Groupe de travail de Géométrie non-linéaire des espaces de Banach

A. Dalet, M. De La Salle, G. Lancien, S. Neuwirth, T. Prochazka, L. Sánchez González

2012-2013

Table des matières

1 Tony. Un résultat de Bourgain concernant le cotype de la famille $Lip(\Omega_n)^*$ où Ω_n sont les cubes de Hamming **2**

2 Tony. Cotype de $Lip(\ell_1)^*$ d'après Bourgain **4**

3 Gilles. Problèmes de déblais et remblais dans le plan d'après Naor et Schechtman **6**

4 Stefan. Produits de Riesz et ensembles de Sidon **12**

5 Aude. $\mathcal{F}([0,1]^2)$ ne se plonge ni uniformément ni grossièrement dans L_2, d'après Naor et Schechtman **15**

6 Gilles. Plongements uniformes et grossiers dans les espaces réflexifs d'après N. Kalton **18**

7 Tony. Plongements uniformes et grossiers dans les espaces réflexifs et la propriété Q d'après N. Kalton **22**

Étude de l'article de M. Mendel, A. Naor : *Nonlinear spectral calculus and super-expanders* **27**

8 Mikael. Propriété (T) renforcée et construction explicite de superexpanseurs (=expanseurs n'admettant aucun plongement grossier dans un espace de Banach K-convexe), d'après Vincent Lafforgue **27**

9 Mikael. Propriété (T) renforcée et construction explicite de superexpanseurs, d'après Vincent Lafforgue (suite) **32**

10 Gilles. Introduction à la théorie spectrale des graphes réguliers et des graphes expanseurs, plan de l'article **37**

Tony. Conservation de la constante de Poincaré par rapport aux produits de graphes (d'après Mendel et Naor) **46**

Luis. Metric Markov Cotype implies the decay of Poincaré constants **46**

11 Aude. Super-reflexive Banach spaces have metric Markov cotype **46**

12 Aude. Super-reflexive Banach spaces have metric Markov cotype (suite) **52**

13 Mikael. Every B-convex Banach space is K-convex, after Pisier **57**

14 Mikael. Every B-convex Banach space is K-convex, after Pisier (suite) **61**

15 Gilles. Construction of Mendel and Naor's base graphs **68**

16 Tony. Discretization of $e^{-t\Delta}$ w.r.t. Poincaré inequalities (proof of Lemma 7.2) **74**

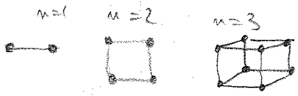
---

17 Luis. Sup-partitions of unity and smooth approximation on Banach spaces **78**

Tony
5/10/2012

Définition : $\Omega_n = \{-1,1\}^n$ muni de $d(x,y) = \sum |x_i - y_i|$ est le <u>cube de Hamming</u> de dim n

- C'est $\{-1,1\}^n$ vu comme partie de ℓ_1
- C'est un graphe muni de sa distance canonique :



- On peut le construire par récurrence : $\{-1,1\}^n$ et $\{-1,1\}^n$ et on connecte les sommets correspondants

Théorème (Bourgain 1986) : $\exists K>0\ \forall k \geq 0\ \exists f(k)\ \ell_1^k$ est K-isomorphe à un sous-espace K-complémenté de $\mathcal{L}ip_0(\Omega_{f(k)})$

[on posera $0 = (-1,-1,\dots,-1)$]

Corollaire : ℓ_∞^k est K-isomorphe à un sous-espace K-complémenté de $\mathcal{L}ip_0(\Omega_{f(k)})^*$

En particulier, $\ell_\infty^k \underset{\text{unif}}{\subseteq} \mathcal{L}ip_0(\ell_1)^*$ et le cotype de $\mathcal{L}ip_0(\ell_1)^*$ et de $\mathcal{F}(\ell_1)$ sont triviaux.

Démonstration : ⓐ Soit $p_k : \mathcal{L}ip_0(\Omega_{f(k)}) \longrightarrow \underbrace{\mathcal{L}ip_0(\Omega_{f(k)})}_{Y_k}$ la projection telle que $P(Y) \simeq \ell_1^k$.

Alors $p_k^* : Y_k^* \longrightarrow Y_k^*$ est une projection telle que $P_k^*(Y_k^*) \simeq \ell_\infty^k$

ⓑ $i : \Omega_{f(a)} \underset{\text{isométrie}}{\hookrightarrow} \ell_1$

$r : \mathcal{L}ip_0(\ell_1) \longrightarrow Y_k$ où on pose $0 = (\underbrace{-1,\dots,-1}_{f(a)\text{ fois}}, 0, 0, \dots)$

et $r^* : Y_k^* \longrightarrow \mathcal{L}ip_0(\ell_1)^*$ est une isométrie.

ⓒ résulte du théorème de Maurey et Pisier : $\ell_\infty^n \underset{\text{unif}}{\subseteq} X \Rightarrow$ cotype $X = \infty$

De plus, $\mathcal{F}(\ell_1)^{**} = (\mathcal{L}ip_0(\ell_1))^*$ et de manière générale, si cotype $X^{**} = \infty$, alors cotype $X = \infty$: X^{**} est finiment représenté dans X.

[Rappel : • Soit M un espace métrique pointé et $0 \in M$. $\mathcal{F}(M)$ est le prédual de $\mathcal{L}ip_0(M)$ correspondant à la fermeture des Diracs dans $\mathcal{L}ip_0(M)^*$. C'est l'espace Lipschitz-libre de M]

[• On dit que X a cotype q si $\exists C>0$ $\left(\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i\|^q\right)^{\frac{1}{q}} \geq C\left(\sum_{i=1}^n \|x_i\|^q\right)^{\frac{1}{q}}$]

Mendel et Naor ont défini le cotype q pour les espaces métriques.

Une tentative précédente avait été de définir le cotype non-linéaire par le cotype de $(\mathcal{L}ip_0(M))^*$.

Mais on voudrait que cela coïncide avec le cotype pour les espaces de Banach.

On ℓ_1 a cotype 2 et $(\mathcal{L}ip_0(\ell_1))^*$ a cotype ∞.

Preuve du théorème : on va prouver que $\ell_1^k \underset{K_2, C}{\subseteq} \mathcal{L}ip_0\left(\mathbb{Z}_{j(k)}^{2^k}\right)$ où $\mathbb{Z}_j = \mathbb{Z}/j\mathbb{Z} = \{e^{2i\ell\pi/j}\}$

ou plutôt, que $\ell_1^k \underset{K_3, C}{\subseteq} \mathcal{L}ip_0(\mathbb{T}^{2^k})$ muni de la distance normalisée

[ici, X^n est muni de $d(x,y) = \sum_{i=1}^{n} d(x_i, y_i)$.],

ou plutôt, que $\ell_1^k \underset{K_3 C}{\subseteq} \underbrace{W^{1,\infty}(\mathbb{T}^{2^k})}_{W}$, l'espace de Sobolev de toutes les f$^{\text{ns}}$ mesurables bornées telles que $\sup_{1 \le s \le 2^k} \|\partial_s f\|_\infty < \infty$

$\uparrow$ dérivée partielle distributionnelle.

Plus précisément, $\exists P_k : W \to W$, $\|P_k\| \le K_3$, tel que $P_k(W) \subset \underset{-N \le m \le N}{\mathrm{span}}\, e^{imt} = W'_N \subset \mathcal{L}ip_0$

Soient F_m les noyaux de Fejér sur $\mathbb{T}^{2^k}$.

Pour $f \in \mathcal{L}ip_0(\mathbb{T}^{2^k})$, $f * F_m \in W'_m$

et $f \in W^{1,\infty}(\mathbb{T}^n)$, $f * F_n \xrightarrow{\text{spf}} f$

Suite de l'argument : Soit $(k_j) \subset \mathbb{N}$ croissant assez vite et fixons $k = J$ $= \dim \ell_1^k$

Soit $\sigma : \{1, \dots, 2^J\} \to \Omega_J$, $((\sigma(s))_{s=1}^{2^J}$ suite de tous les J-uplets.

$$P_J : W \to W$$
$$f \mapsto \sum_{j=1}^{J} \hat{f}(k_j \sigma_j(1), \dots, k_j \sigma_j(2^J))\, e^{2\pi k_j \sum_{s=1}^{2^J} \sigma_j(s) t_s}$$

P se factorise en
$$\begin{tikzcd} W \arrow[rr, "P"] \arrow[dr, bend right, "P_1"'] & & W \\ & \ell_1 \arrow[ur, bend right, "P_2"'] & \end{tikzcd}$$
où $P_1 : f \mapsto (\hat{f}(a_j))_{j=1}^{J} \in \ell_1$

$$P_2 : \ell_1 \to W$$
$$e_j \mapsto \frac{1}{k_j} e^{2\pi a_j \sum_{s=1}^{2^J} \sigma_j(s) t_s}$$

Tony 26/10/2012

Article de Bourgain [1986] : suite du 5/10/2012

Proposition (Bourgain 1986) : $\exists K_1\ \forall k\in\mathbb{N}\ \ell_1^k \overset{K_1}{\simeq} X \overset{K_1}{\underset{c}{\subseteq}} \mathrm{Lip}_0(\mathbb{T}^{2^k})$ où $\mathbb{T}^{2^k}$ est le ℓ^1-produit de $(\mathbb{T},d)$, d distance normalisée

On a vraiment besoin de la complémentation : il suffit de mq $\exists K_2\ \forall k\in\mathbb{N}\ \ell_1^k \overset{K_2}{\simeq} X \overset{K_2}{\underset{c}{\subseteq}} W^{1,\infty}(\mathbb{T}^{2^k})$

où $W = W^{1,\infty}(\mathbb{T}^{2^k}) = \{f \text{ nulle en } 1 : \forall 1\le s\le 2^k\ \partial_s f\in L_\infty(\mathbb{T}^{2^k})\}$. En effet, soit F_N le noyau de Fejér de $\mathbb{T}^{2^k}$: $F_N(\vartheta)=\prod_{s=1}^{2^k}\sum_{j=-N}^{N}\frac{N-|j|}{N}e^{2i\pi j\vartheta_s}$. Si on considère C_N l'opérateur de convolution par F_N sur $\mathrm{Lip}_0(\mathbb{T}^{2^k}) \to W$, alors $\|C_N\|=1$ et $C_N\circ i \xrightarrow[\text{ponctuellement}]{\text{simplement}} Id_W$ (i.e. $(F_N * f)' = F_N * f' \to f'$), où $i: W \hookrightarrow \mathrm{Lip}_0(\mathbb{T}^{2^k})$

Si on a

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "p"'] & \mathrm{Lip}_0(\mathbb{T}^{2^k}) \arrow[d, "C_N"] \\ X & X \subseteq W \end{tikzcd}$$

il existe N tel que $\|C_N\circ i|_X - Id_W|_X\|<\varepsilon$ et donc $\|iC_N|_{i(X)} - Id_{Lip}|_{i(X)}\|<\varepsilon$ et $\|(iC_N|_{i(X)})^{-1}\| < \frac{1}{1-\varepsilon}$ et $\|C_N|_{i(X)}^{-1}\|<\frac{1}{1-\varepsilon}$

Soit $(k_i)_{i=1}^{\infty}\subset\mathbb{N}$ une suite croissant assez vite et fixons $k=J$ et $\sigma:\{1,\dots,2^J\}\xrightarrow{\text{bijection}}\{-1,1\}^J$

Soit $P_1(f) = \left(k_j \,\mathrm{Im}\,\hat f(k_j\sigma_j)\right)_{j=1}^J$ et $\sigma_j = (\varepsilon_s)_{s=1}^{2^J}$

$$\begin{tikzcd} W \arrow[rr, "P"] \arrow[dr, "P_1"'] & & W \\ & \ell_1^k \arrow[ur, "P_2"'] & \end{tikzcd}$$

$P_2 e_j := \frac{i}{k_j}\left(\underbrace{e^{2\pi i k_j(\sigma_j\cdot\vartheta)}}_{\chi_j} - \underbrace{e^{-\cdots}}_{\chi_{-j}}\right)$ où $\sigma_j\cdot\vartheta = \sum_{s=1}^{2^J}\sigma_j(s)\vartheta_s$. Alors $\|P_2e_j\|=4\pi$

Comme les e_j sont les points extrémaux de la boule unité de ℓ_1^k, $\|P\|=4\pi$. On a aussi $P_1P_2e_j = e_j$

En effet, on a $\widehat{\chi_j}(k_i\sigma_i) = \begin{cases}1 & \text{si } i=j\\ 0 & \text{sinon}\end{cases}$ et $\mathrm{Im}\,\hat f(k_j\sigma_j) = \frac{\hat f(k_j\sigma_j)-\hat f(-k_j\sigma_j)}{2i}$ car f à valeurs réelles.

Cherchons à majorer $\|P_1\|$: Définissons
$$A:\mathbb{T}^{2^J}\to\mathbb{R}^+,\quad \vartheta\longmapsto\prod_{j=1}^J(1+\cos 2\pi k_j\sigma_j\cdot\vartheta) = \prod_{j=1}^J\left(1+\frac{\chi_j(\vartheta)+\chi_{-j}(\vartheta)}{2}\right)$$

C'est un produit de Riesz : $A\ge 0$, $\int_{\mathbb{T}^{2^k}} A(\vartheta)=1$.

Lemme : Soit $k_1>0$ et $k_j > 2\sum_{i<j}k_i$. Alors $\varepsilon=(\varepsilon_i)\in\{0,-1,1\}^{<\omega}\longmapsto\sum_{i=1}^{|\varepsilon|}\varepsilon_ik_i$ est "presque injective" et nulle ssi $\varepsilon=0$

Preuve : par récurrence sur $|\varepsilon|$: si $|\varepsilon|=1$ ✓, si $|\varepsilon|=j-1$ c'est vrai pour, soit $\sum_{i<j}\xi_ik_i \pm k_j = \sum_{i<j}\varepsilon_ik_i$.

Alors $\pm k_j = \sum_{i<j}(\varepsilon_i-\xi_i)k_i$ et donc $k_j\le 2\sum_{i<j}k_j$ ↯.

On voit alors que $\|f*A\|_W\le\|f\|_W$ car $\partial_s(f*A)(\vartheta) = (\partial_s f)*A(\vartheta)$.

et $f*A(\vartheta) = \hat f(0) + \sum_{j=1}^J\{D_{j,+}(\vartheta)\chi_j(\vartheta) + D_{j,-}(\vartheta)\chi_{-j}(\vartheta)\}$ où $D_{j,\pm}(\vartheta) = \frac12\sum_{\varepsilon\in\{0,\pm1\}^{j-1}}\underbrace{\frac{\prod_{i<j}\chi_{\varepsilon_i i}(\vartheta)}{2^{\|\varepsilon\|_1}}\times\hat f(\varepsilon_i k_i\sigma_i \pm k_j\sigma_j)}_{L_{\pm j,\varepsilon}(\vartheta)}$

En effet, $f*\chi_j = \hat f(j)\chi_j$.

Observation : $L_{j,\varepsilon} = \overline{L_{-j,-\varepsilon}}$ car f réelle.

$$\partial_s(f*A)(\vartheta) = (-2\pi)\sum_{j=1}^J\sum_{\varepsilon}{}'\left(\underbrace{\overbrace{k_j\sigma_j(s)}^{\text{correspond à la }\Sigma\text{ qu'on veut majorer}}}_{a_j} + \underbrace{\overbrace{\sum_{i<j}\varepsilon_ik_i\sigma_i(s)}^{\text{perturbation}}}_{b_{j,\varepsilon}}\right)\underbrace{\mathrm{Im}\,L_{j,\varepsilon}\chi_j}_{c_{j,\varepsilon}} \quad (*)$$

Bourgain dit ici que $f * A \approx \sum 2\pi k_j \sigma_j(s) \left(D_{j,+} \chi_j - D_{j,-} \chi_{-j}\right)$

- Si $\left| \sum_{j=1}^{J} \sum_{\varepsilon} k_j \sigma_j(s) \operatorname{Im} L_{j,\varepsilon}(\vartheta) \chi_j(\vartheta) \right|$ atteint son max en ϑ, alors tous les termes de cette double somme sont positifs à condition que k_j croît vite (croisse) ["on a affaire à un ensemble de Sidon"] : $\circledast \geq \frac{1}{2} \left| \sum_{j=1}^{J} \sum_{\varepsilon}' \underbrace{k_j \sigma_j(s)}_{a_j} \underbrace{\operatorname{Im} L_{j,\varepsilon} \chi_j(\vartheta)}_{c_{j,\varepsilon}} \right|$ (#)

En effet, par l'hypothèse sur les k_j, $a_j c_j = |a_j c_{j,\varepsilon}| > 2 |b_{j,\varepsilon} c_{j,\varepsilon}|$ pour tous j, ε.

Donc $a_j c_{j,\varepsilon} + b_{j,\varepsilon} c_{j,\varepsilon} \geq \frac{1}{2} a_j c_{j,\varepsilon}$

Or (#) $= \left| \sum_{j=1}^{J} k_j \sigma_j(s) \operatorname{Im} D_{j,+} \chi_j(\vartheta) \right| = \sum' k_j \underbrace{\left| \operatorname{Im} D_{j,+} \chi_j(\vartheta) \right|}_{(**)}$

et si on sait que $(**) \geq \int_{\mathbb{T}^{2^k}} \left| \operatorname{Im} D_{j,+} \underbrace{\chi_j(\vartheta)}_{1} \right| dt$, alors on aurait que $\geq \left| \int_{\mathbb{T}^{2^k}} \operatorname{Im} D_{j,+} \right|$; or $\int \operatorname{Im} D_{j,+} = \hat{f}(k_j \sigma_j) - \hat{f}(-k_j \sigma_j)$

et (#) serait $\geq \underbrace{\sum' k_j \left| \hat{f}(k_j \sigma_j) - \hat{f}(-k_j \sigma_j) \right|}_{\|P_1 f\|_1}$

Quand on passe des ℓ_1^n à ℓ_1, on note que ℓ_1 est complémenté dans W, mais on n'en sait rien quant à $L_{p_0}(\mathbb{T}^{2^k})$!

La suite : • version quantitative du non-plgt grossier des espaces L-libres de $c_0(\ell_p)$ Naor-Giladi-Oleszkiewicz

- obstruction "spectral gap", plgt de graphe, matrice d'adjacence dont la première vaut $\lambda_1 = 1$, et $1 - \lambda_2$ est le spectral gap $\to$ Mendel-Naor.
- $\to$ reconstruction de graphes expanseurs qui ne se plgt dans aucun superréflexif.

19/10/2012 (1)

Gilles

D'après Naor-Schechtman :

"Planar earthmover is not in L_1"

<u>Problème du transport optimal</u> (X,d) métrique

$\mu = \sum_{i=1}^{n} a_i \delta_{x_i}$ avec $a_i > 0$

$\nu = \sum_{j=1}^{m} b_j \delta_{y_j}$ avec $b_j > 0$ où $\sum a_i = \sum b_j = M$.

On va considérer $d(x_i, y_j)$ comme le coût du transport de x_i vers y_j

$\pi(x_i, y_j) \geq 0$ la masse transportée de x_i vers y_j

Ainsi $\sum_i \pi(x_i, y_j) = b_j$ et $\sum_j \pi(x_i, y_j) = a_i$: on va noter $\pi \in E_{\mu,\nu}$ "plan de transfert"

On a $E_{\mu,\nu} \neq \emptyset$ puisque $\pi(x_i, y_j) = \frac{a_i b_j}{M} = \mu \otimes \nu$ marche

Le pb est de minimiser $I(\pi) = \sum_{i,j} \pi(x_i, y_j)\, d(x_i, y_j)$

Si $\tau(\mu,\nu)$ est le minimum, on définit ainsi une distance sur $\mathcal{S}_1(X) = \mathrm{conv}\{\delta_x : x \in X\}$ $(M=1)$

Cela remonte au mémoire <u>Sur les déblais et remblais</u> de Gaspard Monge (1780)

Espace Lipschitz libre : $0 \in X$: $\mathrm{Lip}_0(X) = \{f : X \xrightarrow{\mathrm{Lip}} \mathbb{R} \mid f(0) = 0\}$

Alors $\delta_x : \mathrm{Lip}_0(X) \to \mathbb{R}$, $f \mapsto f(x)$ est une f.l. On pose $\mathcal{F}(X) = \overline{\mathrm{vect}}^{\|\cdot\|}\{\delta_x : x \in X\}$ dans $\mathrm{Lip}_0(X)^*$

Alors $\mathrm{Lip}_0(X)$ est le dual de $\mathcal{F}(X)$

Ex : $L_\infty(\mathbb{R}) \to \mathrm{Lip}_0(\mathbb{R})$, $f \mapsto \left(x \mapsto \int_0^x f(t)\,dt\right)$ isométrie surjective w^*-continue

et ainsi $\mathcal{F}(\mathbb{R}) \equiv L^1(\mathbb{R})$

Nous allons alors interpréter le résultat de Naor-Schechtman comme : $\mathcal{F}(\mathbb{R}^2) \not\hookrightarrow L_1$.

Autre description de $\mathcal{F}(X)$:

Soit $m \in \mathcal{F}(X)$ de la forme $m = \sum_{\text{finie}} m(x)\delta_x$ avec $\sum m(x) = 0$

[Weaver appelle cela une molécule.] Notons $\mathcal{M}$ leur espace.

On peut écrire $m = \mu_+ - \mu_- = \mu - \nu$ avec μ et ν comme précédemment !

Une molécule élémentaire est $m_{x,y} = \delta_x - \delta_y$.

Alors m se décompose en molécules élémentaires : à chaque plan de transfert correspond une telle décomposition $\sum_{i,j} \pi(x_i, y_j) m_{x_i, y_j}$

La norme d'Arens-Eels est $\|m\|_{Æ} = \inf \left\{ \sum_{i,j} \pi(x_i, y_j) d(x_i, y_j) : \pi \in E_{\mu,\nu} \right\}$
$= \tau(\mu, \nu) = \tau(m_+, m_-)$

- Soit $m = \sum \pi(x_i, y_j) m_{x_i, y_j}$. Si $f \in \mathrm{Lip}_0(X)$,
$$|\langle m, f\rangle| = \left|\sum_{i,j} \pi(x_i, y_j)\left(f(y_j) - f(x_i)\right)\right| \leq \left(\sum_{i,j} \pi(x_i, y_j) d(x_i, y_j)\right) \mathrm{Lip}_0 f$$
En prenant l'inf en π, $|\langle m, f\rangle| \leq \|m\|_{Æ} \mathrm{Lip}_0(f)$ et donc $\|m\|_{\mathcal{F}(X)} \leq \|m\|_{Æ}$

- Soit $\phi \in (\mathcal{M}, \|\cdot\|_{Æ})^*$ et posons $f(x) = \phi(\delta_x - \delta_0)$: $f(0) = 0$ et
$|f(x) - f(y)| = |\phi(\delta_x - \delta_y)| \leq \|\phi\| \|\delta_x - \delta_y\|_{Æ} \leq \|\phi\| . d(x,y)$ et donc $\mathrm{Lip} f \leq \|\phi\|_{Æ^*}$

- Si m est de nouveau de la forme ci-dessus,
$$\langle m, f\rangle = \sum_{i,j} \pi(x_i, y_j)\left(f(x_i) - f(y_j)\right) \leq \left(\sum_{i,j} \pi(x_i, y_j) d(x_i, y_j)\right) \|\phi\|_{Æ}$$
On a $\langle m, f\rangle = \phi(m)$ et par Hahn-Banach, on trouvera f qui fait aussi bien : donc $\|m\|_{\mathcal{F}(X)} \geq \|m\|_{Æ}$

On a démontré une variante élémentaire du théorème de Kantorovitch-Rubinstein :
$$\boxed{(\mathcal{P}_1(X), \tau) \longrightarrow (\mathcal{F}(X), \|\cdot\|_{\mathcal{F}(X)}) \text{ est une isométrie.}}$$

Le théorème de Naor et Schechtman en est une forme quantitative :

Th Soit $X = \{0, \dots, n-1\}^2 \subseteq \mathbb{R}^2$ euclidien et $F : \mathcal{S}_1(X) \longrightarrow L_1$ est telle que pour $\mu, \nu \in \mathcal{S}_1(X)$ $\tau(\mu,\nu) \leq \|F(\mu) - F(\nu)\|_1 \leq L\,\tau(\mu,\nu)$.

Alors $L \geq C\sqrt{\log n}$

Cor $\mathcal{F}(\mathbb{R}^2) \not\hookrightarrow L_1$.

Fixons n.

n.b. : $\delta_0 = 0_{\mathcal{F}(X)}$ et donc $\delta_x = \delta_x - \delta_0$ est une molécule

Lemme 1 : $\exists N\ \exists T : (\mathcal{M}, \|\ \|_{AE}) \overset{\text{complété de}}{\cong} \mathcal{F}(X) \longrightarrow \ell_1^N$ linéaire $\forall m \in \mathcal{M}$ $\|m\|_{AE} \leq \|Tm\|_{\ell_1^N} \leq 2L\|m\|_{AE}$

Dém : Soit μ_0 la mesure uniforme et supposons en toute généralité que $F(\mu_0) = 0$

Soit $\mu \in \mathcal{M}$ telque $\mu = \mu_+ - \mu_-$: $\mu_+ = \sum a_i \delta_{x_i}$
$\mu_- = \sum b_j \delta_{y_j}$ $x_i \neq y_j$

Soit $\pi \in E_{\mu^+,\mu^-}$: $\sum_{i,j}' \pi(x_i, y_j)\, \underbrace{d(x_i, y_j)}_{\geq 1} \geq \sum_{i,j}' \pi(x_i, y_j) = \mu^+(X) = \mu^-(X)$

Ainsi $\|\mu\|_{AE} \geq \|\mu^+(X)\| \geq \max|\mu(x)|$

Ainsi $\psi : B_{\mathcal{M}} \longrightarrow \mathcal{S}_1(X)$
$\mu \longmapsto \sum' \frac{\mu(x)+1}{n^2} \delta_x$ où $|\mu(x)| \leq 1$
et donc $\sum' \frac{\mu(x)+1}{n^2} = 1$ car $\sum' \mu(x) = 0$.

Si $\mu, \nu \in B_{\mathcal{M}}$, $\tau(\psi(\mu), \psi(\nu)) = \frac{1}{n^2} \sup_{f \in B_{Lip_0(X)}} \left|\sum_x' (\mu(x) - \nu(x)) f(x)\right|$

$= \frac{1}{n^2} \|\mu - \nu\|_{AE}$

Ainsi $\tau(\psi(\mu), \psi(\nu)) = \frac{1}{n^2} \tau(\mu, \nu)$

On considère $h : B_{\mathcal{M}} \xrightarrow{n^2\psi} \mathcal{S}_1(X) \xrightarrow{F} L_1$. Alors $h(0) = 0$ et

$\tau(\mu,\nu) \leq \|h(\mu) - h(\nu)\|_1 \leq L\,\tau(\mu,\nu)$.

Pour étendre h à tout l'espace : $\mu \longmapsto ((k\,h(\frac{\mu}{k}))_k)_{\mathcal{U}}$ On a distorsion$(\tilde{h}) \leq L$:
$\tau(\mu,\nu) \leq \|\tilde{h}(\mu) - \tilde{h}(\nu)\| \leq L\,\tau(\mu,\nu)$.

$\mathcal{M}$ est séparable et donc $\tilde{h}(\mathcal{M}) \subseteq Z \equiv L_1$.

Par Heinrich-Mankiewicz ('80), il existe $U : \mathcal{M} \longrightarrow L_1^{**}$ linéaire telque

$\forall \mu \in \mathcal{M}$ $\|\mu\| \leq \|U\mu\| \leq L\|\mu\|$ (1)

[Il existe des points de G-différentiabilité préfaible de $\tilde{h}$ telsque $D\tilde{h}(\mu_0)$ vérifie (1)] $\mathcal{M} \underset{L}{\hookrightarrow} L_1^{**}$

Comme $\dim M < \infty$, la réflexivité locale donne $M \underset{3/2 L}{\hookrightarrow} L_1$

De plus, $\forall F \subset L_1 \; \exists G \subset L_1$ $G \underset{1+\varepsilon}{\sim} \ell_1^N$ tel que $F \subseteq G$.

$\rightarrow$ ce sont les fonctions étagées qui "composent" ℓ_1^N

Donc $M \underset{2L}{\overset{\sim}{\hookrightarrow}} \ell_1^N$.

Idée de la suite : $T : M \to \ell_1^N$ a un adjoint $T^* : \ell_\infty^N \to Lip_0(X)$ tel que $\|T^*\| \leq 2L$

$Lip_0(X) \subset \ell_2(X)$: on regarde $\mathcal{F} \circ T^*$ ($\mathcal{F}$ transformée de Fourier) $T(B_{\ell_\infty^N}) \supset B_{Lip_0(X)}$

Étape 1 : utiliser $\|T^*\| \leq 2L$ pour montrer que $\mathcal{F}(T^*(B_{\ell_\infty^N})) \subseteq \{y \in \ell_2(X) : |y| \leq 4\}$

Étape 2 : utiliser $T(B_{\ell_\infty^N}) \supset B_{Lip_0(X)}$ pour prouver que $\|x\|_2 \geq c n \sqrt{\log n}$ avec $\|x\|_2 \leq 4Ln$.

Ccl : On obtient $L \geq C\sqrt{\log n}$.

Soit $W = \{f : X \to \mathbb{R} \mid f(0) = 0\}$ muni de $\|f\|_W = \sum_{i=0}^{n-1} \sum_{j=0}^{n-2} |f(i,j+1) - f(i,j)| + n \sum_{i=0}^{n-2} |f(i+1,0) - f(i,0)| + n \sum_{i=0}^{n-2} |f(0,j+1) - f(0,j)|$

C'est un espace de Sobolev discret.

Notons $I_1 : Lip_0(X) \to W$ identité

$I_2 : W \to \ell_2(X)$

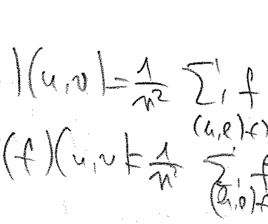
$F : \ell_2(X) \to \ell_2(X)$ défini par $F(f)(u,v) = \frac{1}{n^2} \sum_{(k,l) \in X} f(k,l) e^{\frac{2i\pi}{n}(ku+lv)}$

$\mathcal{F} : \ell_2(X) \to \ell_2(X)$ où $\mathcal{F}(f) = \operatorname{Im} F(f)(u,v) = \frac{1}{n^2} \sum_{(k,l) \in X} f(k,l) \sin \frac{2\pi}{n}(ku+lv)$

Lemme 2 : (a) $\|I_1\| \leq 4n(n-1)$

(b) $\|I_2\| \leq \frac{1}{2}$

(c) $\|\mathcal{F}\| \leq \frac{1}{n}$.



Dém : (a) compter les termes.

