3e cours

Definition: Let $\lambda \in]0,1]$: We say that X is a W_λ-space if
$$f \in C^1_u(B_X) \Rightarrow f \in C_{wsc}(\lambda B_X)$$

Fact: If X is a W_1-space and $\ell_1 \not\hookrightarrow X$. Then, for all B-space Y and ultrafilters $\mathcal{V}$,
$$T \in C^1_u(B_X; Y),\ x^{**} \in \lambda B_{X^{**}},\ x_\alpha \xrightarrow[\alpha \to \mathcal{V}]{} x^{**} \Rightarrow T(x_\alpha) \xrightarrow[\alpha \to \mathcal{V}]{w^*} x_\alpha.$$

Proof: (Recall that $\tilde{T}$ was defined through a special ultrafilter $\mathcal{U}$.)

In fact, $\tilde{T}$ is uniformly w^*-to-w^*-continuous.

Fix any $f \in Y^*$: $f \circ T \in C^1_u(B_X)$, so that it is C_{wsc}-continuous on λB_X

We proved last time that $\ell_1 \not\hookrightarrow X \Rightarrow f \circ T \in C_{wu}(\lambda B_X)$.

Therefore there are $\varphi_1, \dots, \varphi_n$ functionals so that $|\varphi_i(x) - \varphi_i(y)| < \delta$ implies $|f \circ T(x) - f \circ T(y)| < \epsilon$. So these objects witness that $\tilde{T}$ satisfies the same

Example: every Schur space X is a W_1-space. E.g., ℓ_1 is W_1.
↳ every wsC seq is norm convergent.

Th: Let $\ell_1 \not\hookrightarrow X$ and $T \in C^k_u(B_X; Y)$, $k \geq 1$. Then TFAE:

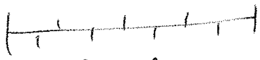
1. $T \in C_{wsC}$
2. $T \in C_{wu}$
3. $dT \in C_{wu}(B_X; \mathcal{L}(X;Y))$ and $dT(x) \in \mathcal{L}_K(X;Y)$.
4. $dT \in C_K(B_X; \mathcal{L}(X;Y))$ and $dT \in \mathcal{L}_K$.
5. $T \in C_K$
6. $d^jT \in C_{wu}(X; \mathcal{P}(^jX;Y))$ and $d^jT(x) \in \mathcal{P}_K(\quad)$ for $j<k$.
7. $d^jT \in C_K(X; \mathcal{P}(^jX;Y))$ . . . .

Fact: Suppose $T \in C^1_u(B_X; Y)$ with a modulus $\omega(\delta)$ of continuity of dT, i.e. $\|x-y\| < \delta \Rightarrow \|dT(x) - dT(y)\| \leq \omega(\delta)$. Then $T(x) - T(y) = \langle dT(y), x-y \rangle + \eta$ as soon as $|\eta| \leq \underbrace{\omega(\|x-y\|) \cdot \|x-y\|}_{\text{a quantitative } o(\|x-y\|)}$.

Theorem: If K is scattered, then $C(K)$ is a W_1-space.

Ex: c_0

Lemma: Let $x, y \in \ell_\infty^n$ with $\|x\|_\infty, \|y\|_\infty < \eta$; then there is a partition $A \cup B = \{1, \dots, n\}$, $A \cap B = \emptyset$ such that $\left|\sum_{i\in A} x_i - \sum_{i \in B} x_i\right|, \left|\sum_{i\in A} y_i - \sum_{i\in B} y_i\right| \leq 2\eta$

→ illustration:  for one vector, this is trivial

→ You can prove this for k vectors (2η to be replaced by kη)

Trick:

Consider $L: \ell_\infty^n \longrightarrow \mathbb{R}^2$. $L(u) = (u\cdot x, u \cdot y)$

This operator has a large kernel: $\exists v \in \ker L\ \forall i$ (but 2) $|v_i| = 1$.

