(1) 20/6/2012

Last course
Smoothness and weak continuity

We saw that

$C^1_u(B_{c_0}) \subset C_{wsc}(B_{c_0})$ and

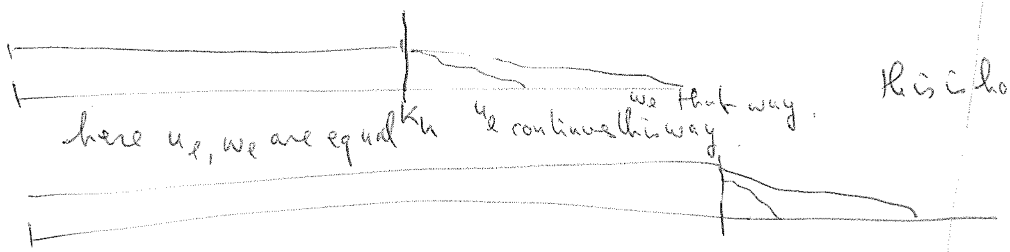
Theorem: If $X = C(K)$, K scattered, then $X \in W_1$, i.e. $C^1_u(B_X) \subset C_{wsc}(B_X)$, i.e. f maps w-Cauchy sequences into norm convergent sequences.

W_2 has been defined because at some points, W_1 is out of reach.

Indications for c_0: by contradiction, construction of $\{u_n\}, \{w_n\} \subset B_X$ that are part of a single w-Cauchy seq., and $f \in C^1_u(B_X)$.

Assume by contradiction that $\exists \beta > 0$ $f(u_n) > 2\beta$ and $f(w_n) = 0$ or viceversa.

Assume wlog that there are intervals $I_n = [0, K_n]$ st $u_\ell|_{I_n} = w_\ell|_{I_n}$ for $\ell \geq n$.

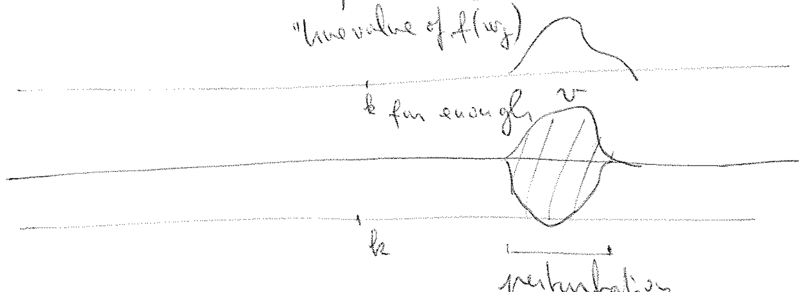


This is how wC sequences look like...

Lemma: Let (w_n) be w-Cauchy in c_0.
~~For all $\beta > 0$~~ There exists an infinite subset $M \subset \mathbb{N}$ and a $k \in \mathbb{N}$ st. for $v \in c_{00}$, supp $v \subset (k, \infty)$ the set $\{j \in M : |f(w_j + v) - 2\beta| > \beta\}$ is finite.

"true value of $f(w_j)$"

Let's suppose $f(w_j) = 2\beta$, $f(u_j) = 0$.



The statement is more complicated than the proof! By contradiction, suppose $\forall k$ $\forall M \subseteq \mathbb{N}$ infinite $\exists v$ finitely supported supp $v \subset (k, \infty)$ $\#\{j : |f(w_j + v) - 2\beta| > \beta\} = \infty$.

Inductive procedure: ~~This will~~ Choose be v_1 for $k = 1$, $M_1 = \mathbb{N}$.
Let M_2 be the subset given by the assumption
Let K_2 be such that supp $v_2 \subset [0, K_2]$.
Let M_3 ...

then for every $j \in M_{n+1}$ $|f(w_j + v_{e_h}) - 2\beta| > \beta$ for $h < n$.

[I will also achieve that $M_1 \supset M_2 \supset M_3 \supset \dots$]

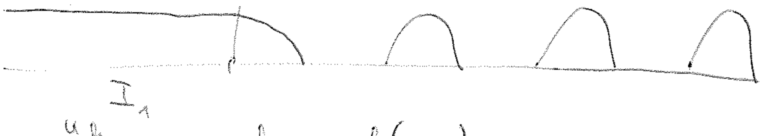
By shifting w_j to the origin, this is equivalent to having $f(0)$, $f(e_n)$ $0 \leftrightarrow w_j$, as e_n is equivalent to v_{e_n}.

then $f(0) = 2\beta$ and $f(e_n)$ differs by β.

this can only happen a limited # of times by our quantitative information.