(b) $|f(u,v)| \leq \sum_{l=0}^{v-1} |f(0,l+1) - f(0,l)| + \sum_{k=0}^{u-1} |f(k+1,v) - f(k,v)| = \cancel{A(v)}$

$\leq \sum_{l=0}^{n-2} |f(0,l+1) - f(0,l)| + \sum_{k=0}^{n-2} |f(k+1,v) - f(k,v)| = A(v)$.

et aussi $|f(u,v)| \leq \sum_{k=0}^{n-2} |f(k+1,0) - f(k,0)| + \sum_{l=0}^{n-2} |f(u,l+1) - f(u,l)| = B(u)$

Alors $\|f\|_2^2 = \sum_{u,v} |f(u,v)|^2 \leq \sum_{u,v} B(u) A(v) \leq \frac{1}{4} \left[\sum B(u) + \sum A(v)\right]^2$

Or $\sum_{v=0}^{n-1} A(v) = n \sum_{l=0}^{n-2} |f(0,l+1)-f(0,l)| + \sum_{v=0}^{n-1}\sum_{h=0}^{n-2} |f(h+1,v)-f(h,v)|$

Dem $\sum_{u=0}^{n-1} B(u) = n \sum_{k=0}^{n-2} |f(k+1,0)-f(k,0)| + \sum_{u=0}^{n-1}\sum_{l=0}^{n-2} |f(u,l+1)-f(u,l)|$

et donc $\|f\|_2^2 \leq \frac{1}{4}\|f\|_W^2$.

(c) exprime le fait que σ est multiple d'un unitaire.

Maintenant, si Y, Z sont des Banach, et $A : Y \longrightarrow Z$, $\pi_1(A) = \inf\{K>0 : (*)\}$

$(*)$ $\forall y_1, \dots y_m \in Y \ \exists y^* \in S_{Y^*}$ $\sum \|Ay_j\|_Z \leq K \sum_{j=1}^m |y^*(y_j)|$

Lemme 3 : $\pi_1(I_1) \leq 4n(n-1)$, $I_1 : Lip_0(X) \longrightarrow W$, et donc $\pi_1(\sigma^{-1} I_2 \sigma I_1 \sigma^*) \leq 4n^2$

Dém : Si $f_1, \dots, f_m \in Lip_0(X)$, $\sum_{j=1}^m \|f_j\|_W = \sum_{s=0}^{n-1}\sum_{t=0}^{n-2} \sum_j \big(|f_j(s,t+1)-f_j(s,t)|$
$+ |f_j(t+1,s)-f_j(t,s)|\big)$
$+ n\sum_{t=0}^{n-2}\sum_j [|f_j(0,t+1)-f_j(0,t)| + |f_j(t+1,0)-f_j(t,0)|]$

Considérons alors $y^* = \delta_{(s_0,t_0+1)} - \delta_{(s_0,t_0)} \in S_{Lip_0(X)^*}$ $\leq 4n(n-1)\sum_j |f_j(s_0,t_0+1)-f_j(s_0,t_0)|$ par exemple.

On a bien $\sum_{j=1}^m \|f_j\|_W \leq 4n(n-1) \sum |y^*(f_j)|$ pour un certain s_0, t_0.

Lemme 4 (Factorisation de Pietsch dans un cadre très particulier) Si $A : \ell_\infty^N \longrightarrow Y$, il existe $(\sigma(k))_{k=1}^N$, $\sigma(k) \geq 0$, $\sum_{k=1}^N \sigma(k) = 1$ telle que si $x \in \ell_\infty^N$, $\|Ax\|_Y \leq \pi_1(A) \underbrace{\sum \sigma(k)|x(k)|}_{\|x\|_{L_1(\sigma)}}$

Dem : Si $x_1, \dots, x_m \in \ell_\infty^N$, $\sum_{j=1}^m \|Ax_j\|_Y \leq \pi_1(A) \sup_{x^* \in S_{\ell_1^N}} \sum |x^*(x_j)|$

car $B_{\ell_1^N} = \operatorname{conv}(\pm e_k : k \leq N)$ $= \pi_1(A) \sup_{k \leq N} \sum |x_j(k)|$

$$\begin{tikzcd}
\ell_\infty^N \arrow[rr] \arrow[dr, "Id"'] & & Y \\
& L^1(\sigma) \arrow[ur] &
\end{tikzcd}$$

Or si on prend $x_j = e_j$, $\sum_{j=1}^N \|Ae_j\|_Y \leq \pi_1(A) \times 1$

$\sigma(k) = \dfrac{\|Ae_k\|_Y}{\sum_{j=1}^N \|Ae_j\|_Y}$, on obtient

$\|Ax\|_Y = \|\sum x(k) A(e_k)\|_Y \leq (\sum \|Ae_j\|_Y) \times \sum \sigma(k)|x(k)|$
$\leq \pi_1(A) \times \sum \sigma(k)|x(k)|$

Lemme 5 : bornitude encore : si $R : (\mathbb{R}^N, \|\ \|_\sigma) \longrightarrow \ell_2$ et $f : \{1, \dots N\} \to [0,\infty[$,

alors $\exists\, x \in \ell_2^+$ $|g| \leq f \Rightarrow |Rg| \leq x$ avec $\|x\|_2 \leq \|R\|\, \|f\|_{L^1(\sigma)}$.

Écrivons $R = [R_{ij}]_{i \in \mathbb{N}}^{j=1,\dots,N}$: $(Rf)_i = \sum_{j=1}^N R_{ij} f(j)$

⑧

Considérons $R^*: \ell_2 \to (\mathbb{R}^N, \|\ \|_\sigma^*)$, où $\|\varphi\|_\sigma^* = \max \left|\frac{\varphi(j)}{\sigma(j)}\right|$.

Ici, $\|R\| = \|R^*\| = \sup_{\sum |f(i)|^2 \le 1} \max_{j \le N} \left|\frac{R^* f(j)}{\sigma(j)}\right| = \sup \max \left|\frac{\sum R_{ij} f(i)}{\sigma(j)}\right|$

$= \max_j \frac{1}{\sigma(j)} \sup_{\sum |f(i)|^2 \le 1} |\sum_i R_{ij} f(i)| = \max_j \frac{1}{\sigma(j)} (\sum_i |R_{ij}|^2)^{1/2}$

Posons $x_i = \sum_j |R_{ij}| |f(j)|$: $|(Rg)_i| = |\sum_j R_{ij} g(j)| \le x_i$ et $\|x\|_2 = \left[\sum_i (\sum_j R_{ij} f(j))^2\right]^{1/2}$

$\le \sum_j \left(\sum_i |R_{ij} f(j)|^2\right)^{1/2}$

Car $\|\ \|_{\ell_p(\ell_q)} \le \|\ \|_{\ell_q(\ell_p)}$ si $p \ge q$.

Ici, c'est simple puisque $\|\ \|_{\ell_\infty(J, \ell_2(I))} \le \|\ \|_{\ell_2(I, \ell_\infty(J))}$.

$\sup_j \left(\sum_i |a_{ij}|^2\right)^{1/2} \ge \left(\sum_i \sup_j |a_{ij}|^2\right)^{1/2}$

$\sup_j \sum_i |a_{ij}|^2 \ge \sum_i \sup_j |a_{ij}|^2$

fin : $\|x\|_2 \le \sum_j (\sum_i R_{ij} f(j))^2)^{1/2}$

$\le \sum_{j=1}^N \sigma(j) f(j) \frac{1}{\sigma(j)} \left(\sum_i |R_{ij}|^2\right)^{1/2}$

$\le \|R\| \sum_{j=1}^N \sigma(j) f(j) = \|R\| \|f\|_\sigma$.

Preuve du théorème : $A = \mathcal{F} \circ I_2 \circ I_1 \circ T^* : \ell_\infty^N \to \ell_2(X)$: $\pi_1(A) \le 4nL$

Alors $T^*(B_{\ell_\infty^N}) \supset B_{\mathrm{Lip}_0(X)}$. Soit $(k,l) \in \{0, \dots, n-1\}^2$ et $(u,v) \in \{1, \dots, n-1\}^2$

Alors $\varphi_{u,v}(k,l) = \frac{1}{u+v} \sin\left(\frac{2\pi}{n}(ku + lv)\right)$ satisfait $\mathrm{lip}\, \varphi_{u,v} \le \frac{2\pi}{n}$.

implique qu'existe $\Phi_{u,v} \in B_{\ell_\infty^N}$ avec $\|\Phi_{u,v}\|_\infty \le \frac{2\pi}{n}$ et $T^*(\Phi_{u,v}) = \varphi_{u,v}$

et pour $j \le N$ $|\Phi_{u,v}(j)| \le \frac{2\pi}{n}$ $\|f\|_\sigma = \frac{2\pi}{n}$ et $f(j) = \frac{2\pi}{n}$.

On va en déduire qu'existe $x \in \ell_2^+(X)$ tel que pour tout u,v $|A(\Phi_{u,v})| \le x$

avec $\|x\|_2 \le 4nL \times \frac{2\pi}{n} = 8\pi L$: il suffit de combiner les lemmes 4 et 5. (inégalité ponctuelle)

$\forall u,v \; |x_{u,v}| \ge A(\phi_{u,v})(u,v)$

$A(\phi_{u,v})(u,v) = \mathcal{F}(\varphi_{u,v})(u,v) = \frac{1}{n^2} \sum_{(k,l) \in X} \frac{1}{u+v} \sin^2\left(\frac{(ku+lv) 2\pi}{n}\right)$

$= \frac{1}{u+v} \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} \left[-\frac{1}{4} e^{i\frac{2\pi}{n}(2ku+2lv)} - \frac{1}{4} e^{-\frac{i2\pi}{n}(2ku+2lv)} + \frac{1}{2}\right]$

$= \begin{cases} \frac{1}{2}(u+v) & \text{si } (u,v) \ne (\frac{n}{2}, \frac{n}{2}) \\ 0 & \text{sinon} \end{cases}$

Ainsi $(8\pi L)^2 \ge \|x\|_2^2 \ge \frac{1}{4} \sum_{u=0}^{n-1} \sum_{v=0}^{n} \frac{1}{(u+v)^2} \ge c \log n$

Q ouverts $\ell_\infty^n \overset{c}{\hookrightarrow} \mathcal{F}(\ell_1)$

$\ell_\infty^n \overset{c}{\hookrightarrow} \mathcal{F}(\mathbb{R}^2)$

Produits de Riesz et ensembles de Sidon.

$\{z\in\mathbb{C} : |z|=1\} =$ 

Définition : Soit G groupe abélien compact ($G=\mathbb{T}$) et Γ son dual, noté additivement ($\Gamma=\mathbb{Z}$). Soit $\Lambda\subset\mathbb{Z}$. Λ est Sidon de constante S si pour tout polynôme trigonométrique $f=\sum_{\lambda\in\Lambda}\hat{f}(\lambda)e_\lambda$ ($e_\lambda(z)=z^\lambda$) on a

$$(*)\qquad \|f\|_{A(\mathbb{T})} \equiv \sum |\hat{f}(\lambda)| \le S \sup_{z\in\mathbb{T}} |f(z)| \equiv S\,\|f\|_{\mathcal{C}(\mathbb{T})}$$

Caractérisation parmi d'autres : Λ est Sidon si $\widehat{M(\mathbb{T})}\big|_\Lambda = \ell^\infty(\Lambda)$
$\widehat{L^1(\mathbb{T})}\big|_\Lambda = c_0(\Lambda)$

$$(**)\qquad \forall (\varphi_\lambda)\in\ell^\infty(\Lambda)\ \exists \mu\in M(\mathbb{T})\quad (\forall\lambda\in\Lambda\ \hat{\mu}(\lambda)=\varphi_\lambda) \text{ et } \|\mu\|_{M(\mathbb{T})}\le S\,\|(\varphi_\lambda)\|_{\ell^\infty(\Lambda)}$$

"toute suite bornée peut être interpolée sur Λ par une mesure"

"toute suite nulle à l'infini ——— fonction intégrable"

La caractérisation qui va nous servir ici est :

$$(***)\qquad \forall (z_\lambda)_{\lambda\in\Lambda}\in\mathbb{T}^\Lambda\ \exists\mu\in B_{M(\mathbb{T})}\quad \forall\lambda\in\Lambda\quad \mathrm{Re}\,(z_\lambda\hat{\mu}(-\lambda))\ge\frac{1}{S}.$$

$(**)\Rightarrow(***)$ Soit $(z_\lambda)\in\mathbb{T}^\Lambda$: il existe μ tel que $\|\mu\|\le S$ et $\hat{\mu}(-\lambda)=\overline{z_\lambda}$.
Alors $\mathrm{Re}\,(z_\lambda\hat{\mu}(-\lambda))=1$ donc $\frac{\mu}{\|\mu\|}$ fait ce qu'on demande.

$(***)\Rightarrow(*)$ Soit f polynôme trigonométrique porté par Λ et $(z_\lambda)_{\lambda\in\Lambda}\in\mathbb{T}^\Lambda$ tel que $\hat{f}(\lambda)=z_\lambda|\hat{f}(\lambda)|$. Soit $\mu\in B_{M(\mathbb{T})}$ tel que $\mathrm{Re}(z_\lambda\hat{\mu}(-\lambda))\ge\frac{1}{S}$.

Alors
$$\begin{aligned}\|f\|_{\mathcal{C}(\mathbb{T})} \ge \mathrm{Re}\int_{\mathbb{T}} f\,d\mu &= \mathrm{Re}\sum_{\lambda\in\Lambda}\hat{f}(\lambda)\hat{\mu}(-\lambda)\\ &= \mathrm{Re}\sum_{\lambda\in\Lambda} z_\lambda|\hat{f}(\lambda)|\,\hat{\mu}(-\lambda)\\ &= \sum_{\lambda\in\Lambda}|\hat{f}(\lambda)|\,\mathrm{Re}\left(z_\lambda\hat{\mu}(-\lambda)\right)\ge\frac{1}{S}\sum_{\lambda\in\Lambda}|\hat{f}(\lambda)|\end{aligned}$$

$(*)\Rightarrow(**)$ Soit $\varphi\in\ell^\infty(\Lambda)$. Soit ℓ la forme linéaire sur le sous-espace des polynômes trigonométriques portés par Λ défini par $\ell(f)=\sum_{\lambda\in\Lambda}\varphi(\lambda)\hat{f}(\lambda)$: alors sa norme sur cet espace est bornée par $\|\varphi\|_{\ell^\infty(\Lambda)}\sup_{\|f\|_{\mathcal{C}(\mathbb{T})}\le 1}\sum|\hat{f}(\lambda)| = S\|\varphi\|_{\ell^\infty(\Lambda)}$. On l'étend en une f.l. sur $\mathcal{C}(\mathbb{T})$, c'est-à-dire en une mesure ν bornée par cela : $\ell(f)=\int f\,d\mu$. Donc $\hat{\nu}(-\lambda)=\ell(e_\lambda)=\varphi(\lambda)$.

$\left[\text{n.b. : si on considère } \mu(z) = \nu(-\bar{z}),\right.$ on a $\left.\int f(z)\,d\mu(z) = \int f(-z)\,d\nu(z)\right]$ ⑦

Rappel de Toni : Λ est qi [quasi-indépendant] si pour toute suite $\varepsilon = (\varepsilon_\lambda)_{\lambda\in\Lambda}$ à support fini et valeurs $\{-1,0,1\}$ on a $\sum'_{\lambda\in\Lambda} \varepsilon_\lambda \lambda \neq 0$ sauf si $\varepsilon = 0$.

Tout ensemble qi est Sidon. On va utiliser les

Produits de Riesz : ① $R_F = \prod_{\lambda\in F}\left(1 + \frac{e_\lambda + e_{-\lambda}}{2}\right)$ pour F fini. On a $R_F \geq 0$!

Dans ce produit, on va exactement trouver tous les produits de termes qui sont pour chaque $\lambda \in F$, soit 1 soit $\frac{e_\lambda}{2}$ soit $\frac{e_{-\lambda}}{2}$.

Si on code le choix fait pour chaque $\lambda \in F$ par $\varepsilon_\lambda = 0$, $\varepsilon_\lambda = 1$, $\varepsilon_\lambda = -1$, le terme s'écrit $\left(\frac{1}{2}\right)^{|\varepsilon_\lambda|} e_{\varepsilon_\lambda \lambda}$. Donc $R_F = \sum'_{\varepsilon\in\{-1,0,1\}^F} \left(\frac{1}{2}\right)^{\sum'|\varepsilon_\lambda|} e_{\sum \varepsilon_\lambda \lambda}$

et la qi indique que $\widehat{R_F}(0) = 1$: $\sum' \varepsilon_\lambda \lambda \neq 0$ sauf si $\varepsilon = 0$: donc R_F est une proba !
Donc, pour chaque $\gamma \in F$, $\widehat{R_F}(\gamma) = \sum_\varepsilon \left\{\left(\frac{1}{2}\right)^{\sum'|\varepsilon_\lambda|} : \gamma = \sum' \varepsilon_\lambda \lambda\right\} \leq 1$. et $\sum'_{\varepsilon \text{ à plusieurs termes}} \leq \frac{1}{2}$

Maintenant compliquons un peu la donne : pour $z = (z_\lambda) \in \mathbb{T}^\Lambda$, on considère
$$R_{F,a,z} = \prod_{\lambda\in F}\left(1 + a\,\frac{z_\lambda e_\lambda + \bar{z}_\lambda^{-1} e_{-\lambda}}{2}\right)$$
avec a un paramètre qu'on va optimiser, $0 \leq a \leq 1$

Comme avant, $R_{F,z}$ va être une proba : $\widehat{R_{F,z}}(0) = 1$ et regardons $\widehat{R_{F,a,z}}(\gamma)$ pour $\gamma \in \Lambda$: $R_{F,a,z} = \sum_\varepsilon \left(\frac{a}{2}\right)^{\sum'|\varepsilon_\lambda|} \left(\prod z_\lambda^{\varepsilon_\lambda}\right) e_{\sum \varepsilon_\lambda \lambda}$

Soit $\gamma \neq 0$ et (ε_λ) tel que $\gamma = \sum \varepsilon_\lambda \lambda$. Alors $\varepsilon_\lambda \neq 0$ (sinon passer le γ de l'autre côté) et si $\varepsilon_\gamma = 1$, $\varepsilon_\lambda = 0$ pour $\lambda \neq \gamma$ (simplifier γ). Si $\varepsilon_\gamma = -1$, il y a nécessairement d'autres termes ε_λ non nuls, à moins que $\gamma = -\gamma$ (penser aux Rademacher !)

Donc, si $\gamma \neq -\gamma$, $\widehat{R_{F,a,z}}(\gamma) = \frac{a}{2} z_\lambda + \widehat{r_{F,z}}(\gamma)$ avec
$$|\widehat{r_{F,z}}(\gamma)| \leq \sum'_{\substack{\varepsilon \text{ à support de}\\ \text{plus d'un point}}} \left\{\left(\frac{a}{2}\right)^{\sum'|\varepsilon_\lambda|} : \gamma = \sum' \varepsilon_\lambda \lambda\right\} \leq a^2 \sum' \left\{\left(\frac{1}{2}\right) : \right\} \leq \frac{a^2}{2}.$$

Considérons maintenant $R^*_{F,a,z} = \int_0^{2\pi} R_{F,ze^{it}}\, e^{-it} \frac{dt}{2\pi}$ moyenne !
$$= \sum' \left(\frac{a}{2}\right)^{\sum|\varepsilon_\lambda|} \left(\prod z_\lambda^{\varepsilon_\lambda}\right) e_{\sum \varepsilon_\lambda \lambda} \int_0^{2\pi} \underbrace{\prod e^{it\varepsilon_\lambda}}_{e^{it\sum \varepsilon_\lambda}} e^{-it} \frac{dt}{2\pi}$$
$$= \sum'_{\sum \varepsilon_\lambda = 1} \left(\frac{a}{2}\right)^{\sum|\varepsilon_\lambda|} \left(\prod z_\lambda^{\varepsilon_\lambda}\right) e_{\sum \varepsilon_\lambda \lambda}$$

C'est une mesure de norme ≤ 1 et $\widehat{R^*_{F,a,z}}(\gamma) = \frac{a}{2} z_\lambda + \widehat{r^*_{F,z}}(\gamma)$ avec
$$|\widehat{r^*_{F,z}}(\gamma)| \leq \widehat{r^*_{F,1}}(\gamma) \leq \sum' \left\{\left(\frac{a}{2}\right)^{\sum'|\varepsilon_\lambda|} : \sum \varepsilon_\lambda = 1,\ \gamma = \sum' \varepsilon_\lambda \lambda\right\} \leq a^3 \sum' \{\ \} \leq \frac{a^3}{2}.$$
et donc ε a au moins un support de 3 termes !

Donc $\mathrm{Re}\left(\bar{z}_\gamma \widehat{R^*_{F,a,z}}(\gamma)\right) \geq \frac{a}{2} - \frac{a^3}{2}$ optimisé quand $a = \frac{1}{\sqrt{3}}$ et $S \leq 3\sqrt{3}$

③

* 1^{er} produit de Riesz. Pour (φ_λ) dans $B_{\ell^\infty(\Lambda)}$, soit

$$R_{F,\varphi} = \prod_{\lambda\in F}\left(1 + \frac{\varphi_\lambda e_\lambda + \overline{\varphi_\lambda}\, e_{-\lambda}}{2}\right)$$

C'est une probabilité ;

Regardons quand, pour $\gamma \neq 0$, $\sum_{\lambda\in F} \varepsilon_\lambda \lambda = \gamma$.

Si Λ satisfait la propriété $\sum_{\lambda\in\Lambda} \varepsilon_\lambda \lambda = 0 \Rightarrow \varepsilon = 0$ pour toute suite Λ à support fini telle que $\varepsilon_\lambda \in \{-1,0,1\}$ et, pour au plus un d'entre eux, $\varepsilon_\lambda = 2$, alors en fait $R_{F,\varphi}$ interpole !

C'est ainsi qu'on montre que si $\Lambda = (\lambda_n) \subset \mathbb{N}^*$, $\lambda_n \geqslant 3\lambda_{n-1}$ alors Λ est Sidon de constante ≤ 2. $|\sum \varepsilon_\lambda \lambda| \geqslant 3\lambda_{max-1} - 2\lambda_{max-1} - \lambda_{max-2} - \dots$

$$= \lambda_{max-1} - \lambda_{max-2} - \dots > 0$$

Les quasi-indépendants sont une classe plus large qui peut s'exprimer par : le # d'écriture d'un élt de Γ comme somme d'élts distincts de Λ est au plus unique : ex : $\Lambda = (2^n)$ dans $\mathbb{Z}$,

ou $\lambda_n \geqslant 2\lambda_{n-1}$:

$$\sum \varepsilon_\lambda \lambda = 0$$
$$\lambda_{max} - \lambda_{max-1} - \dots = 0$$
$$\lambda_{max-1} - \dots = 0$$
$$\dots$$
$$\lambda_{min} = 0.$$

⚠ Λ ne peut contenir 0.

16/11/2012
Aude

$$\mathcal{F}([0,1]^2) \underset{u,g}{\not\hookrightarrow} L_2$$

Plan : a) Théorème (Aharoni, Maurey, Mityagin, 85) : $X \underset{u}{\hookrightarrow} H \Leftrightarrow X \simeq L_0$ pour les e. m. linéaires.

(Randrianarivony, 05) $X \underset{g}{\hookrightarrow} H \Leftrightarrow X \simeq L_0$

b) On va montrer qu'il existe $Y \subseteq \mathcal{F}([0,1]^2)$ avec $Y \equiv \ell_1(\mathcal{F}([0,1]^2))$

c) Théorème (Kalton, 85) : $\ell_1(X) \simeq L_0 \Leftrightarrow X \simeq L_1$

d) Théorème (Naor et Schechtman, 07) : Si il existe $F : \hat{S}_{\{0,\dots,n\}^2} \to L_1$ telle que si $\mu, \nu \in \hat{S}_{\{0,\dots,n\}^2}$, $\|\mu-\nu\| \leq \|F(\mu)-F(\nu)\| \leq C\|\mu-\nu\|$, alors $C \geq L\sqrt{\log n}$.

Conclusion : Alors $\hat{S}_{\{0,\dots,n\}^2} \underset{L}{\not\hookrightarrow} L_1$ unif^t en n

$$\begin{array}{l} \hat{S}_{\{0,\dots,n\}^2} \underset{L}{\not\hookrightarrow} L_1 \\ \quad \cap \\ \mathcal{F}(\{0,\dots,n\}^2) \underset{L}{\not\hookrightarrow} L_1 \text{ ——} \\ \quad \parallel \\ \mathcal{F}(\{0,\tfrac{1}{n},\dots,1\}^2) \underset{L}{\not\hookrightarrow} L_1 \text{ ——} \\ \quad \cap \\ \mathcal{F}([0,1]^2) \underset{L}{\not\hookrightarrow} L_1 \text{ et donc } \not\simeq L_1. \end{array}$$

b) Montrons qu'il existe $Y \subseteq \mathcal{F}([0,1]^2)$ tel que $Y \equiv \ell_1(\mathcal{F}([0,1]^2))$

Soit $\{S_j\}_{j=1}^{\infty}$ une suite de carrés disjoints de $[0,1]^2$ telle que

si $j \neq \ell$, $d(S_j, S_\ell) = \min\limits_{\substack{a \in S_j \\ b \in S_\ell}} d(a,b) \underset{(1)}{>} \max(\operatorname{diam} S_j, \operatorname{diam} S_\ell)$

Posons $Y := \{\mu \in \mathcal{F}([0,1]^2),\ \operatorname{supp}\mu \subseteq \bigcup_{j=1}^{\infty} S_j$ et $\mu(S_j) = 0$ pour tout $j\}$

$\longrightarrow$ Si $\mu \in Y$ et $\operatorname{supp}\mu \subseteq \bigcup S_j$, S_j disjoints, il existe μ_j de support $\subseteq S_j$ avec $\mu_j(S_j) = 0$.

Montrons que $\|\mu\| = \sum \|\mu_j\|$. On peut supposer μ à support fini, i.e. $\mu = m$ molécule.

Donc $m = \sum\limits_{\text{finie}}' a_i \delta_{x_i} - \sum\limits_{\text{finie}}' b_k \delta_{y_k}$ avec $a_i, b_k \geq 0$ et $\sum a_i = \sum b_k$.

On a alors $m = \sum m_j$ avec $m_j = \underbrace{\sum\limits_{x_i \in S_j}' a_i \delta_{x_i}}_{m_j^+} - \underbrace{\sum\limits_{y_k \in S_j}' b_k \delta_{y_k}}_{m_j^-}$ avec $\sum\limits_{x_i \in S_j}' a_i = \sum\limits_{y_k \in S_j}' b_k$.

Soient des $\pi_j \in E_{m_j^+, m_j^-}$: $\pi_j(x_i, y_k) \geq 0$, $\sum_k' \pi_j(x_i, y_k) = a_i$ $\sum_i \pi_j(x_i, y_k) = b_k$

Alors $\pi = \sum \pi_j \in E_{m^+, m^-}$. i.e $\sum_y \pi_j(x,y) = m_j^+(x)$, $\sum_x \pi_j(x,y) = m_j^-(y)$.

On a vu qu'on a $\|m\| = \inf_{\pi \in E_{m^+, m^-}} \sum_{i,k} \pi(x_i, y_k) d(x_i, y_k)$

Ainsi $\|m\| \leq \sum_{i,k} \sum_j \pi_j(x_i, y_k) d(x_i, y_k) = \sum_j \sum_{i,k}$; en passant à l'inf, on obtient

$\leq \sum_j' \|m_j\|$.

Réciproquement, si $\pi \in E_{m^+, m^-}$, posons pour A $\quad \sigma_j(A) = \pi(A^{\cap S_j} \times \bigcup_{l \neq j} S_l)$

$\sigma_j'(A) = \pi(\bigcup_{l \neq j} S_l \times A^{\cap S_j})$

"les masses qui rentrent et qui sortent" Ainsi $\sigma_j(S_j) = \sigma_j'(S_j)$.

Définissons un nouveau plan de transfert à support dans $\bigcup (S_j \times S_j)$:

$$\tilde{\pi} = \overbrace{\pi\, \mathbb{1}_{\bigcup(S_j \times S_j)}}^{\pi_1} + \overbrace{\sum_j' \frac{1}{\sigma_j(S_j)} \sigma_j \otimes \sigma_j'}^{\pi_2} \quad \text{Donc } \operatorname{supp} \tilde{\pi} \subset \bigcup (S_j \times S_j)$$

si $x \in S_{j_0}$

De plus $\tilde{\pi} \in E_{m^+, m^-}$: $m^+(x) = \sum_y \pi(x,y) = \underbrace{\sum_{y \in S_{j_0}}' \pi(x,y)}_{\sum_{y \in S_{j_0}} \pi_1(x,y)} + \underbrace{\sum_{y \notin S_{j_0}}' \pi(x,y)}_{\text{i.e. } \sum_{y \in S_{j_0}} \pi(x,y)\,?}$

En fait, $\sum_{y \in S_{j_0}}' \pi_2(x,y) = \sum_{y \in S_{j_0}}' \sum_j' \frac{1}{\sigma_j(S_j)} \sigma_j(x) \sigma_j'(y)$

$= \sum_{y \in S_{j_0}}' \frac{1}{\sigma_{j_0}(S_j)} \sigma_{j_0}(x) \sigma_{j_0}'(y) = \sigma_{j_0}(x) = \sum_{y \notin S_{j_0}}' \pi(x,y)$

Ce qu'on a fait, c'est qu'on a annulé le transport entre les S_j : localisation à l'intérieur des S_j.

Donc $\tilde{\pi}_j = \tilde{\pi}|_{S_j} \in E_{m_j^+, m_j^-}$ et montrons que $\sum' \|m_j\| \leq \sum_{(x,y) \in \bigcup S_j \times \bigcup S_j}' \pi(x,y) d(x,y)$:

$\sum_j' \|m_j\| \leq \sum_j \sum_{x,y}' \tilde{\pi}_j(x,y) d(x,y) = \sum_j' \sum_{x,y} (\pi_1(x,y) + \pi_2(x,y)) d(x,y)$

Or $\sum_{x \in S_j}' d(x,y) \sigma_j(x) = \sum_{x \in S_j} \sum_{z \notin S_j}' d(x,y) \pi(x,z) \leq \sum_{x \in S_j}' \sum_{z \notin S_j}' d(x,z) \pi(x,z)$

et si $x, y \in S_j$, $d(x,y) \leq \operatorname{diam} S_j < \min_{\substack{a \in S_j \\ b \notin S_j}} d(a,b) \leq d(x,y)$

En somme, $\frac{1}{\sigma_j(S_j)} \sum_{x,y \in S_j}' d(x,y) \sigma_j(x) \sigma_j'(y)$

$\leq \frac{1}{\quad} \underbrace{\sum_{y \in S_j}' \sigma_j'(y)}_{\sigma_j'(S_j) = \sigma_j(S_j)} \sum_{\substack{x \in S_j \\ z \notin S_j}} d(x,z) \pi(x,z) = \sum_{\substack{x \in S_j \\ z \notin S_j}}' d(x,z)$

et $\sum_j' \|m_j\| \le \sum_{(x,y)\in \cup(S_j\times S_j)}' \pi(x,y)d(x,y) + \sum_j' \sum_{\substack{x\in S_j\\ t\notin S_j}} d(x,t)\pi(x,t) = \sum_{\substack{x\in \cup S_j\\ y\in\cup S_j}}' d(x,y)\pi(x,y)$

En passant à l'infimum sur π, $\sum_j' \|m_j\| \le \|m\|$.