→ Choose $v \in \text{Extremal}(\ker L) \cap B_{\ell_\infty^n}$: If more than two coordinates have $|v_i| \neq 1$. Then $v_{i_1}, v_{i_2}, v_{i_3}$ generate a 3-dim vectorspace in the kernel. But then v is not an extremal point.

$\rightarrow \exists\, x = (x_1, x_2, x_3, 0, \dots)\ L(x) = (0,0)$

$V \pm \varepsilon x \longrightarrow (0,0)$

Let $A = \{i : v_i = 1\}$, $B = \{i : v_i = -1\}$. Then $\sum_i v_i x_i = 0$,

so that $\left\|\sum_{i\in A} x_i - \sum_{i\in B} x_i\right\| \leq 2\eta$.

Lemma: Let $\omega(\cdot)$ be a fixed modulus function for the uniform continuity of df

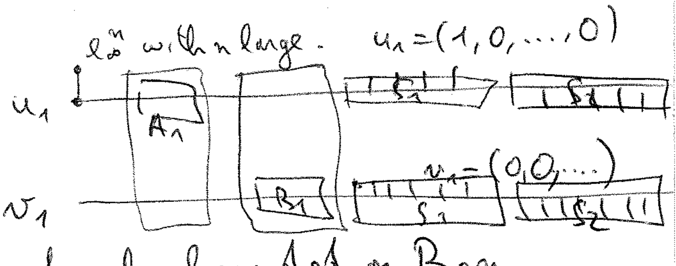
$f: B^+_{\ell_\infty^n} \longrightarrow \mathbb{R}$, $f(0) = 0$, $df(0) = 0$, f symmetric $[\, f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$ for $\pi \in S_n\,]$

Then $\forall \varepsilon > 0\ \exists n\ \forall f\ \forall i\ |f(e_i)| < \varepsilon$

"increasing dimension"

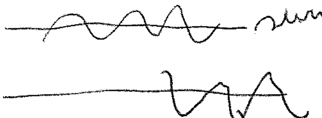
Proof: Idea: $df(u_1)$

$df(v_1) = 0$ because $v = 0$.



df is uniformly continuous and uniformly bounded on $B_{\ell_\infty^n}$

Then $\|df(x)\| < K$ for $x \in B$: K depends on $\omega(\delta)$



derivative normed by ℓ^1-norm.
How many coordinates (K) have $|\ | > \delta$? $A \cdot \delta < K$, i.e. $A < \frac{K}{\delta}$.

$\ell^n \ni \begin{cases} df(u_1) = ? \\ df(v_1) = 0 \end{cases}$ A_1 the set of i where $|df(u_1)_i| > \delta$
B_1 ———— $|df(v_1)_i| > \delta$, i.e. $B_1 = \emptyset$

By the lemma, there are two sets S_1, S_2 in $\{1,\dots,n\}\setminus(A_1\cup B_1)$ such that $\left|\sum_{i\in S_1} df(v_1)_i - \sum_{i\in S_2} df(v_1)_i\right| \le \delta$
and the same for u.

Now take $w_1 = \begin{cases} (w_1)_i = \rho \text{ if } i\in S_1 \\ (w_1)_i = -\rho \text{ if } i\in S_2 \\ \quad = 0 \text{ otherwise} \end{cases}$

We have $f(v_1+w_1) = f(v_1) + \langle df(v_1), w_1\rangle + \text{error terms} = f(v_1) + \rho\cdot 2\delta + \dots$ ← very small.