We may now finish the proof:

Assume that the index set $\mathbb{N}$ was replaced by the M of the previous lemma



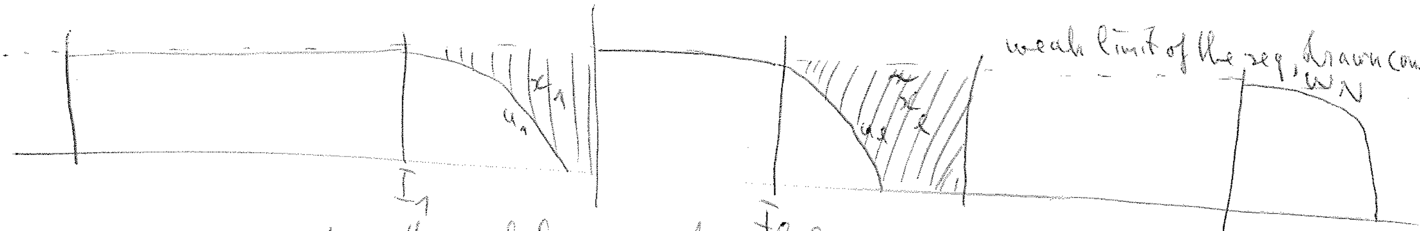
I_1

u_1 where $f(u_n) = 0$

w_1 $f(w_n) = 2\beta$

By the main lemma, adding these blocks will make us achieve the same value:

$\exists \ell \ \forall u \ \operatorname{supp} u \subset [\ell, \infty[\ f(u_1 + u) = f(u_1)$



- you can keep the whole w_N and cancel x_1 : $w_N - x_1$
- you can look at u_1 + far-away block :

$f(\qquad\qquad) = f(u_1) = 0$

$\exists M \subset \mathbb{N}$, $\exists h$ $h \in I_\ell$ $|f(w_j + h) - f(w_j)| < \beta$ except for finitely many j.

Therefore $f(w_N - x_1) = f(w_N) \pm \beta$ ↯

this proves that ~~for~~ any wC seq must be convergent.

Thus $c_0 \in W_1$. For any $C(K)$, here is what I do: reduce first to $C([0,\alpha])$ with α a countable ordinal. Then reason similarly.

By known machineries, it extends to any $C(K)$, K scattered.

This was the 1st round

Next round: for Banach-space valued:

Theorem: Let K be scattered, $X = C(K)$, Y arbitrary B-space. Let $T \in C^1_u(B_X; Y)$.

Then TFAE: (1) $T \in C_{wsc}(B_X;Y)$ (2) $T \in C_{wsC}(B_X;Y)$ (3) $T \in C_K(\ ;\ )$

(4) $T \in C_{wK}(\ ;\ )$ (5) $T \in C_{wu}(\ ;\ )$

(6) $T|_{\{x_i\}}$ is convergent for every $\{x_i\} \sim \{e_i\}$ canonical basis of c_0

(7) $d\tilde{T}(x^{**}) \in \mathcal{L}_K(X^{**}, Y^{**})$ (8) $d\tilde{T}(x^{**}) \in \mathcal{L}_{wK}$

Def: Let us say that X has $\circledast$ if $C^1_u(B_{c_0}; X) \subset C_{wsC}(B_{c_0}; X)$

Theorem: X has $\circledast$ if any of the following is true;

(†) (1) $c_0 \not\hookrightarrow X^{**}$ ($\Leftrightarrow \ell_1 \not\hookrightarrow X^*$) (3) X has PCP

(2) X is a dual space and $c_0 \not\hookrightarrow X$ (4) X has (u) and $c_0 \not\hookrightarrow X$

(5) X is weakly seq. complete. (6) $\cdot X$ has cotype $\cdot$

Consider $\mathcal{L}(\ell_\infty; X) = \mathcal{L}_{wK}(\ell_\infty; X)$ provided $c_0 \not\hookrightarrow X$ (Rosenthal)

Thus wK is for free...

Main problem: Is it so that X has $\circledast \Leftrightarrow c_0 \not\hookrightarrow X$?

$\rightarrow$ If $T: c_0 \xrightarrow{C^1_u} X$, is T compact?

3rd level of results: Take $X = C(K)$ for any K compact

(4th level: $\mathcal{L}_{\infty,\lambda}$ spaces)

Theorem: Let K be compact, $X = C(K)$, $\{x_n\}$ wC in B_X, $\varepsilon > 0$

then there is $Z \hookrightarrow X^{**}$ s.t. $Z \overset{\text{isometrically}}{\cong} C[0,\alpha[$, α a countable ordinal

and there is $\{y_n\} \subset Z$ wC s.t. $\|x_n - y_n\| < \varepsilon$ for all n.