On a donc montré que $Y = \left(\bigoplus_j \mathcal{F}(S_j)\right)_{\ell_1}$; or $\mathcal{F}(S_j) \equiv \mathcal{F}([0,1]^2)$.

Donc $Y \equiv \ell_1(\mathcal{F}([0,1]^2))$ et ainsi $\mathcal{F}([0,1]^2) \underset{u.g}{\hookrightarrow} L_2$.

d) Théorème de Naor et Schechtman.

En effet, si $\mathcal{F}(\{0,\dots,n\}^2) \underset{u.g}{\hookrightarrow} L_2$ uniformément en n,

alors aussi $\mathcal{F}(\{\frac{0}{n},\dots,\frac{n}{n}\}^2) \underset{u.g}{\hookrightarrow} L_2$ uniformément en n.

et en prenant la limite, $\mathcal{F}([0,1]^2) \underset{u.g}{\hookrightarrow} (L_2)_{\mathcal{U}} = L_2$. (densité de $\{\frac{0}{n},\dots,\frac{n}{n}\}^2$ dans $[0,1]^2$)

Il n'existe pas de fonctions $f_n : \mathcal{F}_{\{0,\dots,n\}^2} \to L_2$ telles que $\sqrt{\|\mu-\nu\|} \le \|f_n(\mu)-f_n(\nu)\|$
$\le C\sqrt{\|\mu-\nu\|}$.

Notons $S \subset \mathcal{F}_{\{0,\dots,n\}^2}$ la boule unité de $\mathcal{F}(\{0,\dots,n\}^2)$.

Si de telles fonctions existent, alors $\exists\, g_n : \mathcal{F}(\{0,\dots,n\}^2) \to \ell^2$

telles que pour $\mu,\nu\in S$, $\sqrt{\|\mu-\nu\|} \le \|g_n(\mu)-g_n(\nu)\| \le C\sqrt{\|\mu-\nu\|}$

Soit $\mathcal{U}$ un ultrafiltre sur $\mathbb{N}$ et $\tilde{g}_n : \mathcal{F}(\{0,\dots,n\}^2) \to (L_2)_{\mathcal{U}}$
$\mu \longmapsto \lim_{\mathcal{U}} \sqrt{j}\, g_n(\frac{\mu}{j})$

Si $\mu,\nu\in \mathcal{F}_{\{0,\dots,n\}^2}$, et j tel que $\|\frac{\mu}{j}\|, \|\frac{\nu}{j}\| \le 1$,

$\sqrt{\|\mu-\nu\|} = \sqrt{j}\sqrt{\|\frac{\mu}{j}-\frac{\nu}{j}\|} \le \sqrt{j}\,\|g_n(\frac{\mu}{j}) - g_n(\frac{\nu}{j})\| \le C\sqrt{j}\sqrt{\|\frac{\mu}{j}-\frac{\nu}{j}\|} = C\sqrt{\|\mu-\nu\|}$

$\sqrt{\|\mu-\nu\|} \le \|\tilde{g}_n(\mu)-\tilde{g}_n(\nu)\| \le C\sqrt{\|\mu-\nu\|}$

Il y a ici une question ouverte

23/11/2012

Plgts grossiers uniformes ds les Banach réflexifs (d'après Kalton)

Th : Tout e. métrique <u>stable</u> se plonge dans un superréflexif

Déf : (M,d) est stable si pour $(x_n),(y_n)$ bornées dans M, $\mathcal{U},\mathcal{V}$ ultrafiltres libres sur $\mathbb{N}$

on a $\lim\limits_{n\to\mathcal{U}} \lim\limits_{m\to\mathcal{V}} d(x_n,y_m) = \lim\limits_{m\to\mathcal{V}} \lim\limits_{n\to\mathcal{U}} d(x_n,y_m)$

<u>ou</u> : dès que toutes les limites ci-dessous existent, alors $\overset{(*)}{\lim\limits_{n}} \lim\limits_{m} d(x_n,y_m) = \lim\limits_{m}\lim\limits_{n} d(x_n,y_m)$.

<u>Remarque :</u> $(*)$ si les limites sont infinies.

→ quitte à extraire, on a $d(x_n,0) \to \infty$.

- $\forall m$ $\lim\limits_{n} d(x_n,y_m) = \infty$ et donc $\lim\limits_{m}\lim\limits_{n} d(x_n,y_m) = \infty$
- Deux cas : (i) $\forall n$ $\lim\limits_{m} d(x_n,y_m) = \infty$: alors $\lim\limits_{n}\lim\limits_{m} d(x_n,y_m) = \infty$

  (ii) $\exists n_0$ $\lim\limits_{m} d(x_{n_0},y_m) = C < \infty$ et

  $d(x_n,y_m) \geq d(x_n,x_{n_0}) - d(x_{n_0},y_m) \geq d(x_n,x_{n_0}) - C$

  et $\lim\limits_{m} d(x_n,y_m) \geq d(x_n,x_{n_0}) - C$ et donc $\lim\limits_{n}\lim\limits_{m} d(x_n,y_m) = \infty$.

Donc : si une des suites vérifie $d(x_n,0) \to \infty$, alors les deux limites doubles valent ∞.

<u>Exemples</u> : • Si (M,d) métrique propre (i.e. boules de M relativement compactes), alors (M,d) est stable. (E evn de dim finie)

- Si M est un espace de Hilbert, $\|x_n-y_m\|^2 = \|x_n\|^2 + \|y_m\|^2 - 2\langle x_n,y_m\rangle$ et la compacité faible des bornés de H montre que H est stable
- L^1 est stable : cela repose sur ce que $e^{-\|x-y\|_1}$ est de type positif et $\|x-y\|_1$ est de type conditionnellement négatif.

  Ainsi il existe $U : L^1 \to B_H$ tel que $e^{-\|x-y\|_1} = \langle U(x),U(y)\rangle$.
- La stabilité passe aux sev fermés et donc si $1 \leq p \leq 2$, L^p est stable.
- Si E Banach stable, et $1 \leq p < \infty$, alors $L^p(E)$ est stable.
- c_0 n'est pas stable : soit $s_m = (\overbrace{1,\dots,1}^{m},0,0,\dots)$ et $e_n = (0,\dots,0,\overset{n}{1},0,\dots)$,

  $\lim\limits_{m}\lim\limits_{n} \|s_m+e_n\|_\infty = 1$ et $\lim\limits_{n}\lim\limits_{m} \|s_m+e_n\| = 2$.

<u>Théorème</u> (Kalton 2007): Soit (M,d) métrique stable. Alors il existe X réflexif et $\phi: M \to X$ plongement grossier et uniforme: $M \xhookrightarrow[\text{unif}^t]{\text{fortem}^t} X$.

<u>Idée</u>: · pour $a \in]0,1[$, soit $d_a = \begin{cases} d^a & \text{si } d \le 1 \\ d & \text{si } d \geq 1 \end{cases}$, i.e. $d_a = \mathrm{Max}(d, d^\alpha)$.

· Construire $W \subset B_{Lip_0(M,d_a)}$ tel que W soit w-rel^t-compact et pour $x \neq y \in M$, $\sup_{f\in W} |f(x)-f(y)| \geq \omega(d(x,y))$.

C'est ce qui permet de conclure: fin de la preuve: $S: \ell_1(W) \longrightarrow Lip_0(M,d_a)$

$$(\xi_f)_{f\in W} \longmapsto \sum_{f\in W} \xi_f f$$

$S(B_{\ell_1(W)}) \subset \overline{\text{abs-conv}}(W)$ et w-compact.

et Davis-Figiel-Johnson-Pełczyński (1974): il existe Y réflexif et T, U tels que

$$\begin{tikzcd} \ell_1(W) \arrow[rr, "S"] \arrow[dr, "T"'] & & Lip_0(M,d_a) \\ & Y \arrow[ur, "U"'] & \end{tikzcd}$$

et $\|T\|, \|U\| \le 1$. Diagramme dual:

$$\begin{tikzcd} & Lip_0(M,d_a)^* \arrow[rr, "S^*"] & & \ell_\infty(W) \\ M \arrow[ur, "\delta", "\text{isométrie}"'] & & Y^*=X \arrow[ul, "U^*"] \arrow[ur, "T^*"'] & \end{tikzcd}$$

et on posera $\phi = U^* \circ \delta : (M, d_a) \to X$

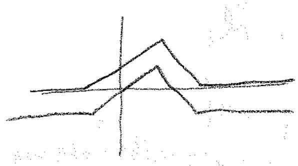
On aura $\|\phi(x) - \phi(y)\|_X \geq \|T^*\phi(x) - T^*\phi(y)\|_{\ell_\infty(W)}$

$$\geq \|S^*(\delta(x)-\delta(y))\|_{\ell_\infty(W)}$$

$$\geq \sup_{f\in W} |\langle \delta(x)-\delta(y), f\rangle| = \sup_{f\in W} |f(x)-f(y)| \geq \omega(d(x,y)).$$

Il faut donc construire ce W à la fois w-rel cpct et suffisamment riche pour "normer": $\tilde{M} = \{(p,q) \in M \times M : p \neq q\}$

$(p,q) \in \tilde{M}$ et $g_{pq}(x) = [\mathrm{Max}(d(p,q) - d(x,q), 0) - \mathrm{Max}(d(p,q) - d(0,q), 0)]$



Alors $|g_{p,q}(x) - g_{p,q}(y)| \le \mathrm{Min}(d(p,q), d(x,y))$

Puis posons $f_{p,q} = \begin{cases} g_{p,q} & \text{si } d(p,q) \le 1 \\ \dfrac{g_{p,q}}{d(p,q)^{1-a}} & \text{si } d(p,q) > 1 \end{cases}$

Alors $f_{p,q} \in B_{Lip_0(M,d_a)}$ car $g_{p,q}$ est 1-Lipschitz pour (M,d)

On pose alors $W = \{f_{p,q} : (p,q) \in \tilde{M}\}$

Si $p \neq q \in M$, $|g_{p,q}(q) - g_{p,q}(p)| = d(p,q)$ et $|f_{p,q}(q) - f_{p,q}(p)| = \begin{cases} d(p,q) & \text{si } d(p,q) \le 1 \\ d^a(p,q) & \text{si } d(p,q) > 1 \end{cases}$

et $\operatorname{Min}(d(x,y), d^a(x,y)) \leq \|\phi(x) - \phi(y)\| \leq \operatorname{Max}(d^a(x,y), d(x,y))$

Prouvons maintenant que W est f.r.c. :

$$V = \left| \frac{f_{p,q}(x) - f_{p,q}(y)}{d_a(x,y)} \right| \leq \begin{cases} d(p,q)^{1-a} & \text{si } d(p,q) \leq 1 \\ \frac{1}{d(p,q)^{1-a}} & \text{si } d(p,q) > 1 \end{cases} \quad (\dagger)$$

$$\leq \begin{cases} d(x,y)^{1-a} & \text{si } d(x,y) \leq 1 \\ \frac{d(p,q)}{d(x,y)} & \text{si } d(x,y) > 1. \end{cases} \quad (\ddagger)$$

On a $V \longrightarrow 0$ quand $d(p,q) \longrightarrow 0$ ou $+\infty$
ou $d(x,y) \longrightarrow 0$ ou $+\infty$.

(Il y a une seule astuce : $\min(a,b) \leq \cancel{a^\alpha b^{1-\alpha}}$)

Eberlein–Shmulyan : la compacité relative faible est déterminée par les suites :
il suffit de vérifier que toute suite $(f_n) = (f_{p_n,q_n})_n$ avec $(p_n,q_n) \in \tilde{M}$ admet une sous-suite faiblement convergente.
Or $E = \overline{\text{vect}}\, \{f_n : n \geq 0\}$ est séparable $\subset Lip_0(M, d_a)$.
Donc $\exists M_0 \subset M$ dénombrable contenant 0 et $p_n, q_n \in M_0$ tels que
$\|f\|_{Lip_0(M,d_a)} = \|f\|_{Lip_0(M_0,d_a)}$ (*)

Par un argument diagonal, on peut supposer $\begin{cases} \forall x \in M_0 \quad f_n(x) \longrightarrow f(x) & \text{(limite simple)} \\ d(p_n,q_n) \longrightarrow r \in [0,+\infty] \\ \forall u \in M_0 \quad \lim_n d(q_n,u) \text{ existe dans } [0,+\infty]. \end{cases}$

M_0 est suffisamment riche pour que $\|f\|_{Lip}$ s'y calcule.
Soit $\tilde{M}_0 = (M_0 \times M_0) \cap \tilde{M}$ et $V : Lip_0(M, d_a) \longrightarrow \ell_\infty(\tilde{M}_0)$, $h \longmapsto \left(\frac{h(x)-h(y)}{d_a(x,y)}\right)_{(x,y) \in \tilde{M}_0}$ est une isométrie par (*).

$f \in B_{Lip_0(M,d_a)}$ et on veut montrer que $f_n \xrightarrow{w} f$. Il suffit de montrer que $Vf_n \xrightarrow{w} Vf$.

Argument d'algèbre de Banach : $B = \ell_\infty(\tilde{M}_0) \equiv C(S_B)$, S_B spectre de B.
Soit A la sous-algèbre de B engendrée par les constantes, $V(E)$, f, et, pour chaque $u \in M_0$, $(x,y) \longmapsto \operatorname{arctan} d(x,u)$
$(x,y) \longmapsto \operatorname{arctan} d(y,u)$
$(x,y) \longmapsto \operatorname{arctan} d(x,y)$.

A est une algèbre séparable, $A \equiv (C(S_A), \sigma(A^*,A))$ et $K = (S_A, \sigma(A^*,A))$ est un compact séparable. De plus, $i : \tilde{M}_0 \longrightarrow S_A$, $(x,y) \longmapsto [i(x,y) : A \longrightarrow \mathbb{R},\ u \longmapsto u(x,y)]$

et $\tilde{M}_0$ est dense dans K car sinon $\overline{\tilde{M}_0}^{\sigma(A^*,A)} \neq K$ et Urysohn donne f continue sur K, i.e. $f \in A$, telle que $f|_{\tilde{M}_0} = 0$ et $f \neq 0$.

De plus, $i|^{i(\tilde{M}_0)} : \tilde{M}_0 \to i(\tilde{M}_0)$ est un homéomorphisme sur son image

---

Il suffit de démontrer que pour $\xi \in K$ $Vf_n(\xi) \to Vf(\xi)$

[th de Riesz + th de convergence dominée].

Soit $\xi \in K$ et $(x_m, y_m) \in \tilde{M}_0$ tels que $(x_m, y_m) \to \xi$.

Or $r = \lim d(p_n, q_n) \in [0, \infty]$; si $r = 0$ ou $r = +\infty$, alors $\|Vf_n\| \to 0$.
→ c'est l'inégalité $(\dagger)$ pour f_n.

Donc $\|f_n\| \to 0$ et comme $\|Vf_n - f_n\| \to 0$, $f = 0$.

Notons $t = \lim d(x_m, y_m)$. On a $(Vf_n)_{(x_m,y_m)} \xrightarrow[m]{} 0$ si $t = 0$ ou $+\infty$, c'est l'inégalité $(\dagger)$ pour f_n. Donc $Vf_n(\xi) = 0$ puisque $Vf_n \in C(K)$.

De même, $Vf(x_m, y_m) \to 0$ et $Vf(\xi) = 0$.

On peut ainsi supposer $0 < r < \infty$ et $0 < t < \infty$. L'interversion de limites arrive.

$$f_n(x_m) = \min\left(1, \frac{1}{d(p_n,q_n)^{1-a}}\right) \times [\quad], \text{ où}$$

$$[\quad] = [\operatorname{Max}(d(p_n,q_n) - d(q_n,x_m), 0) - \operatorname{Max}(d(p_n,q_n) - d(q_n,0), 0)]$$

M est stable et donc $\lim_m \lim_n f_n(x_m) = \lim_n \lim_m f_n(x_m)$

Donc $Vf(\xi) = \lim_m Vf(x_m, y_m) = \lim_m \lim_n \dfrac{f_n(x_m) - f_n(y_m)}{d_a(x_m, y_m)}$ car $f_n \to f$ simplement.

$$= \lim_n \lim_m \frac{f_n(x_m) - f_n(y_m)}{d_a(x_m, y_m)} = \lim_n Vf_n(\xi)$$

Rem : Florent Baudier : espace métrique propre (M, d), X e. Banach sans cotype.
Alors $M \xrightarrow{\text{f.u.}} X = (\sum l_\infty^n)_{l_2}$ réflexif

Tony
30/11/2012

Plongements / espaces réflexifs / Propriété Q.

Plan : - une condition nécessaire pour plongements grossiers et uniformes d'espaces métriques dans des espaces réflexifs.

Déf.: Soit $f : M \longrightarrow N$. On note $\omega_f(t) = \sup\{d(f(x),f(y)) : d(x,y) \le t\}$

$\varphi_f(t) = \inf\{ \qquad \ge \}$

N.B.: • Alors ω_f et φ_f sont croissantes et $\varphi_f(t) \le \omega_f(t)$ si $t \in \operatorname{im} d$.

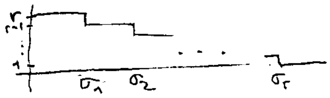
• f est un plongement grossier si ω_f prend des valeurs finies et $\varphi_f \xrightarrow[\infty]{} \infty$

Déf: Soit $\mathbb{M} \subset \mathbb{N}$ infini, $r \in \mathbb{N}$. On pose $\mathcal{S}_r(\mathbb{M}) = \{\sigma \in \mathbb{M} : |\sigma| = r\}$ et le munit d'une structure de graphe dont les arêtes sont les paires $\{\sigma,\tau\}$ telles que si $\sigma = \{\sigma_1 < \dots < \sigma_r\}$ et $\tau = \{\tau_1 < \dots < \tau_r\}$ on a $\sigma_1 < \tau_1 < \sigma_2 < \tau_2 < \dots < \sigma_r < \tau_r$.

N.B.: ce graphe est connexe. On le munit de la distance géodésique : diam $\mathcal{S}_r(\mathbb{M}) = r$.

D'où vient ce graphe ? Si on pose $f : \mathcal{S}_r(\mathbb{N}) \longrightarrow c_0$

$\sigma \longmapsto \sum_{i=1}^{r} 1_{[1,\sigma_i - 1]}$



• Cette construction a été utilisée pour prouver que c_0 ne se plonge pas grossièrement dans un espace dont tous les duaux sont séparables.

• f est une isométrie.

Déf: Soit $\varepsilon > 0$, $\delta > 0$. Un espace métrique M a $Q(\varepsilon,\delta)$ si $\forall r \in \mathbb{N}\ \forall f : \mathcal{S}_r(\mathbb{N}) \to M$ $\omega_f(1) \le \delta \Rightarrow \exists \mathbb{M} \subset \mathbb{N}$ infini $\forall \sigma,\tau \in \mathcal{S}_r(\mathbb{M})$ $d(f(\sigma),f(\tau)) \le \varepsilon$

N.B.: Il suffit de le vérifier pour $\sigma < \tau$, i.e., $\sigma_r < \tau_1$.

Posons $\Delta_M(\varepsilon) = \sup\{\delta : M \text{ a } Q(\varepsilon,\delta)\}$. Si M est un espace vectoriel normé, $\Delta_M(\lambda\varepsilon) = \lambda\Delta_M(\varepsilon)$ pour $\lambda > 0$ et $\varepsilon > 0$ et donc $\Delta_M(\varepsilon) = \Delta_M(1)\varepsilon$.

Notons alors $Q_M = \Delta_M(1)$: M a la propriété Q si $Q_M > 0$.

On a • $Q_{c_0} = 0$

• X réflexif $\Rightarrow X$ a Q.

Rappel : Si $f : \mathcal{S}_r(\mathbb{N}) \to X$ bornée et $\mathcal{U}$ ultrafiltre non principal, on pose
$$\partial_{\mathcal{U}} f(\sigma_1,\dots,\sigma_{r-1}) = \omega^*\text{-}\lim f(\sigma_1,\dots,\sigma_r) \in X^{**},$$
Généralisé : $\partial_{\mathcal{U}}^k f(\sigma_1,\dots,\sigma_{r-k}) = \underbrace{\partial_{\mathcal{U}} \circ \dots \circ \partial_{\mathcal{U}}}_{k \text{ fois}} f(\sigma_1,\dots,\sigma_{r-k})$ et $\partial_{\mathcal{U}}^r f(\emptyset) = \partial_{\mathcal{U}}^r f \in X^{(2r)}$.

Lemme 1 : Si $f : \mathcal{S}_r(\mathbb{N}) \to \mathbb{R}$, alors $\forall \varepsilon > 0\ \exists M \subset \mathbb{N}\ \forall \sigma \in \mathcal{S}_r(M)\ |f(\sigma) - \partial_{\mathcal{U}}^r f| < \varepsilon$.

Dém : soit $m_1 \in \mathbb{N}$ tel que $|\partial_{\mathcal{U}}^{r-1} f(m_1) - \partial_{\mathcal{U}}^r f| < \varepsilon$. Réitérons.

Si $m_1 < \dots < m_n$ sont choisis dans $\mathbb{N}$, et qu'ils ont la propriété

$\forall \sigma \subset \{m_1,\dots,m_n\}$ avec $s = |\sigma| \le r-1$, alors $|\partial_{\mathcal{U}}^{r-s} f(\sigma) - \partial_{\mathcal{U}}^r f| < \varepsilon$.

Ainsi, pour chaque tel σ, il existe $A_\sigma \in \mathcal{U}$ tel que pour $m \in A_\sigma$ $(m > m_n)$ on ait $|\partial_{\mathcal{U}}^{r-s-1} f(\sigma \cup \{m\}) - \partial_{\mathcal{U}}^r f| < \varepsilon$. Alors $A = \bigcap_\sigma A_\sigma \in \mathcal{U}$ et on pose $m_{n+1} \in A$ avec $m_{n+1} > m_n$.

Lemme 2 : Soit $f : \mathcal{S}_r(\mathbb{N}) \to X$ bornée. Alors $\forall \varepsilon > 0\ \exists M \subset \mathbb{N}\ \forall \sigma \in \mathcal{S}_r(M)$
$$\|f(\sigma)\| \le \|\partial_{\mathcal{U}}^r f\| + \omega_1(f) + \varepsilon.$$

Dém : Soit $g : \mathcal{S}_r(\mathbb{N}) \to S_{X^*}$ telle que $\langle f(\tau), g(\tau) \rangle = \|f(\tau)\|$.

Posons $h : \mathcal{S}_{2r}(\mathbb{N}) \to \mathbb{R}$
$$\sigma \longmapsto \langle f(\sigma_2,\sigma_4,\dots,\sigma_{2r}), g(\sigma_1,\sigma_3,\dots,\sigma_{2r-1}) \rangle.$$

Par le lemme 1, il existe $A \subset \mathbb{N}$ tel que $|h(\sigma) - \partial_{\mathcal{U}}^{2r} h| < \varepsilon$. Notons $A = \{m_1, n_1, m_2, n_2, \dots$ et posons $M = \{m_i : i \in \mathbb{N}\}$. Soit $\sigma \in \mathcal{S}_r(M)$. On a alors $\sigma' \in \mathcal{S}_r(\{n_1, n_2, \dots\})$ choisi judicieusement tel que
$$\|f(\sigma)\| = \langle f(\sigma), g(\sigma) \rangle = \underbrace{\langle f(\sigma'), g(\sigma) \rangle}_{h(\{\sigma_1,\sigma_1',\sigma_2,\sigma_2',\dots,\sigma_r,\sigma_r'\})} + \underbrace{\langle f(\sigma) - f(\sigma'), g(\sigma) \rangle}_{\le \omega_f(1)}$$
$$\le \|\partial_{\mathcal{U}}^r f\| + \omega_f(1) + \varepsilon.$$

Théorème : Soit X réflexif et $r \in \mathbb{N}$, $f : \mathcal{S}_r(\mathbb{N}) \to X$ bornée. Alors $\forall \varepsilon > 0\ \exists M \subset \mathbb{N}$ $\exists x \in X\ \forall \sigma \in \mathcal{S}_r(M)\ \|f(\sigma) - x\| \le \omega_f(1) + \varepsilon$. Ainsi $Q_X \ge \frac{1}{2}$.

Dém : on pose $f' = f - \partial_{\mathcal{U}} f$. Il existe M tel que $\underbrace{\|f'(\sigma)\|}_{\|f(\sigma) - x\|} \le 0 + \omega_f(1) + \varepsilon$.

Soit $\xi > 0$ et $2\delta < \xi$. Soit f telle que $\omega_f(1) \le \delta$: il existe M tel que $\|f(\sigma) - f(\tau)\| \le 2\omega_f(1) + 2\varepsilon < \xi$

Donc $\Delta_X(\xi) \ge \frac{1}{2}\xi$.

Théorème 4 : Soit M métrique. Si $M \underset{u}{\hookrightarrow} X$ réflexif, alors $\forall \varepsilon > 0$ $\Delta_M(\varepsilon) > 0$

Si $M \underset{g}{\hookrightarrow} X$ reflexif, alors $\lim_{\varepsilon\to\infty} \Delta_M(\varepsilon) = \infty$

Dém : $S_r(\mathbb{N}) \xrightarrow{f} M \xrightarrow{h} X$ et f telque $\omega_f(1) \leq \delta$ (et donc $\omega_{h\circ f}(1) \leq \omega_h(\delta)$)

Si h est uniforme ou grossier h est bornée sur les boules.

Par le théorème 3, il existe $M \subset \mathbb{N}$ telque si $\sigma, \tau \in S_r(M)$,

$\varphi_h(d(f(\sigma), f(\tau))) \leq \|h\circ f(\sigma) - h\circ f(\tau)\| \leq 3\,\omega_{h\circ f}(1) \leq 3\,\omega_h(\delta)$. $(*)$

Si h est un plongement uniforme, alors $\forall \varepsilon > 0$ $\exists \varepsilon'$ $\varphi_h(t) < \varepsilon' \Rightarrow t < \varepsilon$

(puisque $\varphi_h \uparrow$, $\varphi_h > 0$, $\varphi_h(t) \xrightarrow[t\to 0]{} 0$). Fixons $\varepsilon > 0$ et soit ε' comme ci-dessus.

Prenons δ telque $3\,\omega_h(\delta) < \varepsilon'$. Alors $\varphi_h(d(f(\sigma), f(\tau))) < \varepsilon$. Donc $\Delta_M(\varepsilon) \geq \delta > 0$.

Si h est un plongement grossier, il suffit de prouver que $\Delta_M(\varepsilon_n) \to \infty$ pour une suite $\varepsilon_n \to \infty$. Soit $(\varepsilon_n) \subset \mathbb{R}^+$ telque $\forall h$ $\forall t > \varepsilon_n$ $\varphi_h(\varepsilon_n) < \varphi_h(t)$

et $\varepsilon_n \to \infty$. Soit δ_n maximal telque $3\,\omega_h(\delta_n) \leq \varphi_h(\varepsilon_n)$. Ainsi, par $(*)$,

$\varphi_h(d(f(\sigma), f(\tau))) \leq \varphi_h(\varepsilon_n)$ et donc $d(f(\sigma), f(\tau)) \leq \varepsilon_n$. Alors $\delta_n \to \infty$

parceque $\varphi_h(\varepsilon_n) \to \infty$.

Corollaire : Si $M \in B$ telque $Q_M = 0$, alors (i) $B_M \underset{u}{\not\hookrightarrow} X$ réflexif et (ii) $M \underset{g}{\not\hookrightarrow} X$ reflexif

$\to$ pour (i), notez que $\Delta_M(\varepsilon) = \Delta_{BM}(\varepsilon)$ pour $\varepsilon \leq 1$.

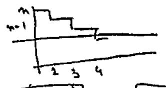
Exemples d'espaces sans propriété Q : c_0, J espace de James, J^*, X non réflexif de type non trivial.

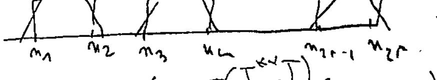
Rappel : les espaces stables ont Q et se plongent dans les réflexifs.

Théorème 5 : Si $X \in B$ avec $Q_X > 0$, on a pour tout $\varepsilon > 0$ et toute $(x_n) \subset X$ bornée avec $x^{**} \in \overline{\{x_n\}}^{\sigma(X^{**}, X^*)}$, une sous-suite $(y_n) \subset (x_n)$ telle que si $n_1 < n_2 < \dots < n_{2r}$, $r \geq 1$

alors $\|\sum_{j=1}^{2r} (-1)^j y_{n_j}\| > (1-\varepsilon)\, Q_X\, d(x^{**}, X)\, r$

Rappel sur J : $\|x\|_J = \sup_{\substack{\sigma \in S_r(\mathbb{N}) \\ r\in\mathbb{N}}} \left(\sum_{i=2}^{r} (x_{\sigma_i} - x_{\sigma_{i-1}})^2\right)^{\frac{1}{2}}$ et $J = \{x : \|x\|_J < \infty \text{ et } \lim_n x_n = 0\}$.

Posons $X_n = \sum_{i=1}^{n} \mathbb{1}_{[1,i]}$ $\mathbb{1}_{[1,n]}$  $= \sum_{i=1}^{n} e_i$

Considérons $\|\sum_{i=1}^{2r} (-1)^i X_{n_i}\|$  $= (2r)^{\frac{1}{2}} \not\geq cr$.

Donc J n'a pas Q. N.B. : $X_n \xrightarrow{\sigma(J^{**}, J^*)} (1, 1, \dots) \in J^{**} \setminus J$.

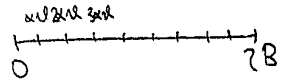
Preuve du théorème 5 : Notons $\vartheta = d(x^{**}, X) > 0$, et $B = \sup \|x_n\|$. Soit $\lambda > 1$ et $\alpha > 0$ tel que $\lambda^{-2} Q_X - 2\alpha B - \alpha > (1-\varepsilon) Q_X$.

Soit (v_n) la sous-suite de (x_n) telle que $d\left(\operatorname{conv}(v_1, \dots, v_m), \operatorname{conv}(v_{m+1}, \dots, v_n)\right) \geq \frac{\vartheta}{\lambda}$ (C'est le critère de James) On fait ainsi : $\|x_1 - x^{**}\| > \frac{\vartheta}{\lambda}$. Par Hahn-Banach on a $f(x_1 - x^{**}) > \frac{\vartheta}{\lambda}$. Soit $V = \{y \in Y^{**} : f(y) > f(x_1) - \frac{\vartheta}{\lambda}\} \supset \{x_n : n \in M_1\}$.

