2\Delta}\int_{2\Delta}^{4\Delta}\sum_{i=1}^{\infty}\int_0^{\infty}\mu'\big(\Omega_{i,R}\,(d(x,\omega_i)=\rho)\big)\,d\nu(\rho)\,dR$$

$$\leq \frac{1}{2\Delta}\int_{2\Delta}^{4\Delta}\sum_{i=1}^{\infty}\int_{[\Delta,5\Delta]}\mathbb{1}_{\{R\in[\ell-t,\ell+t]\}}\,[\,1-\nu([0,\rho])]^{i-1}\,d\nu(\rho)\,]\,dR$$

$$\leq \frac{t}{\Delta}\int_{\Delta}^{5\Delta}\frac{d\nu(\rho)}{\nu[0,\rho]} \leq \frac{t}{\Delta}\ \underset{?}{}\ \frac{\nu(0,5\Delta)}{\nu[0,\Delta]}$$

if $d(x,y) < \Delta$ and $t = d[x,y)$

Def: (Y,d) metric space, $X\subset Y$ subspace, $(\Omega,\mathcal{B},\mu)$ measure space?
Given $K>0$, $\psi:\Omega\times Y\to[0,\infty[$ is a K-gentle partition of unity wrt X if

① $\forall x\in Y\setminus\overline{X}$ $\omega\mapsto\psi(\omega,x)$ is measurable and $\int_\Omega\psi(\omega,x)\,d\mu(\omega)=1$.

② $\forall\omega\in\Omega$ $\forall x\in X$ $\psi(\omega,x)=0$

③ $\exists\gamma:\Omega\to\overline{X}$ $\forall x,y\in Y$ $\int_\Omega d(\gamma(\omega),x)\,|\psi(\omega,x)-\psi(\omega,y)|\,d\mu(\omega)\leq K d(x,y)$.

Th 4.1: $\exists C>0$ $\forall Y>X$ ① Let $n\in\mathbb{Z}$. If Y is an (ε,δ)-thick 2^n-bounded SD wrt X, then Y is a $\frac{C}{\varepsilon\delta}$-gentle partition of unity wrt X.

② Idem for (ε,δ)-padded SD

③ Idem for (ε,δ)-separating SD with $\frac{C}{\varepsilon\delta}$ replaced by $\frac{C(\delta+\varepsilon)}{\varepsilon\delta}$

Proof: Let φ be a 2 Lipschitz map with supp $\varphi\subset[\frac{1}{2},1]$ and $\varphi\equiv1$ on $(1,2]$.
Let $\varphi_n(x)=\varphi\left(\frac{d(x,X)}{\varepsilon 2^{n-3}}\right)$. Let (Ω,μ) be the disjoint union of $\{I\times\Omega_n\}_{n\in\mathbb{Z}}$.

Let $\psi(i,\omega,x)=\frac{1}{s(x)}\,\vartheta^n_\omega(x)\,\varphi_n(x)\,\mathbb{1}_{\Gamma^i_n(\omega)}(x)$, where $\forall n\in\mathbb{Z}$ $\forall x\in Y$ $\omega\mapsto\vartheta^n_\omega(x)\in[0,\infty]$ is μ_n-integrable,
and $S(x)=\sum_{n\in\mathbb{Z}}\sum_{i\in I}\int_{\Omega_n}\vartheta^n_\omega(x)\,\varphi_n(x)\,\mathbb{1}_{\Gamma^i_n(\omega)}(x)\,d\mu_n(\omega)$, and $\psi(\omega,X)=0$ if $x\in\overline{X}$.

We want to compute $\sum_{n\in\mathbb{Z}}\sum_{i\in I}\int_{\Omega_n} d(\gamma(i,\omega),x)\,|\psi(i,\omega,x)-\psi(i,\omega,y)|\,d\mu(\omega)\leq\begin{cases}\frac{C}{\varepsilon\delta}d(x,y) & \text{for ①②}\\ \frac{C(\varepsilon+\delta)}{\varepsilon\delta}d(x,y) & \text{in ③}\end{cases}$

Claim 4.2: Fix $\omega\in\Omega_n$ and assume that $\psi(i,\omega,x)\neq\psi(i,\omega,y)$. Then $d(\gamma(i,\omega),x)\leq b(x,y)+\frac{18}{\varepsilon}\max\{d(x,X),d(y,X)\}$

Proof: $\psi(i,\omega,x)>0$ or $\psi(i,\omega,y)>0$. $d(\gamma(i,\omega),x)\leq d(\gamma(i,\omega),\Gamma^i_n(\omega))+\operatorname{diam}(\Gamma^i_n(\omega))$
$\leq 2d(X,\Gamma^i_n(\omega))+2^n\leq 2d(x,X)+2^n$.

$\varphi_n(x)>0$, $\frac{d(x,X)}{\varepsilon 2^{n-3}}\geq\frac{1}{2}$, so that $d(x,X)\geq\varepsilon 2^{n-4}$ and $2^n\leq\frac{16}{\varepsilon}d(x,X)$

As $0<\varepsilon\leq1$, $2d(x,X)\leq\frac{2}{\varepsilon}d(x,X)$

(a) $\psi(i,\omega,y)>0$ for $y\in\Gamma^i(\omega)$.

Then $d(\gamma(i,\omega),x) \le d(\gamma(i,\omega),y) + d(X,y)$

$$\le d(\gamma(i,\omega),\Gamma_n^i(\omega)) + \mathrm{diam}(\Gamma_n^i(\omega)) + d(x,y)$$

$$\le 2\, d(y,X) + 2^n 7\, d(x,y)$$

and $I \le d(x,y) \sum_n' \sum_i' \int_{\Omega_n} [\psi(i,\omega,x) + \psi(i,\omega,y)]\, d\mu_n(\omega)$

$$+ \frac{18}{\varepsilon} \max\{d(x,X), d(y,X)\} \sum_n' \sum_i' \int_{\Omega_n} |\psi(i,\omega,x) - \psi(i,\omega,y)|\, d\mu_n(\omega)$$

$$\le 2\, d(x,y) + \ldots$$

$d(x,y) < d(\{x,y\},X)$; $d(x,X) \geq d(y,X)$.

$$I \le \frac{2}{\delta(x)} \sum_n' \int_{\Omega_n} \sum_i |\vartheta_\omega^n(x)\varphi_n(x)\mathbb{1}_{\Gamma_n^i}(x) - \vartheta_\omega^n(y)\varphi_n(y)\mathbb{1}_{\Gamma_n^i}(y)|\, d\mu_n(\omega).$$

1st Case: $\vartheta_\omega^n(x) = \Pi_\omega^n(x)$ with $\Pi_\omega^n(x) = \sum_{i\in I} \min\{d(x, X\setminus\Gamma_n^i(\omega)), 2^n\}$

Let n_0 s.t. $\frac{d(x,X)}{\varepsilon 2^{n_0-3}} \in [1,2]$. Then $\rho(x) = \sum_{n\in\mathbb{Z}}' \varphi_n(x) \int_{\Omega_n} \Pi_\omega^n(x)\, d\mu_n(\omega)$

2nd case: $\vartheta_\omega^n(x) = g\left(\frac{\Pi_\omega^n(x)}{\varepsilon 2^{n-1}}\right)$ with $g = \begin{cases} 1 & \text{if } x \geq 2 \\ x-1 & \text{if } 1 \le x \le 2 \\ 0 & \text{if } 0 \le x \le 1 \end{cases}$

3rd case: $\vartheta_\omega^n(x) = 1$