[This is a variation on Zippin's lemma]

$\rightarrow$ this transfers structure in X to structure in $C[0,\alpha[$

"Z has another direction in X^{**} than X but contains a seq. point by point close to the original sequence.

Sketch of proof: assume K is metric. Then $X^* = M(K)$ and $X^{**} = ??$
very huge.

But $\{x_n\}$ is wC, then $x = w^*\text{-}\lim x_n$ lives in X^{**}

Consider $x = \tau_p$-limit of the x_n, x Baire 1-f^n (Borel)

Because K is compact and $\{x_n\}$ is uniformly bounded, ~~Bes~~ the Lebesgue thm yields that if $\mu \in M(K)$ $\int x_n d\mu = \int x d\mu$

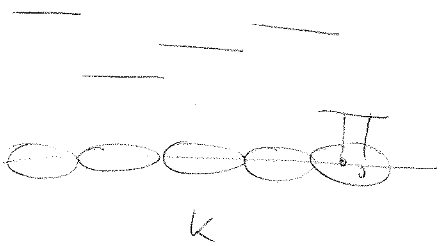
Theorem: $x = \tau_p\text{-}\lim \{x_n\}$: fix $\varepsilon = \frac{1}{n}$; $\Delta = \{\frac{j}{n}; -n \le j \le n\}$

and replace x by a discretely valued-sequence.

Idea: If x is discretely valued: for every $t \in K$, look at $x_n(t)$: it is eventually constant

Then define equivalence class where the x_n are constant.

↳ a workaround to get K to be countable.



Theorem: Let $X = C(K)$, K compact, (1) ~~either~~ $C^1_u(X,Y) \cap C_{wK} \subset C_{wsC}(B_X;Y)$

(2) ~~A~~ if Y has (*) then $C^1_u(B_X;Y) \subset C_{wsC}(B_X;Y)$

N.B.: in the 2nd case, $C^1_u(B_X;Y)$ need not be weakly compact.

Sketch of proof: Let $T \in C^1_u \cap C_{wK}$. Assume by contradiction that there is $\{x_n\}$ wC in X st $\{T(x_n)\}$ is ε-not convergent [not ε-close to a convergent sequence]

1st step: extend T to $\tilde{T}: B_{X^{**}} \to Y^{**}$

By a Zippin-style lemma, there is $\{y_n\}$ and $Z \hookrightarrow X^{**}$, $Z = C[0,\alpha]$
$\to = C(K)$, K scattered.

such that for all k $|\tilde{T}(y_n) - \tilde{T}(x_n)| < \frac{\varepsilon}{4}$

i.e., $\tilde{T} \in C^1_u(Z)$ with same modulus for $d\tilde{T}$

Then $\{\tilde{T}(y_n)\}$ is not $\frac{\varepsilon}{2}$-convergent. End of proof.

Note that T wK $\Rightarrow \tilde{T}$ wK.

There are many other results!

Theorem: If X is $\mathcal{L}_{\infty,\lambda}$, then $C^1_u(B_X;Y) \cap C_{wK} \subset C_{wsC}(\lambda B_X;Y)$

and similarly there is a "(*)"-result

Proof: in fact, by a machinery

X^{**} is complemented in some $C(K)$ space.

$T \rightsquigarrow \tilde{T} \rightsquigarrow \tilde{\tilde{T}} : C(K) \to Y$

Theorem: If X is W_λ and $\ell_1 \not\hookrightarrow X$, assume that Y^* has type.

Then $C^1_u(B_X; Y) \subset C_K(\lambda B_X; Y)$

$\rightsquigarrow$ we have (*) is close to "$\ell_1 \not\hookrightarrow X^*$" and we want to strengthen it to having type.

Corollaries: • Theorem (Bates): $\forall X, Y \; \exists T : X \xrightarrow{\text{onto}} Y$, $T \in C^1$ and Lipschitz.

For higher smoothness, Bates proved (under weak assumptions on X) that

$\forall Y \; \exists T : X \xrightarrow{\text{onto}} Y$ C^∞-smooth.

→ Proof by Hájek: If X^* has type, then $\forall Y$ Banach $\nexists T : X \xrightarrow{\text{onto}} Y$ s.t. T is a polynomial

Ex: $X = \ell_2$, Y separable $\exists P : X \xrightarrow{\text{onto}} Y$ 2-homogeneous.

• There is no C^2-smooth surjection from $W_\lambda \ni X$, $\ell_1 \not\hookrightarrow X$ onto Y s.t. Y^* has type.

Note: (†) resembles Aharoni's Theorem.

$\hookrightarrow \Leftarrow$ X has shipped BCS.