Si $v_1, \dots, v_m$ sont choisis avec $M_1 \supset M_2 \supset \dots \supset M_n$, on sépare $\operatorname{conv}\{v_1, \dots, v_n\} + B_X(\frac{\vartheta}{\lambda})$ de x^{**}. Puis on prend la diagonale.

$r=1$: il existe $\begin{cases} b_r \in \mathbb{R} \\ M_1 \subset \mathbb{N} \end{cases}$ tel que si $\sigma \in \mathcal{P}_r(M_1)$, $b_r - \alpha\vartheta \leq \left\| \sum_{j=1}^{2r} (-1)^j v_{\sigma_j} \right\| \leq b_r$

C'est une conséquence du théorème de Ramsey



On réitère : et on trouve $M_{n+1} \subset M_n \subset \dots \subset M_1$ tels que $\forall \sigma \in \mathcal{P}_{2r}(M_{n+1})$ $b_r - \alpha\vartheta \leq \| \quad \| \leq b_r$

Et il existe ~~$M \in \mathbb{N}$ infini~~ une suite (y_n) extraite de (v_n) telle que $\forall r\ \exists b_r\ \forall \sigma \in \mathcal{P}_{2r}(M)$ avec $\sigma_1 \geq \alpha r$ $b_r - \alpha\vartheta \leq \| \quad \| \leq b_r$. (**) (*)

Fixons $r \in \mathbb{N}$ et posons $f : \mathcal{P}_r([\![\alpha r, \infty[\![) \longrightarrow X$, $\sigma \longmapsto \sum_{k \in \sigma} y_k$. Alors $\omega_f(1)$ car (*)

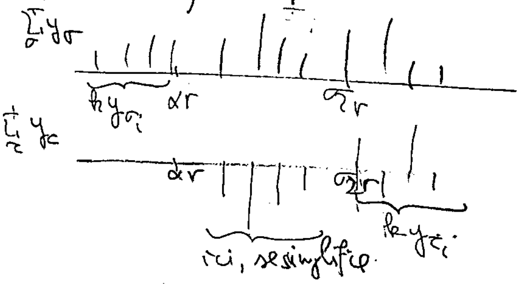
Donc, si on pose $\varepsilon = \lambda b_r Q_X^{-1}$, $b_r < \Delta_X(\varepsilon) = \lambda b_r$. Par la propriété Q, il existe $\sigma < \tau \in \mathcal{P}_r(M)$ tels que $\|f(\sigma) - f(\tau)\| \leq \lambda b_r Q_X^{-1}$.

Or le théorème de James avec poids $\frac{1}{r}$ montre que $r\frac{\vartheta}{\lambda} \leq \|f(\sigma) - f(\tau)\|$

[on pose $f(\sigma) = r \sum_1 \frac{1}{r} y_n$, $f(\tau) = r \sum_1 \frac{1}{r} y_k$ et $\sigma < \tau \Rightarrow \| \underbrace{\sum_1 \frac{1}{r} y_k}_{\in co(1)} - \underbrace{\sum_1 \frac{1}{r} y_k}_{\in co(1)} \| \geq \frac{\vartheta}{\lambda}$

Soit $\sigma \in \mathcal{P}_{2r}(\mathbb{N})$: il existe $\tau \in \mathcal{P}_{2r}([\![\alpha r, \infty[\![)$ et $\beta \in \{-1, +1\}$ tels que

$$\left\| \sum_{j=1}^{2r} (-1)^j y_{\sigma_j} + \beta \sum_{j=1}^{2r} (-1) y_{\tau_j} \right\| \leq 2\alpha r B$$



$$\left\| \sum_{j=1}^{2r} (-1)^j y_{\sigma_j} \right\| \geq \underbrace{\left\| \sum (-1)^j y_{\tau_j} \right\|}_{} - \| \sim\sim - \beta \sim \|$$

$$\geq b r - \vartheta\alpha \quad - 2\alpha B r$$

avec $k \leq \alpha r$.
et $B = \sup \|x_n\| \geq \sup \|y_n\|$.

$$\geq \lambda^{-2} Q r \vartheta - \alpha \vartheta r - 2\alpha B r$$

$$\geq (1-\varepsilon) Q r \vartheta$$

Déf : X a la propriété de Banach-Saks alternée si $\forall (x_n) \subset X$ bornée $\exists (y_n)$ sous-suite de (x_n) $\frac{1}{n} \sum_{k=1}^{n} (-1)^k y_k \longrightarrow 0$.

Th : Si X a cette propriété et que $Q_X > 0$, alors X est réflexif.

Dém : Par le théorème 5, si $(x_n) \subset X$ bornée et x^{**} pt d'accumulation préfaible, il existe (y_n) sous-suite et c tel que $\| \sum_{j=1}^{2r} (-1)^j y_{n_j} \| > c r d(x^{**}, X)$.

Ainsi, pour $n_1 < n_2 < \dots < n_{2r}$, il existe $(n_i) \subset \mathbb{N}$ telle que

$$\left\| \frac{1}{2r} \sum_{j=1}^{2r} (-1)^j y_{n_j} \right\| \geq \frac{\varepsilon}{2}\, d(x^{**}, X)$$

$$\downarrow$$

$$0 \quad \text{et donc } x^{**} \in X.$$

Or, par le théorème d'Eberlein–Shmulyan ou Kaplansky, X est réflexif. ∎

Or Beauzamy a prouvé que si X a un type non trivial, alors X a Banach-Saks alternée.

Corollaire: $\underbrace{X \text{ non-réflexif de type non trivial}}_{\text{de tels espaces existent : cf James, Pisier-Xu}} \Rightarrow Q_X = 0$

Autre résultat: $Q_{J^*} = 0$

En effet, (e_n^*) a un point d'accumulation préfaible $x^{***} \in J^{***}$.

Or $\left\| \sum_{j=1}^{2r} (-1)^j e_{n_j}^* \right\| \leq \sqrt{r}$: en testant contre $x \in J$,

$$\left\langle \sum (-1)^j e_{n_j}^*, x \right\rangle = \sum_{j=1}^{2r} (-1)^j x(n_j) \leq \sum_1^r |x_{n_{2j+1}} - x_{n_{2j}}|$$

$$\leq \sqrt{r}\Big(\sum_1^r | \qquad |^2\Big)^{1/2} \leq \sqrt{r}\, \|x\|_J.$$

L'article montre aussi que si X est quasiréflexif,

X a $Q \iff X^{\omega *}$ dual transfini s'écrit facilement en termes de K.

Q ouverte. Tout espace qui ne se plonge pas dans un réflexif, a-t-il non Q?

i.e., est-ce que $Q \Rightarrow \exists Y$ réflexif $X \hookrightarrow Y$ ou $B_X \xrightarrow[u]{} Y$

Tout réflexif peut-il être plongé dans un stable ?

Que peut-on dire sur les constantes ?

On a montré que X réflexif $\Rightarrow \forall f : S_r(\mathbb{N}) \longrightarrow X \ \exists M \ \forall \sigma, \tau \in S_r(M)$

$$\|f(\sigma) - f(\tau)\| \leq \omega_f(1) + \varepsilon.$$

Or $Q_X > 0 \Rightarrow \forall f : S_r(\mathbb{N}) \longrightarrow X$ ~~$\forall \varepsilon > 0$~~ $\forall \delta < Q_X$ X a $Q(1,\delta)$:

$\forall r \, \forall f : S_r(\mathbb{N}) \longrightarrow X$ avec $\sup(1) < \delta$ $\exists M \ \forall \sigma, \tau \in S_r(M)$ $\|f(\sigma) - f(\tau)\| < 1$

Mikael de la Salle
25/1/13

cf Bader Monod Gelander : pb de pt fixe sur L^1. (Nowak?) (Ryll)

Strengthened property (T) and explicit construction of superexpanders.

Def: $\circ$ (X, d_X), (Y, d_Y) espaces métriques. $i : X \to X$ is a coarse embedding if $\exists\, \varphi_-, \varphi_+ : \mathbb{R}_+ \circlearrowleft$ increasing, $\underbrace{\varphi_- \xrightarrow[+\infty]{} +\infty}_{\text{proper}}$, $\varphi_- \leq \varphi_+$ s.t. $\varphi_- \circ d_X(x,x') \leq d_Y(i(x), i(x')) \leq \varphi_+ \circ d_X(x,x')$.

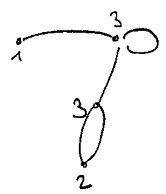
$\circ$ (X_n, d_n) suite d'espaces métriques coarsely embeds in (Y, d_Y) if $\forall n\ \exists i_n : X_n \to Y$ coarse s.t. φ_-, φ_+ are independent par de n.

But: Build graphs, X_n, find conditions on Y Banach space s.t. $(X_n) \overset{\text{coarse}}{\not\hookrightarrow} Y$

Remark: Conditions on Y are needed: $\forall X$ $X \overset{\text{isometric}}{\hookrightarrow} \ell^\infty(X)$.

Hence $\forall X_n$ graphs (finite) $X_n \hookrightarrow Y$ if Y has no cotype.

Expanders: Un graphe (fini) $G = (V, E)$.



The degree of $v \in V$ = # of edges having v as endpoint.

G is d-regular if $\deg v = d$ for all v.

The adjacency matrix of G is $A_G \in M_V(\mathbb{C})$, $(A_G)_{xy} = \frac{1}{d}$ # edges between x, y.

Then $A_G^* = A_G$ and $A_G \underset{(1,1,\dots,1)}{\underset{\shortparallel}{\mathbb{1}}} = \mathbb{1}$. Let $\ell_2^\circ(V) = \mathbb{1}^\perp \subseteq \ell_2(V) = \{(x_v)_{v\in V} : \sum x_v = 0\}$

$A_G : \ell_2^\circ(V) \to \ell_2^\circ(V)$. Define $\lambda_2(G) = \|A_G\|_{B(\ell_2^\circ(V))} \leq 1$ $(= 1 \Leftrightarrow G$ bipartite$)$

Def: A family $G_n = (V_n, E_n)$ of d-regular finite connected graph is a family of expanders if $\sup_n \lambda_2(G_n) < 1$ and $|V_n| \to \infty$.

[If $\{x, x\} \in E$ for all x, then $A_G = \frac{1}{d} I_d + \frac{d-1}{d} A_{G'}$; the smallest eigenvalue of A_G is $\geq \frac{1}{d} - \frac{d-1}{d}$].

Prop: [Gromov] If (G_n) is an expander, then $G_n \not\hookrightarrow H$ Hilbert space.

Proof: By contradiction: if we had $i_n : V_n \to H$, φ_-, φ_+; we can assume $\varphi_+(1)=1$; thus the i_n are 1-Lipschitz.

$\| A_{G_n}|_{\ell_2^0} \otimes id_H \|_{B(\ell_2^0 \otimes_2 H)} = \lambda_2(G_n) \leq 1-\varepsilon$.

Wlog we can assume $\forall n$ $\sum_{v\in V_n}' i_n(v)=0$ if $\xi_n = (i_n(v))_{v\in V_n} \in \ell_2^0 \otimes H$

$\|\xi_n\| \leq \| A_{G_n} \otimes id_H \xi_n \| + \| (A_{G_n} - Id) \otimes id_H \xi_n \|$
$\leq (1-\varepsilon)\|\xi_n\| + \Big\| \underbrace{\Big(\sum_{\substack{y\in V_n \\ y\sim x}}' \frac{\xi_n(y)}{d} - \xi_n(x) \Big)}_{\sum_{\substack{y\in V_n \\ y\sim x}}' \frac{(\xi_n(y)-\xi_n(x))}{d}} \Big\| \leq (1-\varepsilon)\|\xi_n\| + \sqrt{|V_n|}$.

average of elements which have norm ≤ 1 because of 1-Lipschitzness of i_n.

Thus $\|\xi_n\| \leq \frac{1}{\varepsilon}\sqrt{|V_n|} \Leftrightarrow \frac{1}{|V_n|} \sum_{v\in V_n}' \|i_n(v)\|^2 \leq \frac{1}{\varepsilon^2}$.

Take $R>0$. Let $\mathcal{E}_R = \{ v \in V_n : \|i_n(v)\| \leq R \}$.

$\frac{|\mathcal{E}_R^c|}{|V_n|} R^2 \leq \frac{1}{|V_n|} \sum_{v\in V_n}' \|i_n(v)\|^2 \leq \frac{1}{\varepsilon^2}$. Thus $1 - \frac{|\mathcal{E}_R|}{|V_n|} \leq \frac{1}{\varepsilon^2 R^2}$.

If $R = \frac{\sqrt{2}}{\varepsilon}$, we have $|\mathcal{E}_R| \geq \frac{1}{2}|V_n|$. But let us give a lower bound:

Pick $v_0 \in \mathcal{E}_R$: $\|i_n(v) - i_n(v_0)\| \leq 2R$ for all $v \in \mathcal{E}_R$.

Therefore $d(v,v_0) \leq C$ where C is such that $\varphi_-(C) > 2R$.

$\mathcal{E}_R \subseteq \{ v\in V_n : d(v,v_0) \leq C\} \leq \underbrace{1 + d^2 + \ldots + d^C}_{\text{by induction: } v_0 \text{ has } \leq d \text{ neighbours, etc.}}$

Remark: the proof is very flexible; more generally, if the G_n are graphs, with $|V_n| \to \infty$ d-regular, there is K s.t. $A_n \in M_{V_n}(\mathbb{C})$ with $\begin{smallmatrix} A_n = A_n^* \\ A_n \mathbb{1} = \mathbb{1} \end{smallmatrix}$, a Banach space Y s.t. $\sup_n \|A_n\|_{\ell_2^0(Y)} \leq \delta < 1$ and the $(A_n)_{x,y} = 0$ for x,y with $d(x,y) > K$.

n.b.: $\ell_2^0(Y) = \{ (y_v)_{v\in V_n} : \Sigma y_v = 0 \}$

Th (Pisier) If Y is ϑ-Hilbertian, $\vartheta < 1$ [i.e., $Y = (Y_0, Y_1)_\vartheta$ with Y_0 Hilbert space, Y_1 compatible B-sp.]
Then for any sequence of expanders G_n, $G_n \underset{\text{coarsely}}{\not\hookrightarrow} Y$.

Proof: 1) Observation: If $\sup \lambda_2(G_n) \lneq 1$, let $P_n : \ell_2(V_n) \to \ell_2^0(V_n)$ orthogonal proj.
$\|P_n \otimes id\|_{\ell_2(V_n,Y)} \leq 2$ for all B-spaces Y. Consider $A_{G_n} P_n \otimes id$: it has norm $\begin{smallmatrix} \leq \lambda_2(G_n) \text{ on } Y_0 \\ \leq 2 \text{ on } Y_1 \end{smallmatrix}$

Therefore it has norm $\leq \underbrace{\lambda_2(G_n)}_{<1}^{1-\vartheta} 2^{\vartheta}$ on $Y_\vartheta = Y$ (3)

[In fact $A_{G_n} P_n \otimes id$ also has norm $\leq \begin{matrix} \lambda_2(G_n) \text{ on } \ell_2(V_n; Y_0) \\ 2 \quad \text{on } \ell_2(V_n; Y_1) \end{matrix}$.

<u>General case</u>: If $\vartheta < 1$ fixed, there is $K \in \mathbb{N}$ s.t. $\|A^K_{G_n}\|^{1-\vartheta}_{B(\ell_2^0(V_n))} 2^{\vartheta} < 1$.
$$\| A_{G_n}\|^{K(1-\vartheta)}_{B(\ell_2^0(V_n))} 2^{\vartheta}$$

and we use the first step.

this implies $G_n \not\hookrightarrow Y$ by the Remark.

[In his Memoir AMS, Pisier proves Y is ϑ-Hilbertian iff $\forall T: \begin{matrix} L^\infty(\Omega,\mu) \to L^\infty & \text{norm } 1 \\ L^2 \to L^2 & \varepsilon \\ L^1 \to L^1 & \text{norm } 1, \end{matrix}$

then $\|T \otimes id_X\|_{L^2(\Omega, X)} \leq C \varepsilon^{\vartheta}$.

<u>Goal</u>: to construct an explicit family of expanders st $G_n \overset{\text{coarsely}}{\not\hookrightarrow} Y_\vartheta$ with $\vartheta < 1$,
where $Y_\vartheta = (Y_0, Y_1)_\vartheta$, Y_0 has type p, cotype q, $\frac{1}{p} - \frac{1}{q} < \frac{1}{4}$.
Q: What are these spaces? All such spaces have nontrivial type. Converse?

II Property (T) and construction of expanders (Margulis)

<u>Convention</u>: G l.c.g. A representation of G on X is a homomorphism $\pi: G \to B_{inv}(X)$ such that $\forall x$ $g \mapsto \pi(g).x$ is norm continuous.

<u>Def</u> A unitary rep. of G on a Hilbert space H almost has invariant vectors (a.i.v.) if $\exists (\xi_\alpha)$ net, $\|\xi_\alpha\|_H = 1$, $\forall Q \subset G$ compact $\sup_{g\in Q} \|\pi(g)\xi_\alpha - \xi_\alpha\| \underset{\alpha}{\to} 0$
If G is discrete, $\forall g \in G$ $\|\pi(g)\xi_\alpha - \xi_\alpha\| \to 0$

<u>Def</u>: G has (T) if for all unitary rep^s of G with a.i.v., $H^G \neq \{0\}$, where $H^G = \{\xi \in H : \pi(g)\xi = \xi \text{ for all } g \in G\}$

<u>Trivial example</u>: G finite or compact.
<u>Difficult example</u>: $SL(3,\mathbb{Z})$; $SL(3,\mathbb{R})$ [Kazhdan]
$SL(3, \mathbb{Q}_p)$

<u>Th</u> [Margulis] If Γ is a countable discrete group with (T), and S is a finite generating subset in Γ (with $1 \in S$, $s^{-1} \in S$ for $s \in S$)
If there is a sequence (X_n) of finite sets with $|X_n| \to \infty$ and an action $\Gamma \curvearrowright V_n$ that is transitive, define $G_n = (V_n, E_n)$ the graph with edges (x, gx) for $g \in S$:
G_n is a $\delta = |S|$-regular graph, connected; G_n is an expander.

N.B. For $x \in V_n$, the neighbours of x are the gx, $g \in S$. ⑥

<u>Example</u>: $\Gamma = SL(3,\mathbb{Z})$. Take $V_n = SL(3, \mathbb{Z}/n\mathbb{Z})$. Then $\Gamma \curvearrowright V_n : \gamma \cdot a = (\sum_k \gamma_{ik} a_{kj})$

Here $S = ($ diag. matrices w. coeff ± 1; A_{ij} $\begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})$

<u>Proof</u>: 1st observation: for each unitary rep. π of G on H, $H^G = \{0\}$,

$\| \underbrace{\frac{1}{|S|} \sum'_{s \in S} \pi(s)}_{A} \|_{B(H)} < 1$. In fact, otherwise $\exists\, \xi_n \in H$, $\|\xi_n\| = 1$, $\langle A\xi_n, \xi_n \rangle \to \pm 1$

since $s \in S$ $\sum'_{s \in S} \langle \pi(s)\xi_n, \xi_n \rangle \longrightarrow \pm |S|$ and one of the terms is $=1$;

the others are ± 1, so that $\sum'_{s \in S} \langle \pi(s)\xi_n, \xi_n \rangle \longrightarrow |S|$ and $\langle \pi(s)\xi_n, \xi_n \rangle \to 1$

but $\|\pi(s)\xi_n - \xi_n\| \xrightarrow{n \to \infty} 0$ $\lightning$

2nd observation: $\exists \varepsilon > 0$ $\| \frac{1}{|S|} \sum'_{s \in S} \pi(s) \| < 1 - \varepsilon$ for π unitary rep. on H with $H^\Gamma = \{0\}$.

In fact, otherwise, if π_n is a sequence like above,

$\oplus \pi_n : G \longrightarrow \mathcal{U}(\oplus H_n)$ s.t. $\| \frac{1}{|S|} \sum'_s \pi_n(s) \| \to 1$, $\oplus \pi_n$ has no

fixed vector and $\| \frac{1}{|S|} \sum'_{s \in S} \oplus \pi_n(s) \| = \sup_n \| \frac{1}{|S|} \sum'_{s \in S} \pi_n(s) \| = 1$

3rd observation: consider for each n the action of Γ on $\ell^0_2(V_n)$. Since it is

transitive, there is no Γ-fixed vector. Therefore $\underbrace{\| \frac{1}{|S|} \sum_{s \in S} \pi_n(s) \|}_{A_{G_n}}{}_{B(\ell^0_2(V_n))} \le 1 - \varepsilon$.

II Banach space case

Theorem: (Vincent Lafforgue) If $G = SL(3, \mathbb{Q}_p)$ [a topological group], there is $f_n \in C_c(G)$, $f_n \ge 0$, such that for all uniformly bounded representations π of G on a Banach space X with nontrivial type, $\pi(f_n) = \int \pi(g) f_n(g)\, dg \in B(X)$

$\downarrow{}^{n \to \infty}$ in $\| \ \|_{B(X)}$-norm $\qquad$ Haar m. on G

P a projection on X^G;

Ex: If $X^G = \{0\}$, $\|\pi(f_n)\| \longrightarrow 0$;

Ex: If $X = \mathbb{C}$ with trivial action, $\int f(g)\, dg \longrightarrow 1$.

$K = SL(3, \mathbb{Z}_p)$ compact (and open) subgroup of G: If $g_n \in G$, $g_n \to \infty$, $f_n = \mathbb{1}_{K g_n K}$.

If π is a unitary representation of G on H, $H^G = \{0\}$, there is $f_n \in C_c(G)$

with $\|\pi(f_n)\| < \frac{1}{2}$ where $\int f_n = 1$. Thus, for all $\xi \in H$, $\|\xi\| = 1$, $\|\pi(f_n)\xi - \xi\| \ge \frac{1}{2}$

thus $\frac{1}{2} \le \int f_n(g) \sup_{g \in \mathrm{supp}\, f_n} \|\pi(g)\xi - \xi\| \le \sup_{g \in \mathrm{supp}\, f_n} \|\pi(g)\xi - \xi\|$ $\qquad$ $\int f_n(g) \|\pi(g)\xi - \xi\| \ge \frac{1}{2}$.

<u>Th</u> : If $G = SL(3,\mathbb{R})$, the same holds, but only for Banach space Y_θ, Y_θ has type p, cotype q with $\frac{1}{p} - \frac{1}{q} < \frac{1}{4}$.

<u>Corollary②</u>: If Γ is a lattice in $SL(3,\mathbb{Q}_p)$, $\Gamma = SL(3,\mathbb{Z})$, the associated expanders do not $\not\hookrightarrow$ coarsely X for all X as in the Theorem.

N.B.: $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ for the absolute value $\left|\frac{a}{b}\right| = p^{v_p(b) - v_p(a)}$.

$\mathbb{Q}_p = \left\{ \sum'_{n \geq N} a_n p^n : N \in \mathbb{Z},\ a_n \in \{0, \dots, p-1\} \right\}$: in this rep, $|\ | = p^{-N}$
$a_N \neq 0$.

<u>Corollary①</u>: let Γ be a lattice in $G \in \{SL(3,\mathbb{Q}_p), SL(3,\mathbb{R})\}$; then there are $g_n : \Gamma \to \mathbb{R}$ finitely supported s.t. for all isometric representations π of Γ on X as in Th A or Th B, $\pi(g_n) \xrightarrow[B(X)]{n \to \infty}$ a projection on X^G.

<u>Def</u>: a discrete subgroup Γ of a locally compact group G with Haar measure dg is a lattice if $\exists \Omega \subseteq G$ of finite measure s.t. $\Omega\Gamma = \{\omega\gamma : \omega \in \Omega, \gamma \in \Gamma\} = G$.

<u>Ex</u>: $\mathbb{Z} < \mathbb{R}$, $\Omega = [0,1[$; $SL(3,\mathbb{Z}) < SL(3,\mathbb{R})$

<u>Proof of Corollary 1</u>: Since Γ is a lattice in G, there is $f \in L^1(G)^+$ s.t. $\sum'_{\gamma \in \Gamma} f(g\gamma) = 1$ a.s. in g.
In fact, we can assume $\forall g \in G\ \exists! (\omega, \gamma)\ \omega\gamma = g$ and take $f = 1_\Omega$.

<u>Define</u> $g_n(\gamma) = \int_{G \times G} f_n(g_1) f(g_2\gamma) f(g_1 g_2)\, dg_1 dg_2$. Then $\sum'_{\gamma \in \Gamma} g_n(\gamma) = \int_{G \times G} f_n(g_1) f(g_1 g_2)\, dg_1 dg_2$
$g_n \geq 0$,
and put $\tilde{g}_2 = g_1 g_2$: thus it $= \int f_n(g_1) \int f(\tilde{g}_2) = \int f_n(g_1) = 1$

[Here we normalised the Haar measure on Γ so that $\int f = 1$]

Take π isometric representation of Γ on X. Let $X' = \{ \xi : G \to X \text{ s.t. } \xi(g\gamma) = \pi(\gamma^{-1})\xi(g) \}$ with norm$^2 = \int_G f(g) \|\xi(g)\|^2 dg$. $X' \subseteq L^2(G, f\,dg; X)$

then let $\pi' : G \to \mathrm{Isom}(X')$ s.t. $\pi'(g)\xi(x) = \xi(g^{-1}x)$.

Consider $\alpha : X \to X'$
$x \mapsto \alpha(x) = \sum_{\gamma \in \Gamma} f(g\gamma)\pi(\gamma)x$ and $\beta : X' \to X$
$\beta(\xi) = \int_G f(g)\xi(g)\,dg$

We have $\beta \circ \alpha(x) = \int_G f(g) \sum'_{\gamma \in \Gamma} f(g\gamma)\pi(\gamma)x$ and $\beta \circ \pi'(f_n) \circ \alpha(x) = ?$

$\pi'(f_n) \circ \alpha(x) = \int f_n(g_1)\pi'(g_1)\alpha(x) : g \mapsto \int f_n(g_1)\alpha(x)(g_1^{-1}g) = \int f_n(g_1) \sum'_{\gamma \in \Gamma} f(g_1^{-1}g\gamma)\pi(\gamma)x$.

The details went wrong, but the idea was to apply the theorem to π'.

As for Cor. 2, we shall check that $\pi(g_n) \to 0$

① 5/12/13
Michael

<u>Th</u> [Lafforgue] $SL_3(\mathbb{Q}_p)$ has (T) wrt to the class of spaces with type $p > 1$, which is the class of K-convex spaces.

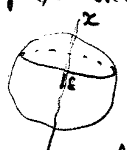
<u>Remark:</u> G cannot have (T) wrt L^1 unless G is compact.

<u>Proof:</u> If you take the trivial rep. π, then $\pi(f_n) = \int_G f_n(g)\pi(g)\,dg = \int f_n(g)\,dg$, so that the net $(f_n) \in C_c(G)$ in the definition of (T) satisfies $\int f_n(g) \to 1$, and $G \curvearrowright L^1(G)$ by left translations, transitively: $L^1(G)^G = \{0\}$, while $\|\lambda(f)\|_{B(L^1(G))} \geq \left|\int f(g)\,dg\right|$ (exercise)

<u>ThA</u> [Lafforgue] $SL_3(\mathbb{R})$ has (T) wrt $\mathcal{E}_{mypt}$, where $\mathcal{E}_{mypt} = \{E : \exists C\ \exists \beta > 0\ \|T_0 - T_\delta\|_{L^2(S^2;E)} \leq C|\delta|^\beta\}$

Here, for $\delta \in [-1, 1]$, $T_\delta : L^2(S^2) \circlearrowleft$ and if f is complex continuous on S^2,

$T_\delta f(x)$ = average of $f(y)$ on $\{y : \langle x, y\rangle = \delta\}$



Second def of T_δ: Let $K = SO_3(\mathbb{R})$, $U = \begin{bmatrix} 1 & 0 \\ 0 & SO_2 \end{bmatrix} \subset K$. Then

$U\backslash K \simeq S^2$
$k \mapsto k^{-1}e_1$
$\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) \in \mathbb{R}^3$

Consider $\lambda : K \to \mathcal{U}(L^2(K))$

Claim: If $k \in K$, $\iint \lambda(u_1 k u_2)\, du_1\, du_2 \in B(L^2(K))$ (Haar prob. measure on $\mathcal{U}$)

$$= \begin{cases} 0 & \text{on } L^2(U\backslash K)^\perp \\ T_\delta & \text{on } L^2(U\backslash K) \end{cases} \text{ if } k_{11} = \delta.$$

<u>Fact</u>. (description of $\mathcal{U}\backslash K/\mathcal{U}$) For $k, k' \in K$, $UkU = Uk'U \Leftrightarrow k_{11} = k'_{11}$.

Proof: $\Rightarrow$ is clear (i.e. $\exists u_1, u_2 \in U$ $u_1 k u_2 = k'$)

$\Leftarrow$ take $k \in K$ with $k_{11} = \delta$: $k = \begin{pmatrix} \delta & \ell \\ c & A \end{pmatrix}$: $\|c\| = \sqrt{1-\delta^2}$, so that

there is $u_1 \in SO_2$ s.t. $u_1 c = \begin{pmatrix} \sqrt{1-c^2} \\ 0 \end{pmatrix}$. Similarly, $u_2 \in SO_2$ s.t. $\ell u_2 = (-\sqrt{1-\delta^2}\ \ 0)$

Then $\begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix} k \begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix} = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \multicolumn{2}{c}{\begin{bmatrix} \delta & 0 \\ 0 & 1 \end{bmatrix}} \\ 0 & & \end{pmatrix}$

you have no choice here

you take δ here.

and you take $x_\delta = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Remark: the representation may be taken to grow slightly.

We prove $E \in \mathcal{E}_{mypt}$ strictly convex, $\pi : G \to B(E)$ isometric representation, then

$\pi(KgK') = \int_{K \times K} \pi(kgk')\, dk\, dk' \xrightarrow[g \to \infty]{\|\cdot\|_{B(E)}} P$ projection on E^G

1^{st} step: Cauchy criterion for $g \longmapsto \pi(KgK)$

Fact: $K \backslash G / K \simeq \Lambda = \{(r,s,t) \in \mathbb{R}^3 : r+s+t=0,\ r \geq s \geq t\}$

$$K \begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix} K \longleftarrow (r,s,t) \qquad \text{(polar decomposition)}$$

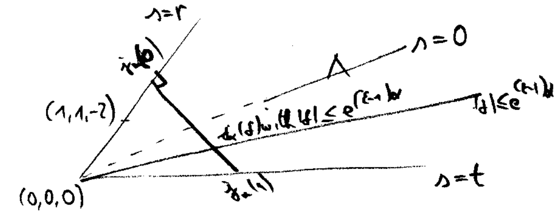
$$K\ g\ K \longmapsto (r = \log \|g\|_{B(\mathbb{R}^3)},\ s,\ t = -\log\|g^{-1}\|)$$

Denote $f : \Lambda \longrightarrow B(E)$

$$\lambda = (r,s,t) \longmapsto f(\lambda) = \pi\left(K \begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix} K\right)$$

N.B.: We are looking for a net of f^n with compact support to prove Th A. Here we get Radon measures KgK. Take $f \in C_c(SL_3(\mathbb{R}))$ with $\int f = 1$, $f \geq 0$. Take $g_n \in G$, $g_n \to \infty$, $f_n \in C_c(G)$. ($SL_3(\mathbb{R}) = G$)

$$f_n(g) = \iint f(k g g_n k')\, dk\, dk'$$



We have here $U \subseteq K \subseteq G$.