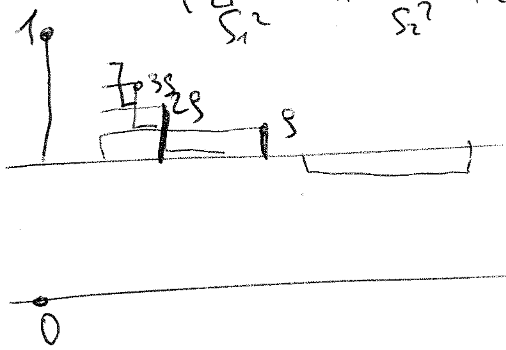
$f(u_1+w_1) = f(u_1) + \langle df(u_1), w_1\rangle + \text{——}$

Take $u_2 = u_1 + w_1$ and consider the derivative in S_1 and S_2:
$v_2 = v_1 + w_1$

We find S_1^2 and $S_2^2 \subset S_1$, $S_1^2 \cap S_2^2 = \emptyset$ s.t. $\left|\sum_{S_1^2} df(u_2)_i - \sum_{S_2^2} df(u_2)_i\right| < \delta$

$\left|\sum_{S_1^2} df(v_2)_i - \sum_{S_2^2} df(v_2)_i\right| < \delta$

Choose w_2: $(w_2)_i = \begin{cases} \rho \text{ if } i\in S_1^2 \\ -\rho \text{ if } i\in S_2^2 \\ 0 \text{ otherwise.} \end{cases}$



There are $k = \frac{1}{\rho}$ steps

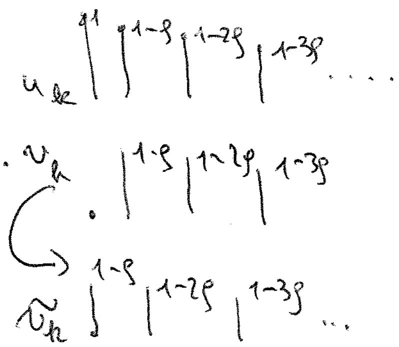
the final vectors are u_k and v_k.

At each step, $f(u_j) - f(u_{j+1}) \approx 0$

In the end, $|f(u_k) - f(u_1)| < \eta$ (k times the error term
$|f(v_k) - f(v_1)| < \eta$ ↳ $o(\rho)$, by uniformity)

the trick: the symmetry: the only coordinates we care about are those that remain: the "staircase vectors"

By symmetry, I can shift this by one.



Thus $\|u_k - \tilde{v}_k\| \le \rho$. This means f has the "same" value at u_k and $\tilde{v}_k$:

$|f(u_k) - f(\tilde{v}_k)| \le \omega(\rho)$

$|f(u_k) - f(u_1)| < \eta$ $\qquad |f(v_1) - f(u_1)| < 2\eta + \omega(\rho)$.

$|f(v_k) - f(v_1)| < \eta$ But this means that all $|f(e_i)| <$ same

$f(\tilde{v}_k) = f(v_k)$

But the only thing that refrains us from continuing our construction would have been that there are not enough coordinates. That is why n needs to be large enough.

Corollary: Let $\omega(\delta)$ be the modulus of continuity of df, $f(0)=0$, $df(0)=0$.

Then $\#\{i : |f(e_i)| > \delta\} < N(\omega(\cdot))$

Proof: Pass from f to $\tilde{f}$ the symmetrization of f.

If $f(e_i) > \delta$ for $i \in S$, then $\tilde{f} = \frac{1}{|S|!} \sum f \circ I_\pi$ with I_π, π permutation on the index set S

....

Using this, we conclude that if $f \in C_u^1(c_0)$ then f is wsC-continuous

Note: for any weakly null sequence $\{u_n\} \subset B_{c_0}$ $f(u_n) \longrightarrow 0$

- bumps that are describing l_∞^n: $f([u_i]_{i=1}^N)$
$\underset{\approx}{\ } l_\infty^n$

Corollary: $C_u^1 \subset C_{wsc}$. (but we want more: C_{wsC}, but we will lose the quantitative information)

$\rightsquigarrow$ argument by transfinite induction: $C(\alpha)$, α ordinal and for every $C(K)$, K scattered.

and then for every $C(K)$, but this is highly non trivial.