<u>Lemma</u>: $\|\pi(Ux_0U) - \pi(Ux_\delta U)\|_{B(E)} \leq C|\delta|^s$

$$\iint \pi(u_1 x_0 u_2)\, du_1\, du_2 .$$

N.B.: If you have any unitary rep. for a compact group, it is weakly contained in the left regular representation: every norm $\leq$ valid for the second is valid for the first. Here, same trick for Banach space. "Fell absorption principle"

<u>Proof</u>: $i : E \longrightarrow L^2(K; E)$

$i(x)(k) = \pi(k^{-1})x$.

$\lambda(k_1) i(x)(k) = i(x)(k_1^{-1}k) = \pi(k^{-1})[\pi(k_1)x] = i(\pi(k_1)x)(k)$

$\lambda(k_1) i(x) = i(\pi(k_1)x)$

$i\big((\pi(Ux_0U) - \pi(Ux_\delta U))x\big) = [\lambda(Ux_0U) - \lambda(Ux_\delta U)]\, i(x)$

$$\|(\pi(Ux_0U) - \pi(Ux_\delta U))x\|_E = \|\underbrace{i(\ \ )}_{=(T_0 - T_\delta)i(x)}\|_{L^2(K;E)}$$

$$\leq \|T_0 - T_\delta\|_{B(L^2(K;E))}$$

"Hölder continuity of exponents"

Embed $(K, U) \hookrightarrow (G, K)$: consider, for $\alpha > 0$, $D_\alpha = \begin{pmatrix} e^\alpha & 0 & 0 \\ 0 & e^{-\frac{\alpha}{2}} & 0 \\ 0 & 0 & e^{-\frac{\alpha}{2}} \end{pmatrix} \in G$,

$i_\alpha : K \longrightarrow G$ $\qquad$ D_α commutes with U, so that

$k \longmapsto D_\alpha k D_\alpha$ $\qquad$ $K\, i_\alpha(u_1 k u_2)\, K = K\, i_\alpha(k) K$ and

$f_\alpha : [-1, 1] \longrightarrow K \backslash G / K$ ($[-1,1] = U \backslash K / U$, $K \backslash G / K = \Lambda$)

i_α factors : gives $U^K/U \to U^G/U$

Lemma: $\|f(j_\alpha(0)) - f(j_\alpha(\delta))\|_{B(E)} \leq C|\delta|^s$

$f(j_\alpha(0)) = \pi(K D_\alpha x_0 D_\alpha K) = \pi(KD_\alpha)[\pi(U x_0 U)]\pi(D_\alpha K)$

$\|\pi(KD_\alpha)\| \leq \int \underbrace{\|\pi(kD_\alpha)\|}_{\leq 1} dk$

Let us compute: $j_\alpha(1)$: $i_\alpha(n_1) = \begin{pmatrix} e^{2\alpha} & & \\ & e^{-\alpha} & \\ & & e^{-\alpha}\end{pmatrix}$, so that $j_\alpha(1) = (2\alpha, -\alpha, -\alpha)$

$i_\alpha(n_0) = \begin{pmatrix} 0 & e^{-\frac{\alpha}{2}} & 0 \\ e^{\frac{\alpha}{2}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so that $j_\alpha(0) = (\frac{\alpha}{2}, \frac{\alpha}{2}, -\alpha)$

Proof: $i_\alpha(n_\delta) = \begin{pmatrix} e^{2\alpha}\delta & -e^{\frac{\alpha}{2}}\sqrt{1-\delta^2} & 0 \\ e^{\frac{\alpha}{2}}\sqrt{1-\delta^2} & \delta e^{-\alpha} & 0 \\ 0 & 0 & e^{-\alpha} \end{pmatrix}$. By definition,

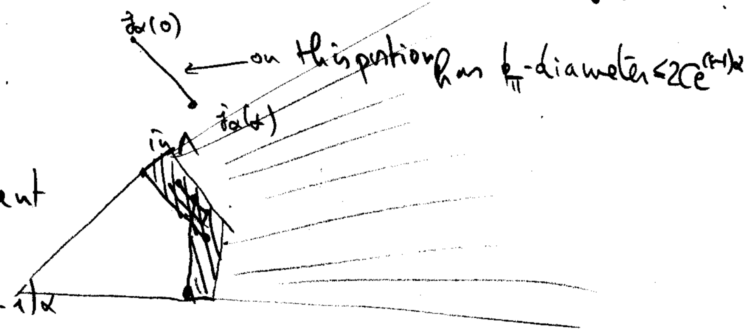
$j_\alpha(\delta) = (r, s, t)$, $r+s+t=0$. $r = \log \|i_\alpha(n_\delta)\|$

$t = -\log \|i_\alpha(n_\delta)^{-1}\|$

$t = -2\alpha$.

If $s = \varepsilon t$, $j_\alpha(\delta) = ((1+\varepsilon)\alpha, -\varepsilon\alpha, -\alpha)$, $e^{2\alpha}|\delta| \leq \|i_\alpha(n_\delta)\| = e^{(1+\varepsilon)\alpha}$

If $d_\pi(\lambda, \lambda') = \|f(\lambda) - f(\lambda')\|_{B(E)}$



Take its symmetric segment and take the region: its d_π-diameter $\leq 6Ce^{(\varepsilon-1)\alpha}$

Then the whole region has d_π-diameter $\leq 6C \sum_n e^{(\varepsilon-1)(1+\varepsilon)^n\alpha}$

Thus the Cauchy criterion is satisfied and $P = \lim \pi(KgK)$ exists

<u>Claim</u>: $\pi(Kg)P = P$ for all $g \in G$.

$$\pi(Kg)P = \lim \pi(Kg)\pi(KhK) = \int_K \pi(Kgkh K)\,dk \quad \text{(} \downarrow \text{ unif for } k \in K \text{, } \infty\text{)}. \text{ Hence } P^2 = P.$$

Indeed, "$\lim_{g\to\infty} \pi(KgK)P = \lim_{g \to\infty} P$", with $\pi(KgK)P = \pi(Kg)P$.

If $x \in E^G$, $Px = x$. If $x \in \operatorname{Im} P$, let us show that $x \in E^G$.

By the claim, for all $g \in G$, $\int_K \pi(kg)\,x\,dk = x$ (average, unit vector; π strictly contractions). By strict convexity, $\pi(kg)x = x$.

If the space was not strictly convex, it would become much more difficult to conclude.

Study of the mysterious class $\mathcal{E}_{myst}$.

<u>Lemma</u>: (Lafforgue) $\|T_0 - T_\delta\|_{B(L^2(K))} \leq 4\sqrt{\delta}$.

Thus $\mathbb{C} \in \mathcal{E}_{myst}$ and every Hilbert space belongs to it.

[Recall if $E = H$ Hilbert space, $L^2(K;H) \simeq L^2(K) \otimes_2 H$: $\langle f\otimes x, g \otimes y\rangle = \langle f,g\rangle_{L^2}\langle x,y\rangle$

$\|T_1 \otimes T_2\|_{B(H_1 \otimes H_2)} = \|T_1\|\|T_2\|$.

Recall $T_\delta = \lambda(U x_\delta U) \in B(L^2(K))$: $\lambda : K \longrightarrow \mathcal{U}(L^2(K))$

$\lambda(k_1)f(k_2) = f(k_1^{-1}k_2)$.

<u>Prop</u>: (Pisier) $\mathcal{E}_{myst} \supseteq \vartheta$-Hilbertian spaces with $\vartheta \in]0,1[$

$= \{[X_0, X_1]_\vartheta : X_0 \text{ Hilbert space}, X_1 \text{ arbitrary}\}$

<u>Proof</u>: $\|T_\delta\|_{B(L^2(K;E))} \leq 1$ for all E: If $f \in L^2(K;E)$, $T_\delta f(k) = \iint f(u_1 x_\delta^{-1} u_2 k)\,du_1 du_2$

$T_\delta = \iint \underbrace{\lambda(u_1 x_\delta u_2)}_{\text{isometry}!} du_1 du_2$

If $X = [X_0, X_1]_\vartheta$, $L^2(K;X) = [L^2(K;X_0), L^2(K;X_1)]_\vartheta$, so that

$\|T_0 - T_\delta\|_{B(L^2(K;X))} \leq \|T_0 - T_\delta\|_{X_0}^{1-\vartheta} \|T_0 - T_\delta\|_{X_1}^{\vartheta} \leq (4\sqrt{|\delta|})^{1-\vartheta} 2^{\vartheta}$

This implies the results by Baptiste Olivier in his talk!

But this is only interesting for fixed point properties, not for coarse embeddings.

For coarse embeddings, we need another result.

<u>Observation</u> (Lafforgue - de la Salle) $\forall p > 1\ \exists C_p$ $\|T_0 - T_\delta\|_{S^p(L^2(K))} \leq C_p |\delta|^{\frac{1}{2} - \frac{2}{p}}$.

<u>Recall</u>: If H is a Hilbert space, $S^p(H) = \{T \in B(H) : \mathrm{Tr}(T^*T)^{p/2} < \infty\}$

$\|T\|_{S^p}^p$

If $p > q$, $S^p \supseteq S^q$

<u>Consequence</u>: $\mathcal{E}_{myst} \supseteq \{E : \text{type } p, \text{ cotype } q, \frac{1}{p} - \frac{1}{q} < \frac{1}{4}\}$.

Recall: E has type p if $\exists\, T_p(E)$ $\|\sum g_i x_i\|_{L^2(\Omega;E)} \leq T_p (\sum \|x_i\|^p)^{\frac{1}{p}}$

g_i i.i.d. $\mathcal{N}(0,1)$

cotype q $\exists\, C_q(E)$ $\geq \frac{1}{C_q(E)} (\sum \|x_i\|^q)^{\frac{1}{q}}$.

Pisier - Xu prove that there are nonreflexive examples

Theorem (König, Retherford, Tomczak-Jägermann): If $u : \ell^2(E) \circlearrowleft$ has norm 1 and rank $\leq n$, then $\|u \otimes id_E\|_{B(\ell^2(E))} \leq K_{pq} T_p(E) C_q(E)\, n^{\frac{1}{p}-\frac{1}{q}}$.

there is a conjecture: $\frac{1}{p}-\frac{1}{q}$ may be replaced by $\max\left(\frac{1}{p}-\frac{1}{2}, \frac{1}{2}-\frac{1}{q}\right)$.

Corollary: if $u : \ell^2 \to \ell^2$, $\|u\|_{S^r} < \infty$, $\frac{1}{r} > \frac{1}{p}-\frac{1}{q}$, then $\|u \otimes id_E\|_{B(\ell^2(E))} \leq K_{r,p,q} T_p(E) C_q(E) \|u\|_{S^r}$.

Proof: write $u = \sum_k' \alpha_k u_k$, with u_k of norm 1, rank $\leq 2^k$, with $\sum' |\alpha_k|^r 2^k \leq 2\|u\|_{S^r}^r$.

$\to$ look at the singular values of u: $(\underbrace{s_1}_{1}, \underbrace{s_2, s_3}_{2}, \underbrace{s_4 \ldots}_{4}, \ldots) \in \ell_r$

take $\alpha_k = s_{2^k}$: u_k has singular values $\frac{1}{s_{2^k}}(s_{2^k}, \ldots, s_{2^{k+1}-1})$

$\longrightarrow$ comparison of weak ℓ_r and ℓ_r.

Then $\|u\|_{B(\ell^2(E))} \leq \sum_k' \alpha_k \|U_k\|_{B(\ell^2(E))} \overset{KRT}{\leq} C \sum' \alpha_k 2^{k(\frac{1}{p}-\frac{1}{q})} \leq C\left(\sum_k' |\alpha_r|^r 2^k\right)^{\frac{1}{r}} \underbrace{\left(\sum_k' 2^{k(\frac{1}{p}-\frac{1}{q}-\frac{1}{r})r'}\right)^{\frac{1}{r'}}}_{<\infty}$

$\frac{1}{p}-\frac{1}{q}-\frac{1}{r}<0.$

We have now a result on ℓ_2: we pass to L^2 with martingale techniques.

$L^2(S^2) \simeq \bigoplus_n H_n$ with $H_n = \{$restrictions to S^2 of homogeneous polynomials on $\mathbb{R}^3$ of $d^\circ n$, harmonic$\}$

$K \curvearrowright L^2(S^2)$. The H_n are irreducible representations and $\dim H_n = 2n+1$.

The T_δ and π_k commute for all $k \in K$ and $\delta \in]-1,1]$: $T_\delta = \bigoplus_n \underbrace{P_n(\delta)}_{\text{Legendre polynomial}} 1_{H_n}$:

This is exactly how Legendre introduced his polynomials.

Thus $\|T_0 - T_\delta\|_{S^p}^p = \sum_n' (2n+1) |P_n(\delta) - P_n(0)|^p$

① Mendel Naor (I)

Title:

I) Basics on ~~expanders~~ spectral analysis of graphs and expanders

Let $G = (V, E)$ be a graph

V is finite, E is a "multisubset" of $V \times V$

$\forall (u,v) \in V\times V$ $E(u,v) = \#$ times (u,v) appears in E

$$E(u,v) = E(v,u)$$

$[E(u,v)]_{(u,v)\in V\times V}$ integer valued symmetric matrix

$\forall u \in V$ $\deg_G(u) = \sum_v E(u,v)$

<u>G is d-regular</u> if $\forall u$ $\deg_G(u) = d$

Then the <u>normalized adjacency matrix</u> is $A_G = \frac{1}{d}[E(u,v)]_{u,v}$

A_G is a symmetric stochastic matrix.

<u>Prop 1</u> Let $A = [a_{ij}]$ $n\times n$ sym. stoch. matrix

Denote $\lambda_n \leq \dots \leq \lambda_1$ its eigenvalues

Then <u>(a)</u> $-1 \leq \lambda_n \leq \dots \leq \lambda_1 \leq 1$.

<u>(b)</u> $\forall x_1, \dots, x_n \in \mathbb{R}$

$$\frac{1}{n^2}\sum_{i,j}(x_i - x_j)^2 \leq \frac{1}{1-\lambda_2} \times \frac{1}{n}\sum_{i,j} a_{ij}(x_i-x_j)^2$$

<u>(c)</u> $\forall x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$

$$\frac{1}{n^2}\sum_{i,j}(x_i - y_j)^2 \leq \frac{1}{1-\lambda}\frac{1}{n}\sum_{i,j} a_{ij}(x_i-y_j)^2$$

where $\lambda = \max(|\lambda_2|, |\lambda_n|)$.

<u>Pf</u> (a) Let $x, y \in \mathbb{R}^n$ $\|x\|_2, \|y\|_2 \leq 1$

$$|\langle Ax, y\rangle| = \left|\sum_i \left(\sum_j a_{ij}x_j\right)y_i\right| = \left|\sum_i\sum_j a_{ij}^{1/2}y_i\, a_{ij}^{1/2}x_j\right|$$

$$\leq \sum_i\sum_j \frac{1}{2}a_{ij}y_i^2 + \frac{1}{2}a_{ij}x_j^2 \leq \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 \leq$$

$+$ $A.\mathbb{1} = \mathbb{1}$

(2) (b) easier than (c)

(c). $\frac{1}{n}\sum_{i,j}(x_i-y_j)^2 = \sum_i x_i^2 + \sum_j y_j^2 - \frac{2}{n}\sum_i x_i\left(\sum_j y_j\right)$

$= \|x\|^2 + \|y\|^2 - 2\langle By, x\rangle \qquad B = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \vdots & & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{bmatrix}$

$\sum_{i,j} a_{ij}(x_i-y_j)^2 = \sum_i \left(\sum_j a_{i,j}\right)x_i^2 + \sum_j\left(\sum_i a_{ij}\right)y_j^2 - 2\sum_i\left(\sum_j a_{ij}y_j\right)x_i$

$= \|x\|^2 + \|y\|^2 - 2\langle Ay, x\rangle$

We want to show that

$$(1-\lambda)\left(\|x\|^2+\|y\|^2-2\langle By,x\rangle\right) \leq \|x\|^2+\|y\|^2-2\langle Ay,x\rangle$$

$$\Leftrightarrow \quad 2\langle (A-B)y, x\rangle \leq \lambda\left[\|x\|^2+\|y\|^2 - 2\langle By,x\rangle\right]$$

$\exists$ o.n.b. $\quad A \sim \begin{bmatrix} 1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} \qquad B \sim \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right]$

$x = x_1 + x_2 \quad y = y_1 + y_2$ in the decomposition $\mathbb{R}^n = [\mathbb{1}] \oplus \mathbb{1}^\perp$

$|2\langle (A-B)y, x\rangle| = |2\langle (A-B)y_2, x_2\rangle| \leq 2\lambda \|x_2\|\|y_2\|$

$\leq \lambda\left(\|x_2\|^2+\|y_2\|^2\right) \leq \lambda\left(\|x_2\|^2+\|y_2\|^2+x_1^2+y_1^2 - 2x_1y_1\right)$

$= \lambda\left(\|x\|^2+\|y\|^2 \pm 2\langle By, x\rangle\right).$ $\square$

<u>Rem</u> 1) λ is optimal in (c) (λ_2 optimal in (b)).

2) We can replace $(\ )^2$ by $\|\ \|_H^2$ in (a) and (b).

Let G d-regular

Proposition 2 (a) $\lambda_2 < 1 \iff G$ connected

(b) $\lambda_n = -1 \iff G$ has a bipartite connected component.

<u>Pf</u> (a). Assume G is disconnected and U is a connected component. Then $A\mathbb{1}_U = \mathbb{1}_U$ $\left\{ A\mathbb{1}_U(v) = \sum_{u\in U} A_{vu} = \begin{matrix} 1 & \text{if } v \\ 0 & \text{if } v \end{matrix} \right.$

- Assume G is connected. Let f so that $A_G . f = f$

Pick v so that $f(v)$ is maximal

Then $\sum_{u,(u,v)\in E} \frac{1}{d} f(u) = f(v)$

$\Rightarrow \forall u$ adjacent to v $\quad f(u) = f(v)$ + G connected $\Rightarrow f = \lambda\mathbb{1}$

(3) (b) $H \subset G$ connected component is bipartite if $H = X \dot\cup Y$ and every edge starting in X ends in Y.

Consider $f = 1_X - 1_Y$ we claim that $Af = -f$

$\forall v \in X \quad Af(v) = \sum_{u \in Y, (u,v) \in E} \frac{1}{d} f(u) = -1 = -f(v)$

also $\forall v \in Y \qquad Af(v) = -f(v)$.

$\forall v \in G \setminus H \qquad f(v) = Af(v) = 0$

Assume $\exists f$ so that $Af = -f \qquad f \neq 0$

Pick v so that $|f(v)|$ is maximal

Then $-f(v) = Af(v) = \sum_{u, (u,v) \in E} \frac{1}{d} f(u)$

$\Rightarrow \forall u$ connected to $v \qquad f(u) = -f(v)$

Same argument works for those u's.

So $\forall u$ in the connected component of v, we have $f(u) = \pm f(v)$ and for adjacent vertices the signes are opposite $\rightarrow$ this connected component is bipartite.

Edge expansion $\qquad G = (V, E) \qquad F \subset V$

∂F = set of edges connecting F to $V \setminus F$

$$h(G) = \operatorname{Inf}_{|F| \leq n/2} \left\{ \frac{|\partial F|}{|F|} \right\}.$$

Examples 1) If G is a n-cycle and F connected with $|F| \sim \frac{n}{2}$ then $\frac{|\partial F|}{|F|} \sim \frac{4}{n} \qquad h(G) \lesssim \frac{4}{n} \to 0$

2) G is the full graph with $|V| = n$.

$|F| = \ell \Rightarrow |\partial F| = \ell(n - \ell) \qquad \frac{|\partial F|}{|F|} = n - \ell$

$|F| \leq n/2 \dashrightarrow h(G) = \frac{n}{2} \rightarrow +\infty$.

Def Let (G_n) be a sequence of finite connected d-regular graphs. (G_n) is a family of expanders if $|V_n| \to +\infty$ and $\exists \varepsilon > 0 \quad \forall n \quad h(G_n) \geqslant \varepsilon$

4

Theorem 3 Let G (connected) be a finite d-regular graph (without loops?)

Then $\frac{1}{2} d(1-\lambda_2) \le h(G) \le d\sqrt{2(1-\lambda_2)}$.

Corollary 4 (G_n) d-regular graphs is a sequence of expanders $\iff$ $\lambda_2(A_{G_n})$ is uniformly bounded away from 1

$\iff \gamma_{G_n} = \frac{1}{1-\lambda_2(A_{G_n})}$ uniformly bounded.

Pf Th 3 ① $h(G) \ge \frac{1}{2} d(1-\lambda_2)$ ~~enough to show that $\frac{1}{1-\lambda_2} \le c$ $\Rightarrow$ G_n expanders~~

Rem: Let $F, H \subset V$

$d\langle A 1_F, 1_H\rangle = \sum_v \left(\sum_{u\sim v} 1_F(u)\right) 1_H(v)$

$=$ # of edges connecting F and H

So if $d\langle A 1_F, 1_F\rangle \le (d-s)|F|$

then $|\partial F| \ge s|F|$.

• Let $F \subseteq V$ $\quad 1_F = \frac{|F|}{|V|} 1_V + f \qquad f \in 1_V^{\perp}$

$\|1_F\|_2^2 = |F|, \quad \left\|\frac{|F|}{|V|} 1_V\right\|_2^2 = \frac{|F|^2}{|V|} \Rightarrow \|f\|_2^2 = |F| - \frac{|F|^2}{|V|}$

$$\langle A 1_F, 1_F\rangle = \left\langle A\frac{|F|}{|V|} 1_V, \frac{|F|}{|V|} 1_V\right\rangle + \langle Af, f\rangle$$

$$\le \frac{|F|^2}{|V|} + \lambda_2\left(|F| - \frac{|F|^2}{|V|}\right)$$

$$= (1-\lambda_2)\frac{|F|^2}{|V|} + \lambda_2|F|$$

If $|F| \le \frac{|V|}{2}$ $\quad \langle A1_F, 1_F\rangle \le (1-\lambda_2)\frac{|F|}{2} + \lambda_2|F| = \frac{1+\lambda_2}{2}|F|$

$$= \left(1 - \frac{(1-\lambda_2)}{2}\right)|F|$$

So $|F| \le \frac{|V|}{2} \Rightarrow |\partial F| \ge d\left(\frac{1-\lambda_2}{2}\right)|F|$

Thus $h(G) \ge \frac{1}{2} d(1-\lambda_2)$. $\square$

② $h(G) \le d\sqrt{2(1-\lambda_2)}$

Let $g \in 1_V^{\perp}$ st $Ag = \lambda_2 g$ $\qquad g \ne 0$

$V^+ = \{v \;\; g(v) > 0\}$ wma $|V^+| \le \frac{|V|}{2}$

Note $g \ne 0$ + $g \perp 1_V \Rightarrow g^+, g^- \ne 0$

(5)

<u>Lemma 1</u> $\langle (Id - A) g^+, g^+ \rangle \leq (1-\lambda_2) \|g^+\|_2^2$

<u>Pf</u> $A g^+ + A g^- = \lambda_2 g^+ + \lambda_2 g^-$ $\quad g^+ \perp g^-$

So $\langle A g^+ ~~+ zA~~ , g^+ \rangle + \langle A g^-, g^+ \rangle = \lambda_2 \langle g^+, g^+ \rangle$

Also $\langle A g^-, g^+ \rangle \leq 0 \Rightarrow \langle A g^+, g^+ \rangle \geq \lambda_2 \langle g^+, g^+ \rangle$.

Fix an orientation of the graphe . For each edge e choose an ~~ed~~ endpoint e^+ and denote the other e^-

For $f \in l_2(V)$ define $\nabla f \in l_2(E)$ by $\nabla f(e) = f(e^+) - f(e^-)$

Define $b_{eu} = \begin{cases} 1 & \text{if } u = e^+ \\ -1 & \text{if } u = e^- \\ 0 & \text{otherwise} \end{cases}$ $\quad B = [b_{eu}]_{E \times V}$

$f = [\,] \in l_2(V)$ $\quad \forall e \in E \quad \sum_{u \in V} b_{eu} f(u) = f(e^+) - f(e^-)$

$$\nabla f = B . f$$

<u>Lemma 2</u> $\|\nabla f\|_2^2 = \langle d (I - A) f, f \rangle$

(So $\|\nabla g^+\|_2^2 \leq d(1-\lambda_2) \|g^+\|_2^2$

<u>Pf</u> $\|\nabla f\|_2^2 = \langle Bf, Bf \rangle = \langle {}^tB B f, f \rangle$

$L = {}^tB B$ $\quad L_{uv} = \sum_{e \in E} b_{eu} b_{ev} = -\#$ appearances of uv $\quad u \neq v$

$L_{uu} = d$

$L = d\,Id - dA$. $\square$

<u>Lemma 3</u> G connected $f : V \to \mathbb{R}$ $\quad m$ median of f

Then $\sum_{v \in V} |f(v) - m| \leq \frac{1}{h(G)} \sum_e |f(e^+) - f(e^-)|$

<u>Pf</u> Assume $m = 0$ and $|V| = 2k+1$ is odd ~~and ho~~

$f_1 \leq \dots \leq f_k \leq f_{k+1} = 0 \leq f_{k+2} \leq \dots \leq f_{2k+1}$ values of f

$L_i^- = \{v, f(v) \leq f_i\}$ $\; i \leq k$ $\quad L_i^+ = \{v, f(v) \geq f_i\}$ $\; i \geq k+2$

$f_i^\Delta = f_{i+1} - f_i$ $\quad f_i^\nabla = f_i - f_{i-1}$

$\sum |f_i| = (f_2 - f_1) + (2f_3 - 2f_2) + \dots + k(0 - f_k) + k(f_{k+1} - 0) + \dots + (f_{2k+1} - f_{2k}$

$$\sum_v |f(v)| = \sum_{i=1}^{k} |L_i^-| f_i^\Delta + \sum_{k+2}^{2k+1} |L_i^+| f_i^\nabla$$

(6) $|L_i^-|, |L_i^+| \le k \le \frac{|V|}{2} \Rightarrow |\partial L_i^-| \ge h |L_i^-|$ $|\partial L_i^+| \ge h |L_i^+|$

So $\sum_{v\in V} |f(v)| \le \frac{1}{h}\left(\sum_i^k |\partial L_i^-| f_i^\Delta + \sum_{k+2}^{2k+1} |\partial L_i^+| f_i^\Delta \right)$

For a fixed edge uv : the contribution of uv to $(\quad)$ is $|f(u) - f(v)|$ ......

<u>End of proof</u> $\quad f(v) = (g^+(v))^2$

$|V^+| \le \frac{|V|}{2} \Rightarrow 0$ is a median of f

So $\sum_v |f(v)| = \|g^+\|_2^2 \le \frac{1}{h} \sum_{e\in E} |g^+(e^+) - g^+(e^-)| |g^+(e^+) + g^+(e^-)|$

So $\quad h \|g^+\|_2^2 \le \left(\sum_e |g^+(e^+) - g^+(e^-)|^2 \right)^{1/2} \left(\sum_e |g^+(e^+) + g^+(e^-)|^2 \right)^{1/2}$

$$\le \|\nabla g^+\|_2 \left(2 \sum_e |g^+(e^+)|^2 + |g^+(e^-)|^2 \right)^{1/2}$$

$$\le \sqrt{2d}\, \|\nabla g^+\|_2 \, \|g^+\|_2$$

Lemma 2 $\Rightarrow \quad h \le d\sqrt{2(1-\lambda_2)}$. $\square$

(Note that $g \ne 0$ $g \perp 1_V \Rightarrow g^+, g^- \ne 0$).

(7)

II) Generalized Poincaré constants

- Let X be an arbitrary set $K: X\times X \to [0,\infty)$ symmetric

$\underline{\gamma(A,K)}$ is the infimum of all $\gamma > 0$ st

$$\forall x_1,\dots,x_n \in X \quad \frac{1}{n^2}\sum_{i,j} K(x_i,x_j) \le \frac{\gamma}{n}\sum_{i,j} a_{ij} K(x_i;x_j) \quad (1)$$

(where A is a $n\times n$ sym stoch. matrix).

$\underline{\gamma^+(A,K)}$ is the infimum of all $\gamma_+ > 0$ st

$$\forall x_1,\dots,x_n \in X \quad \forall y_1,\dots,y_n \in X \quad \frac{1}{n^2}\sum_{i,j} K(x_i;y_j) \le \frac{\gamma_+}{n}\sum_{i,j} a_{ij} K(x_i;y_j)$$

If $G=(V,E)$ is a d-regular graph we denote

$\gamma(G,K) = \gamma(A_G,K) \qquad \gamma^+(G,K) = \gamma^+(A_G,K)$

$$(1) \iff \frac{1}{|V|^2}\sum_{V\times V} K(f(u),f(v)) \le \frac{\gamma}{|V|\times d}\sum_{E} K(f(u),f(v))$$

$$(\forall f: V\to X)$$

$(2) \iff \forall f: V\to X \quad \forall g: V\to X$

$$\frac{1}{|V|^2}\sum_{V\times V} K(f(u),g(v)) \le \frac{\gamma_+}{|V|\times d}\sum_{E} K(f(u),g(v)).$$

$\underline{\text{Clearly}}$ $\gamma \le \gamma^+$.

- $\underline{\text{Gromov's argument for non embeddability}}$

Assume that $G_n = (V_n, E_n)$ are connected d-regular graphs and denote d_{G_n} = geodesic distance on V_n

Assume also $G_n \underset{\text{coar.}}{\hookrightarrow} (Y, d_Y)$ i.e $\exists f_n: G_n \to Y$

i.e. $\exists \alpha,\beta: [0,\infty)\to[0,\infty)$ with $\lim_{t\to\infty}\alpha(t)=\infty$

and $\forall n \ \forall x,y\in V_n \quad \alpha(d_{G_n}(x,y)) \le d_Y(f_n(x),f_n(y)) \le \beta(d_{G_n}(x,y))$ (3)

Finally, assume that $\exists p>0 \ \exists \gamma>0$

$\forall n \quad \forall f: V_n \to Y$

$$\frac{1}{|V_n|^2}\sum_{V_n\times V_n} d_Y(f(u),f(v))^p \le \frac{\gamma}{d|V_n|}\sum_{E_n} d_Y(f(u),f(v))^p \quad (4)$$

in other words $\sup_n \gamma(G_n, d_Y^p) < \infty$

$(3)+(4) \Rightarrow \frac{1}{|V_n|^2} \sum_{V_n \times V_n} \left[\alpha(d_{G_n}(u,v))\right]^p \leq \gamma \beta(1)^p$

Fix $k \in \mathbb{N}$ $\quad u \in V \quad |\{v, d_{G_n}(u,v) \leq k\}| \leq d^k$

$|\{(u,v),\ d_{G_n}(u,v) \leq k\}| \leq |V_n| d^k$

$|V_n| d^k \leq \frac{|V_n|^2}{2} \Leftrightarrow d^k \leq \frac{|V_n|}{2} \Leftrightarrow k \lesssim C_d \log |V_n|$

So at least half of the pairs in $V_n \times V_n$ satisfy

$d_{G_n}(u,v) \geq C_d \log |V_n|$

So $\quad \alpha(C_d \log |V_n|) \leq \gamma \beta(1)^p$

In particular $(|V_n|)_n$ is uniformly bounded.

<u>Example</u> If $G_n = (V_n, E_n)$ is a sequence of expander graphs (d-regular), then $\sup_n \gamma(G_n, \| \|_H^2) < \infty$ (Prop 1)

then $(V_n, d_{G_n}) \underset{c}{\not\hookrightarrow}$ Hilbert space.

III) Main result - Strategy of proof

Théorem 1.1 There exists a sequence $G_n = (V_n, E_n)$ of 3-regular graphs such that $\lim |V_n| = +\infty$ and for every super-reflexive Banach space $(X, \| \|_X)$

$$\sup_n \gamma_+(G_n, \| \|_X^2) < \infty$$

A fortiori $\quad \sup_n \gamma(G_n, \| \|_X^2) < \infty$

They are called super-expanders

They do not embed coarsely in any superreflexive B. space

<u>Step 1</u> Construct base graphs $(H_n(\delta))_{n \in \mathbb{N}, \delta > 0}$

so that $\quad \gamma_+(H_n(\delta), \| \|_X^2) \leq 9^3 \quad (\forall X$ K-convex$)$

$|V(H_n(\delta))| = m_n$

But $H_n(\delta)$ $\quad d_n(\delta)$ - regular

$d_n(\delta) \leq e^{(\log m_n)^{1-\delta}}$

9

Based on analysis of $e^{-t\Delta}$ on $L_p(\mathbb{F}_2^n)$.
(Sections 5 and 7)

- **Step 2** Lower the degree but control γ^+ and γ
  - + Edge completion (Lemma 2.9 easy)
  - + Zigzag products | Section 8
  - + Replacement products
  - + $A \longrightarrow A_m = \frac{1}{m}\sum_{0}^{m-1} A^t$
  - + If (X, d_X) has metric Markov cotype p with exponent 2 then $\gamma_+(A_m, d_X^2) \leq C \max\{1, \frac{\gamma_+(A, d_X^2)}{m^{2/p}}\}$] Sect 7
  - Section 6 [$(X, \|\,\|_X)$ super-reflexive $\Rightarrow \exists p \geq 2$ $(X, \|\,\|_X)$ has MMC p with exponent 2

- **Step 3** Mix everything + diagonal argument → end of proof] Section

<u>Possible organization</u>

(A) Accessible stuff on graph operations
behavior of γ_+ under edge completion
zigzag products
replacement products.

(B) Cesàro Means and Markov cotype

(C) Section 6 SR → Markov cotype

?
(D) Section 5

(E) Construction of the base graphs

(F) Section 4

15/3/2013
Ande

Theorem: Let X be a Banach sp. If $K_p(X)$ is finite, then X has metric Markov cotype p with exponent 2

Pisier. If X is superreflexive, then there is a $p\geq 2$ and an equivalent norm s.t. $K_p(X)$ is finite

Def: $(X, \|\cdot\|_X)$ a normed space: its modulus of convexity is $\delta_X(\varepsilon)=\inf_{\substack{x,y\in S_X\\ \|x-y\|=\varepsilon}} 1-\frac{\|x+y\|}{2}$ for $\varepsilon\in(0,2]$. X is uniformly convex if $\delta_X(\varepsilon)>0$ for $\varepsilon\in]0,2]$.

X has modulus of convexity of power type p if there is $c>0$ st. for $\varepsilon\in]0,2[$ $\delta_X(\varepsilon)\geq c\varepsilon^p$ (here $p\geq 2$). Its modulus of uniform smoothness is $\tau\in]0,\infty[$ s.t.

$\rho_X(\tau)=\sup_{x,y\in S_X}\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1$. X has modulus of smoothness of power type p (m.sm.p.t.p) if $\exists C>0\ \forall\tau\in]0,\infty[\ \rho_X(\tau)\leq C\tau^p$ (here $1\leq p\leq 2$).

Lemma 1 [Lindenstrauss] $\rho_{X^*}(\tau)=\sup_{0\leq\varepsilon\leq 1}\tau\varepsilon-\delta_X(\varepsilon)$ [Fenchel dual]

Prop / Def 2: 1) X has m.sm.p.t.p iff $\exists S\geq 1\ \forall x,y\in X$ $\frac{\|x+y\|^p+\|x-y\|^p}{2}\leq\|x\|^p+S^p\|y\|^p$.
The p-smoothness constant of X is the infimum $S_p(X)$ of such S.

2) X has m.c.p.t.p iff $\exists K\geq 1\ \forall x,y\in X$ $\|x\|^p+\frac{1}{K^p}\|y\|^p\leq\frac{\|x+y\|^p+\|x-y\|^p}{2}$:
the p-convexity cst of X is the infimum $K_p(X)$ of such K.

This can be found in Ball-Carlen-Lieb (1994)

Lemma 3 If $1\leq p\leq q$, $\left(\frac{\|x+y\|^p+\|x-y\|^p}{2}\right)^{1/p}\leq\left(\frac{\|x+y\|^q+\|x-y\|^q}{2}\right)^{1/q}$

Lemma 4: $K_p(X)=S_{\frac{1}{1-\frac{1}{p}}}(X^*)$.

Proof: Let $q=\frac{1}{1-\frac{1}{p}}$, $K=K_p(X)$, $S=S_q(X^*)$

Then 1) $K_p(X)\leq S_q(X^*)$ and for all $x^*,y^*\in X^*$ $\frac{\|x^*+y^*\|^q+\|x^*-y^*\|^q}{2}\leq\|x^*\|^q+S^q\|y^*\|^q$

We want that for all $x,y\in X$ $\|x\|^p+\frac{1}{S^p}\|y\|^p\leq\frac{\|x+y\|^p+\|x-y\|^p}{2}$.

Let $x,y\in X$. Then $u^*,v^*\in S_{X^*}$ s.t. $u^*(x)=\|x\|$ and $v^*(y)=\|y\|$.

Let $x^*=\|x\|^{p-1}u^*$, $y^*=\frac{\|y\|^{p-1}}{S^p}v^*$: then $\|x^*\|^q+S^q\|y^*\|^q=\|x\|^{q(p-1)}+\frac{S^q}{S^{pq}}\|y\|^{q(p-1)}$

$=\|x\|^p+\frac{1}{S^p}\|y\|^p$.

$=\|x\|^{p-1}u^*(x)+\frac{1}{S^p}\|y\|^{p-1}v^*(y)$

$=x^*(x)+y^*(y)$

$=\dots$

$$\dots = \frac{x^*+y^*}{2^{1/q}}\frac{x+y}{2^{1/p}} + \frac{x^*-y^*}{2^{1/q}}\frac{x-y}{2^{1/p}} \le \|\cdot\|\cdot\|\cdot\| + \|\cdot\|\,\|\cdot\|$$

$$\le \left(\left\|\frac{x^*+y^*}{2^{1/q}}\right\|^q + \left\|\frac{x^*-y^*}{2^{1/q}}\right\|^q\right)^{1/q}\left(\left\|\frac{x+y}{2^{1/p}}\right\|^p + \left\|\frac{x-y}{2^{1/p}}\right\|^p\right)^{1/p}$$

by Hölder's inequality.

$$\le \left(\|x^*\|^q + S^q\|y^*\|^q\right)^{1/q}\left(\qquad\right)^{1/p}$$

by hypothesis

$$= \left(\|x\|^p + \frac{1}{S^p}\|y\|^p\right)^{1/q}\left(\qquad\right)^{1/p}$$

Thus $K_p(X) \le S_q(X^*)$.

2) $S_q(X^*) \le K_p(X)$ ....

These are unhomogeneous versions of the usual definition of the constants of smoothness and convexity.

Proof of Prop 2 (1) ($\Leftarrow$) is an exercise.

($\Rightarrow$) If $x, y \in X$, we want to show that $\frac{\|x+y\|^p + \|x-y\|^p}{2} \le \|x\|^p + S^p\|y\|^p$

Assume $\|y\| \le \|x\| = 1$. Set $u = x$ and $v = \frac{y}{\|y\|}$ and $\tau = \|y\|$.

then $\frac{\|u-\tau v\| + \|u+\tau v\|}{2} - 1 = \frac{\|x+y\| + \|x-y\|}{2} - 1 \le C\|y\|^p$.

Set $b = \frac{\|x+y\| + \|x-y\|}{2} \le C\|y\|^p + 1$ and $\beta = \frac{\|x+y\| - \|x-y\|}{\|x+y\| + \|x-y\|}$: $b(1+\beta) = \|x+y\|$, $b(1-\beta) = \|x-y\|$.

Hence $\left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p} - b = b\left[\left(\frac{(1+\beta)^p + (1-\beta)^p}{2}\right)^{1/p} - 1\right] \le b(p-1)\beta^2$

(consider the Taylor expansion in β).

Moreover, $|\beta| = \frac{|\|x+y\| - \|x-y\||}{2b} \le \frac{1}{2b}\|(x+y)-(x-y)\| = \frac{\|y\|}{b} \le \|y\|$ as $b \ge 1$.

Thus $\frac{\|x+y\|^p + \|x-y\|^p}{2} \le \left[b + b(p-1)\beta^2\right]^p \le \left[C\|y\|^p + 1 + b(p-1)\frac{\|y\|}{b}\|y\|\right]^p$

$$= \left[1 + C\|y\|^p + (p-1)\|y\|^2\right]^p \le \left[1 + (C+p-1)\|y\|^p\right]^p$$

$$\le 1 + p(C+p-1)\|y\|^p + \frac{p(p-1)(C+p-1)^2}{2}\|y\|^p$$

(Taylor expansion) and $p<2$!!

$$\le \|x\|^p + S^p\|y\|^p$$

Assume now $\|x\| \le \|y\|$:

$\|x\|^p + S^p\|y\|^p - \|y\|^p - S^p\|x\|^p = (\|x\|^p - \|y\|^p)(1 - S^p) \ge 0$.

(2) $\Leftarrow$ ✓

③ 15/3/2013

② $\Rightarrow$: If there is K s.t. for all $x,y\in X$ $\|x\|^p+\frac{1}{K^p}\|y\|^p\le\frac{\|x+y\|^p+\|x-y\|^p}{2}$, assume that $\forall\varepsilon\in[0,2]$ $\delta_X(\varepsilon)\ge c\,\varepsilon^p$.

Lemma 1: Let $\tau>0$. $\rho_{X^*}(\tau)=\sup_{0\le\varepsilon\le 1}\{\tau\varepsilon-\delta_X(\varepsilon)\}$

$$\le\sup_{0\le\varepsilon<\infty}\{\tau\varepsilon-c\varepsilon^p\}=\tau^q C_q \quad \text{for } q=p'.$$

Part ① shows there is $S\ge 1$ s.t. $\forall x^*,y^*\in X^*$ $\frac{\|x^*+y^*\|^q+\|x^*-y^*\|^q}{2}\le\|x^*\|^q+S^q\|y^*\|^q$ and Lemma 4 shows that for $x,y\in X$ $\|x\|^p+\frac{1}{K^p}\|y\|^p\le\frac{\|x+y\|^p+\|x-y\|^p}{2}$

Remark 5: • For $q\ge p$ and $x,y\in X$, $\left(\|x\|^q+\frac{1}{K^q}\|y\|^q\right)^{1/q}\le\left(\|x\|^p+\frac{1}{K^p}\|y\|^p\right)^{1/p}$

• $K_q(X)\le K_p(X)$

• $S_q(X)\ge S_p(X)$.

Prop 6: For $1\le p\le 2$ and $p\le q<\infty$ and (Ω,μ) measure space, then $S_p(L_q(\mu,X))\le(5pq)^{1/p}S_p(X)$.

Proof: We have $\forall x,y\in X$ $\frac{\|x+y\|^q+\|x-y\|^q}{2}\le\left(\|x\|^p+5pq\,S_p(X)^p\|y\|^p\right)^{q/p}$.

then we can deduce the proposition:

$$\frac{\|f+g\|_q^p+\|f-g\|_q^p}{2}\overset{\text{Lemma 3}}{\le}\left(\frac{\|f+g\|_q^q+\|f-g\|_q^q}{2}\right)^{p/q}=\left(\frac{1}{2}\left\{\int_\Omega\|f(x)+g(x)\|_X^q\,d\mu(x)+\int_\Omega\|f(x)-g(x)\|_X^q\right\}\right)^{p/q}$$

$$\underset{(*)}{\le}\left(\int_\Omega\left(\|f(x)\|^p+5pq\,S_p(X)\|g(x)\|^p\right)^{q/p}d\mu(x)\right)^{p/q}$$

$$=\left\|\,\|f\|_X^p+5pq\,S_p(X)^p\|g\|_X^p\right\|_{q/p}\le\|f\|_q^p+\underbrace{5pq\,S_p(X)^p}_{S^p}\|g\|_q^p.$$

Thus $S_p(L_q(\mu,X))\le(5pq)^{1/p}S_p(X)$.

Proof of (*): Suppose $\|y\|\le\|x\|=1$: (i) $\left|\|x+y\|^p-\|x-y\|^p\right|\le\left((\|x\|+\|y\|)^p-(\|x\|-\|y\|)^p\right)$

$$=\left((1+\|y\|)^p-(1-\|y\|)^p\right)$$
$$\le 2p\|y\|$$

(ii) $\forall\alpha\ge 1$ $\forall\beta\in[-1,1]$ $\left(\frac{(1+\beta)^\alpha+(1-\beta)^\alpha}{2}\right)^{1/\alpha}\le 1+2\alpha\beta^2$.

Thus $\beta\in[0,1]$: $\frac{(1+\beta)^\alpha+(1-\beta)^\alpha}{2}-(1+\beta)^\alpha=\frac{(1-\beta)^\alpha-(1+\beta)^\alpha}{2}\le 0$

then $\left(\frac{(1+\beta)^\alpha+(1-\beta)^\alpha}{2}\right)^{1/\alpha}\le 1+\beta$; if $\beta\ge\frac{1}{2\alpha}$, $1+\beta\le 1+2\alpha\beta^2$; but if $\beta\le\frac{1}{2\alpha}$, let $f(\beta)=(1+\beta)^\alpha+(1-\beta)^\alpha=2+\alpha(\alpha-1)\beta^2+o(\beta^2)$:

Hence $\left(\frac{(1+\beta)^\alpha+(1-\beta)^\alpha}{2}\right)^{1/\alpha}=\left(\frac{f(\beta)}{2}\right)^{1/\alpha}=1+\frac{\alpha-1}{2}\beta^2+o(\beta^2)\le 1+\frac{\alpha}{2}\beta^2\le 1+2\alpha\beta^2$.

Let $b = \frac{\|x+y\|^p + \|x-y\|^p}{2}$

$\beta = \frac{\|x+y\|^p - \|x-y\|^p}{\|x+y\|^p + \|x-y\|^p}$

$\vartheta = \left(\frac{(1+\beta)^{q/p} + (1-\beta)^{q/p}}{2}\right)^{1/q} - 1$

$$\vartheta \underset{(ii)}{\leq} 1 + 2\frac{q}{p}\beta^2 - 1 = 2\frac{q}{p}\left(\frac{\|x+y\|^p - \|x-y\|^p}{2b}\right)^2 \underset{(i)}{<} 2\frac{q}{p}\left(\frac{2p\|y\|}{2b}\right)^2 \underset{\text{because } \|y\| \leq 1 \text{ and } b \geq 1}{\leq} 2qp\|y\|^p$$

Finally,
$$\frac{\|x+y\|^q + \|x-y\|^q}{2} = \frac{\left(\|x+y\|^p\right)^{q/p} + \left(\|x-y\|^p\right)^{q/p}}{2} = \frac{\left(b(1+\beta)\right)^{\frac{q}{p}} + \left(b(1-\beta)\right)^{\frac{q}{p}}}{2}$$
$$= b^{\frac{q}{p}}(1+\vartheta)^{\frac{q}{p}} \underset{\text{def of } S_p(X)}{\leq} \left(1 + S_p(X)^p\|y\|^p\right)^{\frac{q}{p}}\left(1 + 2qp\|y\|^p\right)^{\frac{q}{p}}$$
$$= \left(1 + 2qp\|y\|^p + S_p(X)^p\|y\|^p + 2qp\, S_p(X)\|y\|^{2p}\right)^{q/p}$$
$$\leq \left(1 + 5qp\, S_p(X)^p\|y\|^p\right)^{\frac{q}{p}}.$$

Corollary 7: For $p \geq 2$ and $1 \leq q \leq p$ and (Ω, μ) a normed space,
$$K_p(L_q(\mu, X)) \leq \left(\frac{5pq}{(p-1)(q-1)}\right)^{1-\frac{1}{p}} K_p(X).$$

Prop 8. If X is normed, U a random vector on X s.t. $\mathbb{E}\|U\|_X^p < \infty$,

then $\|\mathbb{E}\, U\|_X^p + \frac{1}{(2^{p-1}-1)K_p(X)^p}\mathbb{E}\left[\|U - \mathbb{E}U\|\right]_X^p \leq \mathbb{E}\|U\|_X^p$.

Proof: (1) If $\mathbb{E}\|U - \mathbb{E}U\|^p = 0$, this is Jensen's inequality.

(2) We can assume $\mathbb{E}\|U - \mathbb{E}U\|^p > 0$. Let $\vartheta = \inf_{\substack{V \in L^p(X) \\ \mathbb{E}\|\,\|^p > 0}} \frac{\mathbb{E}\|V\|^p - \|\mathbb{E}V\|^p}{\mathbb{E}[\|V - \mathbb{E}V\|^p]}$

Then $0 \leq \vartheta \leq \frac{\mathbb{E}\|U\|^p - \|\mathbb{E}V\|^p}{\mathbb{E}\|U - \mathbb{E}U\|^p}$.

We will prove that $\vartheta \geq \frac{1}{(2^{p-1}-1)K_p(X)^p}$. Let $\varphi > \vartheta$. Then let V_0 s.t.

$\mathbb{E}\|V_0\|^p - \|\mathbb{E}V_0\|^p < \varphi\, \mathbb{E}[\|V_0 - \mathbb{E}V_0\|^p]$ and $\quad x = \frac{V_0 + \mathbb{E}V_0}{2}$, $y = \frac{V_0 - \mathbb{E}V_0}{2}$.

then $\left\|\frac{V_0 + \mathbb{E}V_0}{2}\right\|_X^p + \frac{1}{K_p(X)^p}\left\|\frac{V_0 - \mathbb{E}V_0}{2}\right\|_X^p \leq \frac{\|V_0\|_X^p + \|\mathbb{E}[V_0]\|_X^p}{2}$ and

$$\varphi\, \mathbb{E}[\|V_0 - \mathbb{E}V_0\|^p] \underset{i}{\geq} \mathbb{E}[\|V_0\|^p] - \|\mathbb{E}V_0\|^p \geq 2\mathbb{E}\left[\left\|\frac{V_0 + \mathbb{E}V_0}{2}\right\|^p + \frac{1}{K_p(X)^p}\left\|\frac{V_0 - \mathbb{E}V_0}{2}\right\|^p\right] - \|\underbrace{\mathbb{E}[V_0]}_{\mathbb{E}\left[\frac{V_0 + \mathbb{E}V_0}{2}\right]}\|^p$$

$$= 2\mathbb{E}\left\|\frac{V_0 + \mathbb{E}V_0}{2}\right\|^p - \left\|\mathbb{E}\frac{V_0 + \mathbb{E}V_0}{2}\right\|^p + \frac{2}{K_p(X)}\mathbb{E}\left\|\frac{V_0 - \mathbb{E}V_0}{2}\right\|^p$$

$$= \frac{1}{2^{p-1}}\mathbb{E}\left[\|V_0 - \mathbb{E}V_0\|^p\right]\left(\vartheta + \frac{1}{K_p(X)^p}\right)$$

Hence $\varphi > \frac{1}{2^{p-1}}\left(\vartheta + \frac{1}{K_p(X)^p}\right)$ and $\vartheta \geq \frac{\vartheta}{2^{p-1}} + \frac{1}{2^{p-1}K_p(X)^p}$

15/3/2013

Theorem 9 : Doob's martingale inequality:

X Banach space s.t. $K_p(X) < \infty$. Let (M_k) be a martingale wrt $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_{n-1}$. Then $\mathbb{E}\|M_n - M_0\|^p \geq \frac{1}{(2^{p-1}-1)K_p(X)^p} \sum_{k=1}^{n} \mathbb{E}\|M_k - M_{k-1}\|^p$

Une martingale p-intégrable a ses différences p-sommable.

Dem: $\mathbb{E}[\|M_n - M_0\|^p \mid \mathcal{F}_{n-1}] \geq \|\mathbb{E}[M_n - M_0 \mid \mathcal{F}_{n-1}]\|^p + \frac{1}{(2^{p-1}-1)K_p(X)^p} \times$

$\times \mathbb{E}[\|M_n - M_0 - \mathbb{E}[M_n - M_0 \mid \mathcal{F}_{n-1}]\|^p \mid \mathcal{F}_{n-1}]$

$= \|M_{n-1} - M_0\|^p + \frac{1}{(2^{p-1}-1)K_p(X)^p} \mathbb{E}[\|M_n - M_{n-1}\|^p \mid \mathcal{F}_{n-1}]$

Taking expectations:

$$\mathbb{E}[\|M_n - M_0\|^p] \geq \mathbb{E}[\|M_{n-1} - M_0\|^p] + \frac{1}{(2^{p-1}-1)K_p(X)^p} \mathbb{E}[\|M_n - M_{n-1}\|^p]$$

$$\geq \frac{1}{(2^{p-1}-1)K_p(X)^p} \sum_{k=1}^{n} \mathbb{E}[\|M_k - M_{k-1}\|^p] \quad \blacksquare.$$

Cor 10 : $p \geq 2$, $q > 1$: X normed space with $K_p(X) < \infty$; then for every q-integrable martingale $\{M_k\}_{k=1}^{n} \subseteq X$ we have

(1) if $q \geq p$, $\mathbb{E}\|M_n - M_0\|^q \geq \frac{1}{(2^{q-1}-1)K_q(X)^q} \sum_{k=1}^{n} \mathbb{E}\|M_k - M_{k-1}\|^q$

(2) if $1 < q \leq p$, $\mathbb{E}\|M_n - M_0\|^q \geq \left(\frac{(1-\frac{1}{p})(1-\frac{1}{q})}{5}\right)^{q(1-\frac{1}{p})} \times \frac{1}{2K_p(X)^q} \times \frac{1}{n^{1-\frac{q}{p}}} \times$

$\times \sum_{k=1}^{n} \mathbb{E}\|M_k - M_{k-1}\|^q$

(1) is just Theorem 9

(2) Let $x = M_{n-1} - M_0 + \frac{M_n - M_{n-1}}{2}$ in $L_q(\Omega; X)$ and $K_1 = K_p(L_q(\Omega; X))$.

$y = \frac{M_n - M_{n-1}}{2}$

then $(\mathbb{E}\|x\|_X^q)^{\frac{p}{q}} + \frac{1}{K^p}\mathbb{E}[\|y\|_X^q]^{\frac{p}{q}} \leq \frac{\mathbb{E}[\|x+y\|_X^q]^{\frac{p}{q}} + \mathbb{E}[\|x-y\|_X^q]^{\frac{p}{q}}}{2}$,

i.e., $\mathbb{E}\left[\left\| M_{n-1} - M_0 + \frac{M_n - M_{n-1}}{2}\right\|^q\right]^{p/q} + \frac{1}{K^p}\mathbb{E}\left[\left\|\frac{M_n - M_{n-1}}{2}\right\|^q\right]^{\frac{p}{q}}$

$\underset{(i)}{\leq} \frac{1}{2}\left(\mathbb{E}[\|M_n - M_0\|^q]^{\frac{p}{q}} + \mathbb{E}[\|M_{n-1} - M_0\|^q]^{\frac{p}{q}}\right)$

Moreover, (ii) $\mathbb{E}\|M_{n-1} - M_0\|^q = \mathbb{E}\|\mathbb{E}[M_n - M_0 \mid \mathcal{F}_{n-1}]\|^q \leq \mathbb{E}\|M_n - M_0\|^q$

(iii) $\mathbb{E}\|M_{n-1} - M_0\|^q = \mathbb{E}\|M_{n-1} - M_0 + \mathbb{E}[\frac{M_n - M_{n-1}}{2} \mid \mathcal{F}_{n-1}]\|^q$

$= \mathbb{E}\|\mathbb{E}[M_{n-1} - M_0 + \frac{M_n - M_{n-1}}{2} \mid \mathcal{F}_{n-1}]\|^q$

$\leq \mathbb{E}\| M_{n-1} - M_0 + \frac{M_n - M_{n-1}}{2}\|^q$

Hence $\left(\mathbb{E}\|M_{n-1}-M_0\|^q\right)^{\frac{p}{q}} + \frac{1}{(2K)^p}\left(\mathbb{E}\|M_n-M_{n-1}\|^q\right)^{\frac{p}{q}} \underset{(ii)}{\le} \left(\mathbb{E}\left[\|M_{n-1}-M_0+\frac{M_n-M_{n-1}}{2}\|^q\right]\right)^{\frac{p}{q}}$

$$+ \frac{1}{(2K)^p}\left(\mathbb{E}\|M_n-M_{n-1}\|^q\right)^{\frac{p}{q}}$$

$$\overset{(i)}{\le} \frac{1}{2}\left(\left(\mathbb{E}[\|M_n-M_0\|^q]\right)^{\frac{p}{q}} + \left(\mathbb{E}[\|M_{n-1}-M_0\|^q]\right)^{p/q}\right)$$

$$\le \mathbb{E}\left[\|M_n-M_0\|^q\right]^{p/q}$$

Iterating, we get $\mathbb{E}[\|M_n-M_0\|^q]^{\frac{p}{q}} \ge \frac{1}{(2K)^p}\sum_{k=1}^{n}\left(\mathbb{E}\|M_k-M_{k-1}\|^q\right)^{p/q}$

very rough → By Hölder's $\le$, $\left(\sum \mathbb{E}\|M_k-M_{k-1}\|^q\right)^{p/q} \le n^{\frac{p}{q}(1-\frac{q}{p})}\sum_{k=1}^{n}\mathbb{E}\left[\|M_k-M_{k-1}\|^q\right]$

and $\mathbb{E}\left[\|M_n-M_0\|^q\right] \ge \frac{1}{(2K)^q}\times\frac{1}{n^{1-\frac{q}{p}}}\sum \mathbb{E}\|M_k-M_{k-1}\|^q$

Corollary 7: $K \le \left(\frac{5pq}{(p-1)(q-1)}\right)^{1-\frac{1}{p}} K_p(X)$.

Aude Dalet
26/3/2013

Def: • The <u>Cesàro means</u> of $A \in M_n(\mathbb{R})$ are the $\mathcal{A}_m(A) = \frac{\sum_{t=0}^{m-1} A^t}{m}$.

• If $p, q \in]0,\infty[$, (X, d_X) has metric Markov cotype p with exponent q if $\forall x_1, \dots, x_n \in X$ $\exists y_1, \dots, y_n \in X$ $\forall A \in M_n(\mathbb{R})$ stochastic and symmetric

$$\sum_{i=1}^n d_X(x_i, y_i)^q + m^{\frac{q}{p}} \sum_{i,j=1}^n a_{ij} d_X(y_i, y_j)^q \leq C^q \sum_{i,j=1}^n (\mathcal{A}_m(A))_{ij} d_X(x_i, x_j)^q$$

We let $C_p^{(q)}(X, d_X)$ be the inf of such C.

Theorem 11: If $p \geq 2$ and $(X, \|\cdot\|)$ with $K_p(X) < \infty$, then for all $A \in M_n(\mathbb{R})$ symmetric and stochastic and $x_1, \dots, x_n \in X$, there are $y_1, \dots, y_n \in X$ such that for $q > 1$,

$$\max\left(\sum_{i=1}^n \|x_i - y_i\|^q, \left(\frac{\left(\frac{1}{5}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\right)^{1-\frac{1}{p}}}{2^5 K_p(X)} \right)^q m^{\min(1, \frac{q}{p})} \sum_{i,j=1}^n a_{ij} \|y_i - y_j\|^q \right)$$
$$\leq \sum_{i,j=1}^n (\mathcal{A}_m(A))_{ij} \|x_i - x_j\|^q. \quad (*)$$

In particular, for $q = 2$, $\sum \|x_i - y_i\|^2 + m^{\frac{2}{p}} \sum_{i=1}^n \sum_{j=1}^n a_{ij} \|y_i - y_j\|^2 \leq (2^5 K_p(X))^2 \sum_{i,j=1}^n (\mathcal{A}_m(A))_{ij} \times \|x_i - x_j\|^2$

Thus X has metric Markov cotype with exponent 2 and $C_p^{(2)}(X) \leq 2^5 K_p(X)$.

Lemma 12: $\|x+y\|^q \geq \frac{1}{2^{q-1}} \|x\|^q - \|y\|^q$ and $\|x-y\|^q \leq 2^{q-1} (\|x-z\|^q + \|z-y\|^q)$.

Def: • $L_p^n(X) = \{ f : \{1, \dots, n\} \longrightarrow X \}$

• If $A \in M_n(\mathbb{R})$ is symmetric and stochastic, let $A \otimes I_X^n : L_p^n(X) \longrightarrow L_p^n(Y)$

$$f \longmapsto \left(i \mapsto \sum_{j=1}^n a_{ij} f(j) \right)$$

<u>Proof</u> of Theorem 11:

1) The "in particular case"

2) Let $m, n \in \mathbb{N}$, $A \in M_n(\mathbb{R})$ symmetric and stochastic, $x_1, \dots, x_n \in X$, $q > 1$.

Define $f \in L_p^n(X)$ by $f(i) = x_i$, and let $Z_0^{(l)}, \dots, Z_m^{(l)}$ be the Markov chain starting in l with transition matrix A.

If $0 \leq t \leq m$, let $f_t \in L_p^n(X)$ be $f_t = (A^{m-t} \otimes I_X^n)(f)$.

Let $M_t^{(l)} = f_t(Z_t^{(l)}) = (A^{m-t} \otimes I_X^n)(f)(Z_t^{(l)})$.

Then $M_0^{(l)}, \dots, M_m^{(l)}$ is a martingale wrt the filtration induced by $Z_0^{(l)}, \dots, Z_m^{(l)}$.

Let $L = A \otimes I_X^{(n)}$. If $t \geq 1$, $L^t = A^t \otimes I_X^n$:

$$(A \otimes I_X^n)^2 f = (A \otimes I_X^n)\left(i \longmapsto \sum_{j=1}^n a_{ij} f(j)\right) = \left(k \longmapsto \sum\sum a_{ki} a_{ij} f(j)\right)$$
$$= \left(k \longmapsto \sum (A^2)_{kj} f(j)\right) = A^2 \otimes I_X^n (f).$$

$$E\left[M_t^{(\ell)} \mid Z_0^{(\ell)}, \dots, Z_{t-1}^{(\ell)}\right] = E\left[f_t(Z_t^{(\ell)}) \mid Z_{t-1}^{(\ell)}\right] = L^{m-t} E\left[f(Z_t^{(\ell)}) \mid Z_{t-1}^{(\ell)}\right]$$
$$= L^{m-t}\left(\sum_{j=1}^{m} \sum_{i=1}^{n} f(i) \mathbb{P}\left[Z_t^{(\ell)} = i \mid Z_{t-1}^{(\ell)} = j\right] \mathbb{1}_{\{Z_{t-1}^{(\ell)} = j\}}\right)$$
$$= L^{m-t}\left(\sum_j \sum_i f(i)\, a_{ij}\, \mathbb{1}_{\{Z_{t-1}^{(\ell)} = j\}}\right)$$
$$= L^{m-t}\left(\sum_j L(f)(j)\, \mathbb{1}_{\{Z_{t-1}^{(\ell)} = j\}}\right) = L^{m-(t-1)}\left(f(Z_{t-1}^{(\ell)})\right) = f_{t-1}(Z_{t-1}^{(\ell)}) = M_{t-1}^{(\ell)}$$

Thus $\{M_t^{(\ell)}\}$ is a martingale.

Let $K = \begin{cases} (2^{q-1}-1) K_p(X)^q & \text{if } q \geq p \\ \dfrac{2 K_p(X)^q\, m^{1-\frac{q}{p}}}{\left(\frac{1}{5}(1-\frac{1}{n})(1-\frac{1}{q})\right)^{(1-\frac{1}{p})q}} & \text{if } 1 < q < p \end{cases}$: this is $\dfrac{1}{\text{quantity in Cor 10}}$.

Recall Cor. 10: If $p \geq 2$ and $q > 1$, X a normed space with $K_p(X) < \infty$, then for any q-integrable martingale $\{M_k\}_{k=0}^m \subset X$ we have $E\left[\|M_m - M_0\|^q\right] \geq \dfrac{\sum_{k=1}^m E\|M_k - M_{k-1}\|^q}{K}$

Then $K E\|M_m^{(\ell)} - M_0^{(\ell)}\|^q \geq \sum_{k=1}^m E\left[\|M_k^{(\ell)} - M_{k-1}^{(\ell)}\|^q\right]$

i.e., $K E\|M_m^{(\ell)} - M_0^{(\ell)}\|^q \geq \sum_{t=1}^m E\|L^{m-t}(f) Z_t^{(\ell)} - L^{m-(t-1)}(f) Z_{t-1}^{(\ell)}\|^q$

If $\{Z_t\}_{t=0}^{+\infty}$ is the Markov chain with transition matrix A s.t. Z_0 is uniformly distributed on $\{1, \dots, n\}$, then $K E\left[\|f(Z_m) - (L^m f)(Z_0)\|^q\right] = \frac{K}{n} \sum_{\ell=1}^n E\|f(Z_m^{(\ell)}) - L^m f(Z_0^{(\ell)})\|^q$
$$\geq \frac{1}{n} \sum_{\ell=1}^m \sum_{t=1}^m E\|L^{m-t} f(Z_t^{(\ell)}) - L^{m-t+1} f(Z_{t-1}^{(\ell)})\|^q$$
$$= \sum_{t=1}^m E\|L^{m-t} f(Z_t) - L^{m-(t-1)} f(Z_{t-1})\|^q$$

(One could have applied directly Cor. 10 to the martingale $L^{m-t} f(Z_t)$!

Then $K \sum_i \sum_j \frac{1}{n} (A^m)_{ij} \|f(i) - L^m f(j)\|^q = K \sum_{i=1}^n \sum_{j=1}^n \mathbb{P}[Z_0 = j]\, \mathbb{P}[Z_m = i \mid Z_0 = j] \|f(i) - L^m f(j)\|^q$
$$= K E\left[\|f(Z_m) - L^m f(Z_0)\|^q\right]$$
$$\geq \sum_{t=1}^m E\|L^{m-t} f(Z_t) - L^{m-(t-1)} f(Z_{t-1})\|^q$$
$$= \sum_{t=1}^m \sum_{i,j=1}^n \|L^{m-t} f(i) - L^{m-(t-1)} f(j)\|^q \times$$
$$\times\, \mathbb{P}[Z_{t-1} = j]\, \mathbb{P}[Z_t = i \mid Z_{t-1} = j]$$

$$= \sum_{t=1}^{m} \sum_{i,j=1}^{n} a_{ij} \|L^{m-t} f(i) - L^{m-(t-1)} f(j)\|^q \qquad \overset{\text{inequality}}{(2)}$$

If $1 \le i \le n$, let $y_i = \frac{1}{m} \sum_{j=1}^{n} \sum_{t=0}^{m-1} (A^t)_{ij} x_j = \frac{1}{m} \sum_{t=0}^{m-1} L^t f(i)$

then $\frac{1}{m} \sum_{t=1}^{m} L^t f(i) = y_i + \frac{1}{m} L^m f(i) - \frac{1}{m} f(i) = y_i - \frac{1}{m}\left(x_i - L^m f(i)\right)$

$$= y_i - \frac{1}{m}\left(x_i - \sum_{k=1}^{n} (A^m)_{ik} x_k\right)$$

$$= y_i - \frac{1}{m} \sum_{k=1}^{n} (A^m)_{ik} (x_i - x_k)$$

Proof of (*): $\sum \|x_i - y_i\|^q = \sum \|x_i - \frac{1}{m} \sum_j \sum_{t=0}^{m-1} (A^t)_{ij} x_j\|^q = \sum_i^n \|\sum_j \sum_t \frac{(A^t)_{ij}}{m} (x_i - x_j)\|^q$

$$\underset{\text{(convexity)}}{\le} \sum_i^n \sum_j \sum_t \frac{(A^t)_{ij}}{m} \|x_i - x_j\|^q = \sum_i^n \sum_j^n (\mathcal{A}_m(A))_{ij} \|x_i - x_j\|^q$$

Proof of (**): $\sum_{t=1}^{m} \sum_i \sum_j a_{ij} \|L^{m-t} f(i) - L^{m-(t-1)} f(j)\|^q = m \sum_i \sum_j a_{ij} \sum_t \frac{1}{m} \|\text{---}\|^q$

$$\ge m \sum_i \sum_{j=1}^{n} a_{ij} \|\frac{1}{m} \sum_{t=0}^{m-1} L^t f(i) - \frac{1}{m} \sum_{t=1}^{m} L^t f(j)\|^q$$

$$= m \sum_i \sum_j a_{ij} \| y_i - y_j + \frac{1}{m} \sum_{k=1}^{n} (A^m)_{jk} (x_j - x_k)\|^q$$

By lemma 12, $\ge m \sum_i \sum_j a_{ij} \left(\frac{1}{2^{q-1}} \|y_i - y_j\|^q - \|\frac{1}{m} \sum_{k=1}^{n} (A^m)_{jk} (x_j - x_k)\|^q \right)$

$$\underset{\text{convexity}}{\ge} \frac{m}{2^{q-1}} \sum_i \sum_j a_{ij} \|y_i - y_j\|^q - \frac{1}{m^{q-1}} \sum_{i,j=1}^{n} a_{ij} \sum_k (A^m)_{jk} \|x_j - x_k\|^q$$

$$\underset{\text{because } A \text{ is stochastic!}}{=} \frac{m}{2^{q-1}} \sum_i \sum_j^n a_{ij} \|y_i - y_j\|^q - \frac{1}{m^{q-1}} \sum_j \sum_k^n (A^m)_{jk} \|x_j - x_k\|^q \qquad \text{(Ineq. (3))}$$

The left hand side of (2) is $\sum_{i,j=1}^{n} (A^m)_{ij} \|f(i) - L^m f(j)\|^q = \sum_i \sum_j (A^m)_{ij} \|x_i - \sum_{k=1}^{n} (A^m)_{jk} x_k\|^q$

$$= \sum_i \sum_j (A^m)_{ij} \|\sum_{k=1}^{n} (A^m)_{jk} (x_i - x_k)\|^q$$

$$\le \sum_i \sum_j \sum_k (A^m)_{ij} (A^m)_{jk} \|x_i - x_k\|^q$$

$$\underset{\text{Lemma 12}}{\le} \sum_i \sum_j \sum_k (A^m)_{ij} (A^m)_{jk} 2^{q-1} \left(\|x_i - x_j\|^q + \|x_j - x_k\|^q\right)$$

$$= 2^{q-1} \sum_i \sum_j \|x_i - x_j\|^q + 2^{q-1} \sum_j \sum_k (A^m)_{jk} \|x_j - x_k\|^q$$

$$= 2^q \sum_i \sum_j (A^m)_{ij} \|x_i - x_j\|^q \qquad \text{(inequality 4)}$$

Moreover $\sum_i \sum_j (A^m)_{ij} \|x_i - x_j\|^q = \sum_i \sum_j \frac{1}{m} \sum_{t=0}^{m-1} (A^t A^{m-t})_{ij} \|x_i - x_j\|^q$

$$\underset{\text{Lemma 12}}{\le} \sum_i \sum_j \frac{1}{m} \sum_{t=0}^{m-1} \sum_{k=1}^{n} (A^t)_{ik} (A^{m-t})_{kj} 2^{q-1} \left(\|x_i - x_k\|^q + \|x_k - x_j\|^q\right)$$

$$= 2^{q-1}\sum_i\sum_k\left(\frac{1}{m}\sum_{t=0}^{m-1}A^t\right)_{ik}\|x_i-x_k\|^q + \frac{2^{q-1}}{m}\sum_j\sum_k\sum_{t=0}^{m-1}\left(A^{m-t}\right)_{kj}\|x_k-x_j\|^q$$

$$= 2^{q-1}\sum_i\sum_k\left(\mathcal{A}_m(A)\right)_{ik}\|x_i-x_k\|^q + \frac{2^{q-1}}{m}\sum_j\sum_k\sum_{t=1}^{m}\left(A^t\right)_{kj}\|x_k-x_j\|^q$$

$$= \frac{2^{q-1}}{m}\sum_j\sum_k\left(m\mathcal{A}_m(A)+A^m-Id\right)_{kj}\|x_k-x_j\|^q$$

$$= 2^{q-1}\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ik}\|x_i-x_k\|^q + \frac{2^{q-1}}{m}\sum_j\sum_i\left(m\mathcal{A}_m(A)+A^m\right)_{ij}\|x_i-x_j\|^q$$

$$= 2^q\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\|^q + \frac{2^{q-1}}{m}\sum_i\sum_j\left(A^m\right)_{ij}\|x_i-x_j\|^q$$

- 1st case: $\underline{m \geq 2^q}$: then $\frac{2^{q-1}}{m} \leq \frac{1}{2}$ and

$$\sum_i\sum_j\left(A^m\right)_{ij}\|x_i-x_j\|^q \leq 2^q\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\|^q + \frac{1}{2}\sum_i\sum_j\left(A^m\right)_{ij}\|x_i-x_j\|^q$$

then $$\sum_i\sum_j\left(A^m\right)_{ij}\|x_i-x_j\|^q \leq 2^{q+1}\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\|^q \quad \text{(Ineq 5)}$$

- 2nd case: if $m \leq 2^q$, then $\sum_i\sum_j\left(A^m\right)_{ij}\|x_i-x_j\|^q = \sum_i\sum_j\sum_{k=1}^{n} a_{ik}\left(A^{m-1}\right)_{kj}\|x_j-x_k\|^q$

$$\underset{\text{Lemma 12}}{\leq} 2^{q-1}\sum_i\sum_j\sum_k a_{ik}\left(A^{m-1}\right)_{kj}\left(\|x_i-x_k\|^q + \|x_k-x_j\|^q\right)$$

$$= 2^{q-1}\sum_i\sum_k a_{ik}\|x_i-x_k\|^q + 2^{q-1}\sum_j\sum_k\left(A^{m-1}\right)_{kj}\|x_k-x_j\|^q$$

$$= 2^{q-1}\sum_i\sum_j\left(a_{ij}+\left(A^{m-1}\right)_{ij}\right)\|x_i-x_j\|^q$$

$\vdots$

$$\leq 2^{q-1}m\sum_i\sum_j\frac{1}{m}\left(\sum_{t=0}^{m-1}A^t\right)_{ij}\|x_i-x_j\|^q$$

$$\leq 2^{2q-1}\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\| \quad \text{(Ineq 6)}$$

Now $(5)+(6) \Rightarrow (7)$: $\sum\sum\left(A^m\right)_{ij}\|x_i-x_j\|^q \leq 2^{2q}\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\|^q$.

Then $$m\sum_i\sum_j a_{ij}\|y_i-y_j\|^q \underset{(3)}{\leq} 2^{q-1}\sum_{t=1}^{m}\sum_i\sum_j a_{ij}\|L^{m-t}f(i)-L^{m-(t-1)}f(j)\|^q + \frac{2^{q-1}}{m^{q-1}}\sum_i\sum_j\left(A^m\right)_{ij}\|x_i-x_j\|^q$$

$$\underset{(2)}{\leq} 2^{q-1}K\sum_i\sum_j\left(A^m\right)_{ij}\|f(i)-L^m f(j)\|^q + \ldots$$

$$\underset{(4)}{\leq} 2^{2q-1}K\sum_i\sum_j\left(A^m\right)_{ij}\|x_i-x_j\|^q + \ldots$$

$$\underset{(7)}{\leq} \left(2^{2q-1}K + \frac{2^{q-1}}{m^{q-1}}\right)4^q\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\|^q$$

$$\leq 2^{4q}K\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\|^q$$

Thus we obtain $$\sum_i\sum_j\left(\mathcal{A}_m(A)\right)_{ij}\|x_i-x_j\|^q \geq \frac{m}{K2^{4q}}\sum_i\sum_j a_{ij}\|y_i-y_j\|^q$$

It remains to prove that: $\frac{m}{K 2^{4q}} \geq \left(\frac{1}{5}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\right)^{q(1-\frac{1}{p})} \frac{1}{2^{5q} K_p(X)^q} m^{\min(1,\frac{q}{p})}$ (5)

- case $1 < q \leq p$: ✓ by def of K.
- case $q \geq p \geq 2$: then $\min(1,\frac{q}{p}) = 1$ and $\frac{1}{2^{4q}K} = \frac{1}{(2^{q-1}-1)K_p(X)^q 2^{4q}}$

$$= \frac{2^q}{2^{5q} K_p(X)^q (2^{q-1}-1)}$$

and $\frac{2^q}{2^{q-1}-1} \geq 1 \geq \left(\frac{1}{5}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{q}\right)\right)^{q(1-\frac{1}{p})}$

→ Idea: L is bounded in $l_q(X)$!

Lemma 12: strong ≤ then use Hölder! | Rewrite everything as triangular ≤ for this norm for which $\|L\| = 1$!

Now we have to build graphs and prove Theorem 5.1.

Mikael

Subject: Bernoulli r.v. with values in B-space. (1)

"B-convexity implies K-convexity", after Pisier 1982.

Notation: $\underline{D = \{-1,1\}^{\mathbb{N}}}$ equipped with the uniform prob. measure μ.

$\underline{D^n = \{-1,1\}^n}$ for $n<\infty$ integer ——————— μ_n

$\underline{\varepsilon_i}$ will be the i^{th} coordinate $\varepsilon_i : (\varepsilon_j)_{j\geq 1} \longmapsto \varepsilon_i$

The ε_i are i.i.d. with $\mu[\varepsilon_i = 1] = \mu[\varepsilon_i=-1] = \frac{1}{2}$.

The $\underline{w_A}$, $A \in \mathbb{N}$ finite, with $w_A = \prod_{i\in A} \varepsilon_i$, are an o.n.b:

$$\langle w_A, w_B\rangle_{L^2(D)} = \mathbb{E}\, w_A w_B = \mathbb{E}\, w_{A\Delta B} = \prod_{i\in A\Delta B} \varepsilon_i = \begin{cases} 0 & \text{if } A \neq B \\ 1 & \text{otherwise}\end{cases}$$

The $\underline{\text{number operator}}$ is the unbounded operator $\underline{N}$ on $L^2(D)$ s.t. $N w_A = |A| w_A$.

Its domain is $\mathrm{span}\{w_A\}_A$. Then $N = N^*$. It generates the semigroup of the

$\underline{T_t = e^{-tN}} : w_A \longmapsto e^{-t|A|} w_A$: $T_{t+s} = T_t T_s$.

For each t, T_t is a convex combination of projections (P_A) such that

P_A and P_B commute for all A and B, and $\|P_A \otimes id_X\|_{B(L^p(D^n;X))} \leq 1$ for every Banach space X

$\underline{\text{Proof}}$: let P_i be the conditional expectation on $\sigma(\varepsilon_j : j\neq i)$: $P_i f(\varepsilon) = \dfrac{f(\varepsilon) + f(\varepsilon \bar{\varepsilon}_i)}{2}$

P_i and P_j commute, and $\|P_i \otimes id_X\| = 1$. where $\bar{\varepsilon}_i = (1 - 1\ (-1)\ 1 —)$, with -1 at place i

$\underline{\text{Claim}}$: $T_t = \prod_{i=1}^{n} (P_i + e^{-t}(1-P_i))$

In fact, $P_i w_A = \begin{cases} 0 & \text{if } i\in A \\ w_A & \text{otherwise}\end{cases}$. then $\underbrace{\prod_{i=1}^{n} (P_i + e^{-t}(1-P_i))}_{w_A \mapsto \begin{cases} e^{-t} w_A & \text{if } i\in A \\ w_A & \text{otherwise}\end{cases}} w_A = (e^{-t})^{|A|} w_A = T_t w_A$.

Then $T_t = \prod_{i=1}^{n} \big[\underbrace{(1-e^{-t}) P_i + e^{-t} \mathrm{Id}}_{\text{c.c. of } P_i \text{ and } 1}\big] = $ c.c. of $\{P_A\}$, $P_A = \prod_{i\in A} P_i$

$\underline{\text{Theorem}}$: (Pisier) Let X be a complex B-space and $1<p<\infty$. TFAE

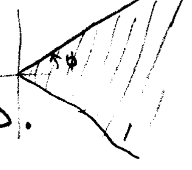
(i) X is $\underline{\text{B-convex}}$: there exists $\varepsilon>0$, X does not contain the ℓ_1^n $(1+\varepsilon)$-uniformly:

$\exists\, \varepsilon>0\ \exists N\in\mathbb{N}\ \forall x_1,\dots,x_N \in X$ of norm 1 $\mathbb{E}\|\sum_{i=1}^{N} \varepsilon_i x_i\| < N(1-\varepsilon)$.

(i') $\forall C>1\ \exists N\ \forall x_1,\dots,x_N \in L_p([0,1];X)$ $\in$ $\|\sum_{i=1}^{N} \varepsilon_i x_i\|_{L_p([0,1];X)} \leq \frac{N}{C}$

(ii) $\exists\, \phi>0\ \forall z\in\mathbb{C}\ |\arg z| \leq \phi$ $\|e^{-zN}\|_{L^p(D^n;X)} \leq M$

independently of n. conflict in notation: we shall write $\mathbb{D}$.



(iii) Let <u>Rad</u> be the orthogonal projection, $\mathrm{Rad}: L^2(D^n) \longrightarrow L^2(D^n)$, $w_A \longmapsto 1_{|A|=1} w_A$

Then $\sup_n \|\mathrm{Rad} \otimes \mathrm{Id}_X\|_{B(L^2(D^n;X))} < \infty$

(Recall Maurey-Pisier 1976: (i) $\Leftrightarrow$ X has nontrivial type: [MP].

$\exists p > 1\ \exists T_p(X)\ \forall n\ \forall x_1, \dots, x_n \in X\ \ \|\sum_i \varepsilon_i x_i\|_{L^2(D_n;X)} \le T_p(X)\left(\sum_i \|x_i\|^p\right)^{1/p}$)

By [MP], (i) $\Rightarrow$ (i') $\leftarrow$ here we need $p \neq 1, \infty$ (at least if $p=2$; otherwise by Khinchin-Kahane)

We shall admit this. We will see $\boxed{(i') \Rightarrow (ii)}$, which is the difficult part.

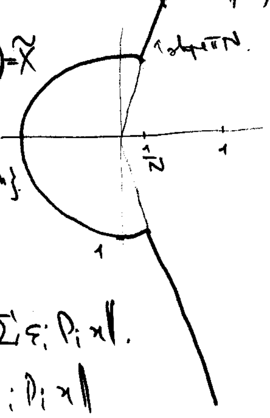
Fix X, p, and $N \geq 256$ s.t. $\mathbb{E}\|\sum_{i=1}^n \varepsilon_i x_i\|_{L^p([0,1];X)} \le \frac{n}{16}$ for all $n \geq N$

(use (i') for $c=32$ and divide n by N and use triangular $\le$)

Idea of the proof: • "Banach space lemma" } look up Maurey in the Handbook of B-spaces
• complex analysis.

<u>Lemma</u>: If $M: L^p([0,1];X) \circlearrowleft$ is a c.c. of commuting norm one projections, then $\|M^n - M^{n+1}\|_{B(L^p(X))} \le \frac{1}{4}$ for all $n \geq N$. This shows that if $t>0$, $\|T_{tn} - T_{t(n+1)}\| \le \frac{1}{4}$.

<u>Lemma 2</u>: Consider the path Γ such that $\|(z-\Delta)^{-1}\|_{L^p(D^n;X)} \le 36\pi N \max(|\mathrm{Im} z|, 1)$



<u>Proof of Lemma 1</u>: Take P a random norm 1 projection on $L^p([0,1];X) = \tilde{X}$ such that $M = \mathbb{E}P$. If $P_1, P_2, \dots$ is an iid sequence of copies of P, the P_i, P_k commute and $M^n = \mathbb{E}(P_1 \cdots P_n)$. Denote $P_B = \prod_{i \in B} P_i$ for $B \subseteq \{1, \dots, n\}$.

Fix $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ and denote $B = \{i : \varepsilon_i = 1\}$, $k = \#B$, $C = \{i : \varepsilon_i = -1\}$, $l = \#C = n-k$.

Assume $k \le l$ and take $x \in \tilde{X}$ of norm 1; $\|P_B(\sum_{i=1}^n \varepsilon_i P_i x)\|_{\tilde{X}} \le \|\sum_i \varepsilon_i P_i x\|$.

Then $P_B P_i = \begin{cases} P_B & \text{if } \varepsilon_i = 1 \\ P_B P_i & \text{otherwise} \end{cases}$ and $\|k P_B x - \sum_{i \in C} P_B P_i x\| \le \|\sum_i \varepsilon_i P_i x\|$

Now take the expected value / P_i: $\mathbb{E}(P_B x) = M^k$ and $\mathbb{E}(P_B P_i x) = M^{k+1}$ if $i \in B$, and $\|\underbrace{k M^k x - l M^{k+1} x}_{l M^k x + (k-l) M^k x}\| \le \mathbb{E}_{P_i}\|\sum_i \varepsilon_i P_i x\|$ if $i \in B$, and

$$\underbrace{l}_{\geq n/2} \overbrace{\|M^k x - M^{k+1} x\|}^{\geq \|M^n x - M^{n+1} x\|} \le |k-l| + \mathbb{E}_{P_i}\|\sum \varepsilon_i P_i x\|,$$ so that

$$\frac{n}{2}\|(M^n - M^{n+1})x\| \le \underbrace{|k-l|}_{|\sum_i \varepsilon_i|} + \mathbb{E}_{P_i}\|\sum_i \varepsilon_i P_i x\|.$$ This is also true if $k \geq l$.

Now average over $\varepsilon_1, \dots, \varepsilon_n \in D^n$: $\frac{n}{2}\|(M^n - M^{n+1})x\| \le \mathbb{E}_{\varepsilon_i}|\sum_i \varepsilon_i| + \frac{n}{16}$ by hypothesis

and $\|M^n - M^{n+1}\| \le \dfrac{\sqrt{n} + \frac{n}{16}}{\frac{n}{2}} \le \frac{1}{4}$ if $n \geq 256$

Proof of Lemma 2 : To show this, we can assume $\dim X$ is finite. We only need to show that $z-\Delta$ is injective to get that it is invertible. Take $x \in L^p(D^n; X)$ and assume that $\varepsilon > 0$ s.t. $\|(z-\Delta)x\| \le \varepsilon$. We shall show that $\varepsilon > \frac{\max(|\operatorname{Im} z|, 1)}{36\pi N}$

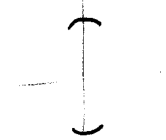
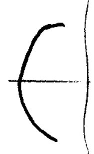
First assume that $\operatorname{Re} z \geqslant 0$. Consider $\phi(t) = T_t x = e^{-t\Delta} x$. Then $\phi'(t) = -\Delta T_t x = T_t(-\Delta x) = \underbrace{T_t\big((z-\Delta)x\big)}_{g(t)} - z T_t x$: $\phi'(t) + z\phi(t) = g(t)$.

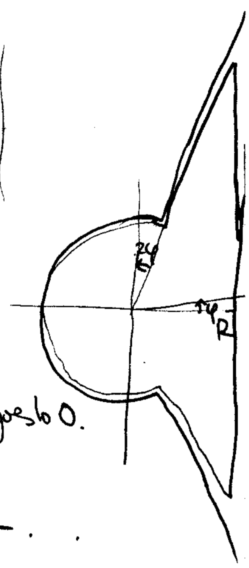
$\phi(t) = e^{-zt}\left(x + \int_0^t e^{zs} g(s)\, ds\right)$ and $\|\phi(t) - e^{-zt}x\| \le \int_0^t \underset{|e^{z(s-t)}|}{1}\, \varepsilon\, ds = \varepsilon t$.

If you consider $z = a+ib$, $\|\phi(t) - e^{-zt}x\| \le \frac{e^{-at}-1}{a}\varepsilon$ if $\operatorname{Re} z \le 0$.

Let $t = \frac{\pi}{|b|}$ and $n = N$: $|e^{-znt} - e^{-z(n+1)t}| = \underbrace{e^{-\frac{an\pi}{|b|}}}_{e^{-1}} \underbrace{\left(1 + e^{-\frac{a\pi}{|b|}}\right)}_{\geqslant 1} \underset{\text{triangular} \le}{\le} \varepsilon(2n+1)t + \frac{1}{4}$. Lemma 1!

Thus $\varepsilon > \frac{(e^{-1} - \frac{1}{4})|b|}{(2n+1)\pi}$

the same which works on  , and then it is easy on 



Proof of (ii) : If $\xi \in \mathbb{C}$ $e^{-\xi z} = \varphi_\xi(z)$; if $a \in \mathbb{R}_+$, $a < R$,

$e^{-\xi a} = \frac{1}{2i\pi} \int_{\Gamma_R} \frac{e^{-\xi z}}{z-a}\, dz$ is the Cauchy formula.

We have to show that the integral on the segment on $\operatorname{Re} z = R$ goes to 0.

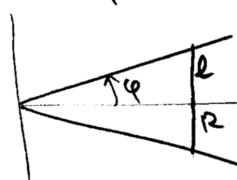
$e^{-\xi a} = \frac{1}{2i\pi} \int_\Gamma \frac{e^{-\xi z}}{z-a}\, dz$ provided that $|e^{-\xi z}| \ll 1$ on $\operatorname{Re} z = R$.

Then $|\arg \xi z| \le \frac{\pi}{2}$ if $|\arg z| = \frac{\pi}{2} - 2\varphi$, ~~(if $z \in R_\varphi(\frac{1}{N} + i\pi)$)~~ $\boxed{\varphi}$ $|\arg \xi| \le \varphi$, $|\arg(z\xi)| \le \frac{\pi}{2} - \varphi$ and $\operatorname{Re}(\xi z) > C_\varphi |\xi z|$.

$e^{-\xi a} = \frac{1}{2i\pi} \int_\Gamma \frac{e^{-\xi z}}{z-a}\, dz$ for all ξ with $|\operatorname{Arg} \xi| \le \varphi$, for all $a > 0$

Thus $e^{-\xi\Delta} = \frac{1}{2i\pi} \int_\Gamma \frac{e^{-\xi z}}{z-\Delta}\, dz$ and $\|e^{-\xi\Delta}\| \le \frac{1}{2\pi} \int_\Gamma e^{-C_\varphi|\xi z|} \frac{36\pi N}{\max(1, |\operatorname{Im} z|)} |dz|$

$\le M$ on this sector.

Proof of (iii) :  R such that the segment ℓ has length $= 2\pi$.

Remark : If $t \in [-\pi, \pi]$, $T_{R+it}(w_A) = e^{-R|A|} e^{-it|A|} w_A$ and $\int_{-\pi}^{\pi} T_{R+it} e^{it} \frac{dt}{2\pi}$ sends $w_A \longmapsto e^{-R|A|} w_A \underbrace{\int_{-\pi}^{\pi} e^{it(1-|A|)} \frac{dt}{2\pi}}_{\substack{1 \text{ if } |A|=1 \\ 0 \text{ otherwise}}}$, so that it is $e^{-R}\mathrm{Rad}$ (projection)

④

Similarly if $m \in \mathbb{N}$, $\mathrm{Rad}_m : w_A \longmapsto \begin{cases} w_A & \text{if } |A| = m \\ 0 & \text{otherwise} \end{cases}$ satisfies

$$e^{-Rm} \, \mathrm{Rad}_m = \int_{-\pi}^{\pi} T_{R+it} \, e^{imt} \frac{dt}{2\pi}, \text{ so that } \|\mathrm{Rad}_m\|_{L^p(D^n;X)} \leq e^{Rm} M$$

To get iii $\Rightarrow$ i: Lemma: $\|\mathrm{Rad}\|_{L^p(D^n;\ell_1)} \geq c\sqrt{n}$ if $p \leq 2$

Proof: We show $\|\mathrm{Rad}\|_{L^p(D^n; L^1(D^n))} \geq \ldots$

Consider $f_n \in L^p(D^n; \underset{\ell_1^{2^n}}{\underset{\uparrow}{L^1(D^n)}})$: $f_n(\varepsilon)(\varepsilon') = \prod_{i=1}^n (1+\varepsilon_i\varepsilon_i') = \begin{cases} 2^n & \text{if } \varepsilon = \varepsilon' \\ 0 & \text{otherwise.} \end{cases}$

We have $\|f_n\| = \left(\int_{D^n} 1^p \, d\varepsilon\right)^{1/p} = 1$ and $\mathrm{Rad}(f_n) = \sum_i \varepsilon_i \varepsilon_i'$

and $\|\mathrm{Rad}\, f_n\|_{L^p(D_n; L^1)} = \mathbb{E}\,|\underset{\substack{\uparrow \\ \text{same law as } \tilde{\varepsilon}_i}}{\sum \varepsilon_i \varepsilon_i'}| \geq c\sqrt{n}$.

7/5/2013

M. De La Salle (1)

Recall: Def: a Banach space X is not B-convex if $\forall \varepsilon>0\ \forall n$ X contains a subspace $(1+\varepsilon)$-isomorphic to ℓ_1^n
($\Leftrightarrow$ an ultrapower of X contains ℓ_1 isometrically)

Notation: $n\in\mathbb{N}$, $D_n=\{-1,1\}^n$ with unif. measure

$D_\infty = D = \{-1,1\}^{\mathbb{N}}$

$i\le n$: ε_i i-th coordinate function, $\varepsilon_i : D_n \to \{-1,1\}$

Walsh functions: if $A\subset\{1..n\}$ (or $A\subset\mathbb{N}$ finite)

$$w_A := \prod_{i\in A}\varepsilon_i : D_n \to \{-1,1\}$$

$(w_A)_{A\subset\{1..n\}}$ ON basis of $L_2(D_n)$

$$\forall t\ge 0\quad T_t : L^p(D_n)\to L^p(D_n) \qquad = e^{-t\Delta},\ \Delta : w_A\mapsto |A|w_A\ \forall A$$
$$w_A \mapsto e^{-t|A|}w_A$$

We say that $\forall p\in(1,\infty)$ T_t is a norm 1 map $L^p(D_n,X)\to L^p(D_n,X)$
$\forall X$ B. sp.
$\forall t\ge 0$

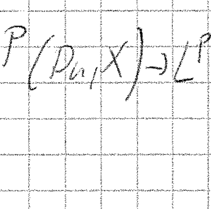
$\forall m\in\mathbb{N}$ $\quad Rad_m : L^p(D_n)\to L^p(D_n)$ $\qquad Rad := Rad_1$

$$w_A \mapsto \begin{cases} w_A & \text{if } |A|=m\\ 0 & \text{otherwise}\end{cases}$$

$$T_t = \sum_{m\ge 0} e^{tm}\, Rad_m$$

Thm (Pisier): X B. sp., $p\in(1,\infty)$,

1) X is B-convex

2) $\exists\theta>0\ \exists M>0$ $\quad \|T_z\|_{B(L^p(D,X))}\le M \quad \forall z\in$ 

3) $\|Rad\|_{L^p(D;X)}<\infty$

Today: Th (Mendel-Naor, 5.1): If X is a B-convex Banach sp.

$\exists A, B, C \in \mathbb{R}_+$, $B > 2$ s.t.

$\forall k \in \mathbb{N}$ $\quad \|T_t\|_{B(L^p_{\geq k}(D; X))} \leq A e^{-Ck \min(t, t^B)}$

$L^p_{\geq k}(D) = \overline{\operatorname{span}}\{W_A, |A| \geq k\} =$ "tail space"

5.2 If X is a B.sp.

if $\exists k \in \mathbb{N}$, $\exists t > 0$ s.t. $\quad \|T_t\|_{B(L^p_{\geq k}(D, X))} < 1 \quad (*)$

then X is B-convex

Proof of 5.2 and of 3) $\Rightarrow$ 1) in Pisier's theorem:

if $(*)$ holds then it holds with X replaced by an ultrapower of X so we only prove that $\|T_t\|_{B(L^p_{\geq k}(D, \ell_1))} = 1 \; \forall k \; \forall p$

$\|Rad\|_{B(L^p(D, \ell_1))} = \infty$

In fact: $\lim_{n \to \infty} \|T_t\|_{B(L^p_{\geq k}(D_n; \overbrace{L^1(D_n)}^{\cong \ell_1^{2^n}}))} = 1$

$\exists c > 0 \quad \|Rad\|_{B(L^p(D_n; L^1(D_n)))} \geq c\sqrt{n}$

Consider $f_n \in L^p(D_n, L^1(D_n))$

$f_n(\varepsilon)(\varepsilon') = \begin{cases} 2^n & \text{if } \varepsilon = \varepsilon' \\ 0 & \text{otherwise} \end{cases}$

$\|f_n\|_{L^p(D_n, L^1)} = 1$

$$f_n(\varepsilon)(\varepsilon') = \sum_{A\subseteq\{1\dots n\}}' W_A(\varepsilon\varepsilon') = \sum_A W_A(\varepsilon) W_A(\varepsilon')$$

check $\qquad$ obvious

③

$$\mathrm{Rad}(f_n) = \sum_{|A|=1}' W_A(\varepsilon) W_A(\varepsilon') = \sum_{i=1}^n{}' \varepsilon_i \varepsilon_i'$$

$$\forall \varepsilon \quad \left\| \sum \varepsilon_i \varepsilon_i' \right\|_{L^1(D_n)} = \left\| \sum_1^n \varepsilon_i \right\|_{L^1(D_n)} \geq c\sqrt{n}$$

CLT

proof of ⊗ for $k=1$: denote $g_n = f_n - 1$

$\in L^p_{\geq 1}(D_n, L^1(D_n))$

$$\|g_n\|_{L^p_{\geq 1}(D_n; L^1(D_n))} \leq 2$$

$$T_t g_n = \left(\sum_A e^{-t|A|} W_A(\varepsilon\varepsilon')\right) - 1$$

$$= \sum_{\delta_1 \dots \delta_n \in \{0,1\}} e^{-t\sum\delta_i} \prod_i (\varepsilon_i\varepsilon_i')^{\delta_i} - 1$$

$$= \prod_{i=1}^n (1 + \varepsilon_i \varepsilon_i' e^{-t}) - 1$$

$$= (1+e^{-t})^k (1-e^{-t})^\ell - 1 \qquad \text{where } k = k(\varepsilon\varepsilon') = \#\{i \leq n, \varepsilon_i\varepsilon_i' = 1\}, \quad \ell = n-k$$

$\forall \varepsilon \in D_n$

$$\|g_n(\varepsilon)\|_{L^1(D_n)} = \frac{1}{2^n} \sum_{\varepsilon' \in D_n} \left| (1+e^{-t})^{k(\varepsilon\varepsilon')} (1-e^{-t})^{\ell(\varepsilon\varepsilon')} - 1 \right|$$

$$= \frac{1}{2^n} \sum_k \binom{n}{k} \left| (1+e^{-t})^k (1-e^{-t})^\ell - 1 \right|$$

$= \ell^1$ - difference of 2 proba measures

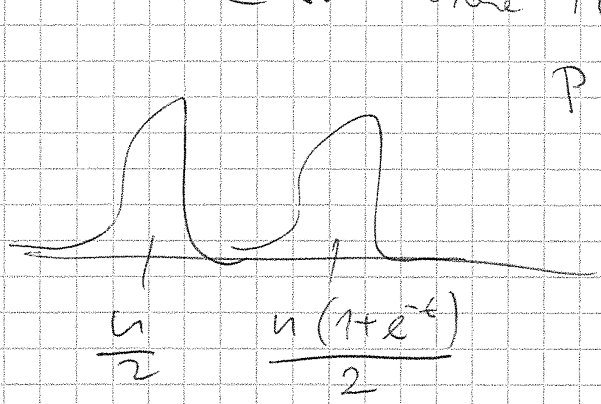
gives mass $\binom{n}{k}/2^n$ to k $\longrightarrow$ unbiased binomial distribution $\sum X_i$: $P(X_i = 1) = P(X_i = \dots) = \frac{1}{2}$

$\binom{n}{k}/2^n \, (1+e^{-t})^k (1-e^{-t})^{n-k}$

the other one is biased binomial distribution (9)

$\sum X_i$ where $P(Y_i = 1) = \left(\frac{1+e^{-t}}{2}\right)$

$P(Y_i = 0) = \left(\frac{1-e^{-t}}{2}\right)$



law of large numbers

$\xrightarrow[n\to\infty]{} 2$ (disjoint supports)

⊗ for k arbitrary: consider $h_n \in L^p(D_{nk}; L^q(D_{nk}))$

$$D_{nk} = \underbrace{D_n \times \dots \times D_n}_{k \text{ times}}$$

$$h_n((\varepsilon^1, \dots, \varepsilon^k))((\varepsilon'^1, \dots, \varepsilon'^k)) := \prod_{i=1}^{k} g_n(\varepsilon^i)(\varepsilon'^i)$$

$h_n \in L^p_{\geq k}(D_{nk}, L^q(D_{nk}))$

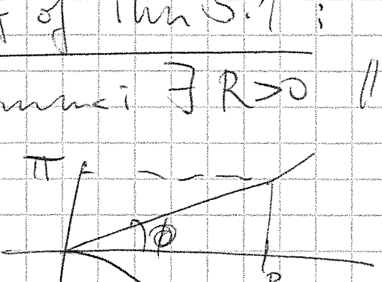
$\|h_n\|_{L^p} = \|g_n\|^k_{L^p} \leq 2^k$

$\|T_t h_n\| = \|T_t g_n\|^k \xrightarrow[n\to\infty]{} 2^k$ $\square$

" $(T_t g_n)^{\otimes k}$ "

Proof of Thm 5.1:

Lemma: $\exists R > 0$ $\|\mathrm{Rad}_m\|_{L^p(D,X)} \leq M e^{Rm}$



$\|T_z\|_{L^p(D,X)} \leq M$

$\forall \theta \in [-\pi, \pi]$ $T_{R+i\theta} = \sum_m e^{-(R+i\theta) m} \mathrm{Rad}_m$ (5)

Fourier coeffs.

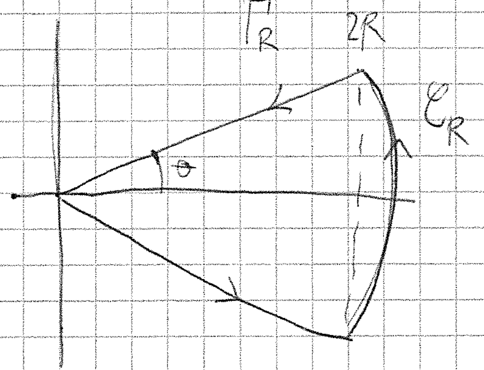
$$= \int_{-\pi}^{\pi} T_{R+i\theta}\, e^{im\theta} \frac{d\theta}{2\pi} \quad \square$$

Proof of Th. 1 for large t: $t \geq 2R$

$$\|T_t\|_{\mathcal{L}(L^p_{\geq k}(D), X)} = \Big\| \sum_{m \geq k} e^{-tm} \mathrm{Rad}_m \Big\|$$

$$\leq \sum_{m \geq k} e^{-tm} M e^{Rm}$$

$$= \frac{M e^{(R-t)k}}{1 - e^{(R-t)}} \leq \frac{M}{1-e^{-R}} e^{-\frac{tk}{2}}$$



$D_r = \mathrm{int}\, \Gamma_R$

Recall $\forall t \in D_R$ $\exists !$ Borel proba measure on Γ_R s.t. $\forall f$ holomorphic on a nbhd of D_r

$$f(t) = \int f(z)\, d\mu_t(z)$$

μ_t is the harmonic measure wrt D_R and t

Lemma 1: $\forall E$ B-sp. $\forall f \in \mathrm{Hol}(\text{nbhd of } D_R; E)$

$$\log \|f(t)\| \leq \int \log \|f(z)\|\, d\mu_t(z)$$

standard proof

Lemma: $\mu_t(\mathcal{E}_R) \geq \left(\frac{t}{r}\right)^{\frac{\pi}{2\phi}}$ $\quad \forall t \in (0, 2R)$

$$r = \frac{2R}{\cos\phi}$$

Conclusion of 5.1: apply Lemma 1 for $z \mapsto T_z$ (6)

$$E = L^p_{\geq k}(D;X)$$

$$\|T_z\|_{L^p_{\geq k}(D;X)} \leq M \;\forall z \in \Gamma_R$$
$$\leq \frac{M}{1-e^{-R}} e^{-Rk} \quad \text{if } z \in \mathcal{E}_k$$

$$\log \|T_t\| \leq \mu_t(\mathcal{E}_k) \log\left(\frac{M}{1-e^{-R}} e^{-Rk}\right) + (1-\mu_t(\mathcal{E}_k)) \log M \leq$$

if k large

$$\leq \left(\frac{t}{r}\right)^{\frac{\pi}{2\phi}} \log\left(\frac{e^{-Rk}}{1-e^{-R}}\right) + \log M \Rightarrow B = \frac{\pi}{2\phi}$$

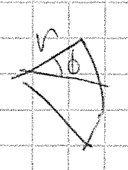
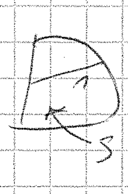
$\square$

Proof of Lemma 2:

~~Use the fact~~

1^{st} reduction: $\Phi = \frac{\pi}{2}$ and $R = 1$

(~~Otherwise~~ $z \mapsto \left(\frac{z}{r}\right)^{\frac{\pi}{2\phi}}$ is a holom. bijection between

 and 

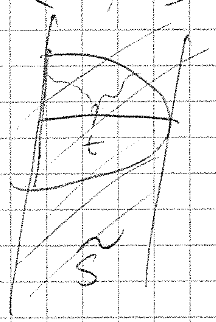
$\mu_r^{r,\phi} \rightsquigarrow \mu_r^{1,\frac{\pi}{2}}$ change of variable)

Use fact: $\forall A \subseteq \partial S$ (open)

$\mu_t(A)$ = probability that a Brownian motion starting from t exits S through A

7

$$\mu_t^S(\mathcal{E}) \geq \mu_t^{\tilde{S}}(\{z, \operatorname{Re} z = 1\}) = t$$



Consider $z \mapsto \bar{z}$ on $\tilde{S}$ $\square$

Gilles

24/5/2013

Construction of base graphs in Mendel-Naor.

Reminder 1: If $K : X\times X \longrightarrow [0,+\infty[$ symmetric kernel and $A=[a_{ij}]$ is a symmetric stochastic matrix, ~~and~~ then

$$\gamma^+(A,K) = \inf\Big\{\gamma^+>0 : \forall x_1,\dots,x_n\in X\ \forall y_1,\dots,y_n\in X \quad \frac{1}{n^2}\sum_{i,j} K(x_i,y_j) \le \frac{\gamma^+}{n}\sum_{i,j} a_{ij}K(x_i,y_j)\Big\}$$

If $G=(X,E)$ is a d-regular graph and $A_G = \frac{1}{d}[E(u,v)]_{u,v\in V}$ the normalized adjacency matrix, then

$$\gamma^+(G,K)=\gamma^+(A_G,K)=\inf\Big\{\gamma^+>0 : \forall f,g : V\longrightarrow X \quad \frac{1}{|V|^2}\sum_{u,v} K(f(u),g(v)) \le \frac{\gamma^+}{|V|\times d}\sum_{(u,v)\in E} K(f(u),g(v))\Big\}$$

$\boxed{\text{Goal}}$: to build $G_n=(V_n,E_n)$ 3-regular graphs so that $|V_n|\to\infty$ and $\forall (X,\|\cdot\|_X)$ superreflexive $\forall p\geq 1$ $\sup_n \gamma^+(G_n,\|\cdot\|_X^p)<\infty$

Preliminaries: $(X,\|\cdot\|_X)$ normed space and $f : \{1,\dots,n\}\longrightarrow X$, $\|f\|_{L_p^n(X)} = \left(\frac{1}{n}\sum_{i=1}^n \|f(i)\|_X^p\right)^{1/p}$

and $A\otimes Id_X : L_p^n(X)\longrightarrow L_p^n(X)$, $f\longmapsto \left(i\longmapsto \sum_j a_{ij}f(j)\right)$ and $L_p^n(X)_0 = \{f\in L_p^n(X) : \sum_{i=1}^n f(i)=0\}$

Remark: $L_p^n(X)_0$ stable under $A\otimes Id_X$ because A is stochastic:
$\sum_i g(i) = \sum_i\left(\sum_j a_{ij} f(j)\right) = \sum_j \left(\sum_i a_{ij}\right) f(j) = \sum_j f(j) = 0$ and
$\lambda_X^{(p)}(A) := \|A\otimes Id_X\|_{B(L_p^n(X)_0)}$

N.B.: If $X=\mathbb{R}$, $p=2$, $\lambda_{\mathbb{R}}^{(2)}(A) = \max_{i\geq 2}|\lambda_i(A)| = \lambda(A)$ and $\gamma^+(A,|\cdot|^2) = \frac{1}{1-\lambda(A)}$
(i.e., $K(x,y)=|x-y|^2$) $= \gamma^+(A,\|\cdot\|_H^2)$

Lemma 6.1.: $\gamma_+(A,\|\cdot\|_X^p)\leq \left(1+\frac{4}{1-\lambda_X^p(A)}\right)^{1/p}$

Remark: there is a converse estimate when X is K-convex: see Lemma 6.6 of the paper.

Proof: $\left(\frac{1}{n^2}\sum\sum\|f(i)-g(j)\|_X^p\right)^{\frac1p} \le \|\bar f-\bar g\|_X + \left(\frac{1}{n^2}\sum_i\sum_j\|f_0(i)-g_0(j)\|_X^p\right)^{\frac1p}$

(1) where $\bar f = \frac1n\sum f(i)$ and $f_0 = f-\bar f \in L_p^n(X)_0$ (This is just the triangular $\le$)

$$\le \|\bar f-\bar g\|_X + \left(\frac{1}{n^2}\sum\sum 2^{p/q}\left(\|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)\right)^{\frac1p}$$

$\left(\frac{a+b}{2}\right)^p \le \frac{a^p+b^p}{2}$
$(a+b)^p \le 2^{p-1}(a^p+b^p)$

$$= \|\bar f-\bar g\|_X + 2^{\frac{p-1}{p}}\left(\frac1n\sum_i\|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)^{\frac1p} \quad (1')$$

and $\|\bar f-\bar g\| = \|\frac1n\sum f(i)-g(i)\|_X = \|\frac1n\sum_i\sum_j a_{ij}f(i)-g(i)\|_X$

$$= \|\frac1n\sum_i\sum_j a_{ij}(f(i)-g(j))\|_X \le \left(\frac1n\sum\sum a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}$$

(this is Jensen's inequality with $\mu(\{(i,j)\}) = \frac1n a_{ij}$

Define $B = \left[\begin{array}{c|c} 0 & A \\ \hline A & 0\end{array}\right]$, $h = f_0\oplus g_0 \in L_p^{2n}(X)_0$

$$(1-\lambda)\|h\|_{L_p^{2n}(X)} = \|h\|_{L_p^{2n}(X)} - \left(\frac{\lambda^p\|f_0\|^p_{L_p^n(X)}+\lambda^p\|g_0\|^p_{L_p^n(X)}}{2}\right)^{\frac1p}$$

$$\le \|h\|_{L_p^{2n}(X)}\left(\frac12\|(A\otimes Id_X)(f_0)\|^p_{L_p^n(X)}+\frac12\|A\otimes Id_X(g_0)\|_X^p\right)^{\frac1p}$$

$$= \|h\|_{L_p^{2n}(X)} - \|(B\otimes Id_X)(h)\|_{L_p^{2n}} \le \|(Id-B\otimes Id_X)(h)\|_{L_p^{2n}(X)}$$

$$= \left(\frac{1}{2n}\sum_i\|f_0(i)-\sum_j a_{ij}g(j)\|_X^p + \frac{1}{2n}\sum_i\|g_0(i)-\sum_j a_{ij}f(j)\|_X^p\right)^{1/p}$$

as A is stochastic and symmetric, $\le \left(\frac1n\sum_i\sum_j a_{ij}\|f_0(i)-g_0(j)\|_X^p\right)^{1/p}$

$$\le \|\bar f-\bar g\|_X + \frac1n\left(\sum_i\sum_j a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}.$$

Let us ~~express~~ summarize: we had to bound (1) and $\|h\| = \left(\frac{1}{2n}\sum_i\|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)^{\frac1p}$.

So $\left(\frac1n\sum_i\|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)^{\frac1p} \le \frac{2^{1/p}}{1-\lambda}\times 2\times\left(\frac1n\sum_i\sum_j a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}$

Finally, (1) $\le \left(1+2^{\frac{p-1}{p}}\times\frac{2^{\frac1p}\times 2}{1-\lambda}\right)\left(\frac1n\sum\sum a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}$.

thus $\gamma^+ \le \left(1+\frac{4}{1-\lambda_X^p(A)}\right)^{\frac1p}$.

Theorem (Corollary 7.4) $\forall\delta\in]0,1[\ \forall p\in]1,\infty[\ \exists n_0(p,\delta)\ \exists\left(H_n^p(\delta)\right)_{n\ge n_0(p,\delta)}$ $d_n^p(\delta)$-regular such that

(i) $|V(H_n(\delta))| = m_n = 2^{D_n}$, $D_n \ge \frac{n}{10}$

(ii) $d_n^p(\delta) \le e^{(\log m_n)^{1-\delta}}$

and $\forall (X,\|\cdot\|_X)$ K-convex Banach space $\exists\delta_0^p(X)\ \forall\delta<\delta_0^p(X)\ \sup\gamma^+(H_n^p(\delta),\|\cdot\|_X^p) \le g^{p+1}$

Lemma 7.3 $\forall K,p>1\ \exists n(K,p)\ \exists \delta(K,p)\in]0,1[\ \exists (H_n(K,p))_{n\geq n(K,p)}$ $d_n^p(K,p)$-regular $d_n(K,p)\leq e^{(\log m_n)^{1-\delta(K,p)}}$ with $m_n=|V(H_n(K,p))|=2^{D_n}$ for $D_n\geq \frac{n}{10}$

and for any X K-convex with $K(X)\leq K$, for all $n\geq n(K,p)$, $\gamma^+(H_n(K,p),\|\cdot\|_X^p)\leq g^{n+1}$

For 7.3 ⟹ 7.4, see the paper.

Reminder 2: If $A\subset\{1,\dots,n\}$ and $W_A:\mathbb{F}_2^n\longrightarrow X$, $x\longmapsto(-1)^{\sum_A x_j}$ and $f\in L^p(\mathbb{F}_2^n,X)$,

$$e^{z\Delta}f=\sum_{A\subset\{1,\dots,n\}}e^{z|A|}\hat{f}(A)\,w_A$$

$$R_z(x)=\prod_{j=1}^n(1+(-1)^{x_j}e^z)=(1-e^z)^{\sum_j x_j}(1+e^z)^{n-\sum x_j}$$

$$=\sum_{A\subset\{1,\dots,n\}}e^{z|A|}(-1)^{\sum_{j\in A}x_j}\quad : e^{z\Delta}f=R_z*f.$$

Then
$$(e^{z\Delta}f)(y)=\sum_{w\in\mathbb{F}_2^n}\left(\frac{1-e^z}{2}\right)^{\sum_j y_j-w_j}\left(\frac{1+e^z}{2}\right)^{n-\sum_j y_j-w_j}\cdot f(w)$$
$$=\sum_{w\in\mathbb{F}_2^n}\left(\frac{1-e^z}{2}\right)^{\|y-w\|_1}\left(\frac{1+e^z}{2}\right)^{n-\|y-w\|_1}f(w)$$
because we just work modulo 2.

In particular, $(e^{-t\Delta}\delta_x)(y)=\left(\frac{1-e^{-t}}{2}\right)^{\|x-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|x-y\|_1}$.

$\tau_t=\frac{1-e^{-t}}{2}$; $\sigma_t^n=\tau_t^{4n\tau_t}(1-\tau_t)^{(1-4\tau_t)n}$ and $e_t^n:\{0,\dots,n\}\longrightarrow\mathbb{N}$, $k\longmapsto\left\lfloor\frac{\tau_t^k(1-\tau_t)^{n-k}}{\sigma_t^n}\right\rfloor$

Lemma 7.2: Action of $e^{-t\Delta}$ and Poincaré inequalities: let $t\in]0,\frac14[$, $p\geq1$, $n\geq 2^{13}$ so that $\tau_t\geq\left(\frac{p\log 18n}{18n}\right)^{1/2}$; $G_t^n=(\mathbb{F}_2^n,E_t^n)$, $V(G_t^n)=\mathbb{F}_2^n$: for all $x,y\in\mathbb{F}_2^n$ the edge (x,y) appears $e_t^n(\|x-y\|_1)$ times: G_t^n is d_t^n-regular, $\frac{1}{3\sigma_t^n}\leq d_t^n\leq\frac{1}{\sigma_t^n}$.

For all metric space (X,d_X),
$$\frac{1}{3|E_t^n|}\sum_{E_t^n}d_X(f(x),g(y))^p\leq\frac{1}{2^n}\sum_{x,y\in\mathbb{F}_2^n}(e^{-t\Delta}\delta_x)(y)\,d_X(f(x),g(y))^p$$
$$\leq\frac{1}{|E_t^n|}\sum_{E_t^n}d_X(f(x),g(y))^p$$

End of talk: 7.2 ⟹ 7.3.

Claim: there's a linear subspace V_n of $\mathbb{F}_2^n$ so that $D_n=\dim(V_n)\geq\frac{n}{40}$, $k_n=\min_{x\in V_n\setminus\{0\}}\|x\|_1\geq\frac{n}{10}$

It's a "good linear code"

④

Proof: Let V be the k-dimensional linear subspace of $\mathbb{F}_2^n$ ~~of dim~~ such that

$$(*) = \left|\{x\in\mathbb{F}_2^n : \exists v\in V\ \|x-v\|_1 < \tfrac{n}{10}\}\right| \le 2^k \times \sum_{\ell<\frac{n}{10}} \binom{n}{\ell} = 2^k A_n$$

We have $A_n = 2^n P[\sum_i X_i < \frac{n}{10}]$ X_i independent Bernoulli "facts of life"

$= 2^n P[\sum_i \varepsilon_i > \frac{4n}{5}]$ ε_i —— Rademacher.

Then $P[\sum_i \varepsilon_i > a] \le e^{-\frac{a^2}{2n}}$ and $A_n \le 2^n e^{-\frac{8n}{25}}$ and $(*) \le 2^n e^{k-\frac{8n}{25}}$

Pick $k\in]\frac{n}{10},\frac{8n}{25}[$ and V k-dimensional and note that $(*) < 2^n \Rightarrow \exists x_0\in\mathbb{F}_2^n\ \forall v\in V\ \|x_0-v\|_1 \ge \frac{n}{10}$.

$V_n = \mathrm{span}(V\cup\{x_0\})$, $\min_{V_n\setminus\{0\}} \|x\|_1 \ge \frac{n}{10}$

Define: $H_n = H_n(k,p)$, $V(H_n) = \mathbb{F}_2^n/V_n^\perp$, $V_n^\perp = \{x\in\mathbb{F}_2^n : \forall y\in V_n\ \sum_i x_i y_i = 0 \bmod 2\}$

and $|V(H_n)| = 2^{D_n}$ and $D_n = \dim V_n \ge \frac{n}{10}$.

The number of edge joining $x+V_n^\perp$ and $y+V_n^\perp$

= —— with one end in $x+V_n^\perp$ and the other end in $y+V_n^\perp$, divided by $|V_n^\perp|$, makes

$$\frac{1}{|V_n^\perp|}\sum_{(u,v)\in V_n^\perp\times V_n^\perp} e_t^n(\|x-y+u-v\|_1) = \sum_{u\in V_n^\perp} e_t^n(\|x-y+u\|_1)$$

and the degree at $\dot{x} = x+V_n^\perp$ is $\sum_{y\in V_n^\perp}\left(\sum_{u\in V_n^\perp}\|x-y+u\|_1\right) = \sum_{z\in\mathbb{F}_2^n} e_t^n(\|x-z\|_1) = d_t^n$ (by 7.2)

Let $\pi:\mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n/V_n^\perp$ be the quotient map and X be a B-space:

If $f\in L^p(\mathbb{F}_2^n/V_n^\perp, X)$, we get $\pi f : \mathbb{F}_2^n \longrightarrow X$, $x \longmapsto f\circ\pi(x)$ $\in L_p(\mathbb{F}_2^n, X)$ constant on cosets.

Claim: let $f\in L^p(\mathbb{F}_2^n/V_n^\perp, X)_0$ (i.e. $\sum_{\dot{x}} f(\dot{x}) = 0$ so that $\widehat{\pi f}(\emptyset)=0$). Then $\pi f\in L_p^{\ge k_n}(\mathbb{F}_2^n, X)$ with $k_n = \min_{x\in V_n\setminus\{0\}} \|x\|_1 \ge \frac{n}{10}$.

Proof: $V_n^{\perp\perp} = V_n$ (exercise). Let $\emptyset \ne A\subset\{1,\dots,n\}$ and $|A| < k_n$. Then $\mathbb{1}_A \notin V_n = V_n^{\perp\perp}$ so that $\exists v\in V^\perp$ $\langle \mathbb{1}_A, v\rangle \ne 0$ $\sum_A v_i = 1 \pmod 2$

$$\widehat{\pi f}(A) = \int_{\mathbb{F}_2^n} \pi f(u)\, w_A(u)\,du = \int_{\mathbb{F}_2^n} \pi f(u+v)\, w_A(u)\,du = \int_{\mathbb{F}_2^n} \pi f(u)\, w_A(u-v)\,du$$

$$= (-1)^{\sum_A v_i}\int_{\mathbb{F}_2^n} \pi f(u)\, w_A(u)\,du = -\widehat{\pi f}(A). \text{ Therefore } \widehat{\pi f}(A) = 0.$$

Remark: Theorem 5.1 $\Rightarrow$ | If f is K-convex with $K(X)\le K$, then there is $A = A(K,p)$, $B = B(K,p) > 2$, $C = C(K,p)$, so that ~~$\forall f\in L_p(\mathbb{F}_2^n$~~ $\|e^{-t\Delta}\|_{B(L_p^{\ge k}(\mathbb{F}_2^n,X))} \le Ce^{-Ak\min(t,t^B)}$

Here $\forall f\in L^p(\mathbb{F}_2^n/V_n^\perp, X)_0$ $\dfrac{\|e^{-t\Delta}\pi f\|_{L_p(\mathbb{F}_2^n,X)}}{\|\pi f\|_{L^p(\mathbb{F}_2^n,X)}} \le Ce^{-Ak_n\min(t,t^B)}$

Pick $t = t(n,p,k) = \left(\dfrac{\log 2C}{k_n A}\right)^{1/B}$: then $Ce^{-Ak_n\min(t,t^B)} \le \frac{1}{2}$.

there is $n(K,p)$ such that for all $n \geq n(K,p)$ $\quad t < \frac{1}{4}$ and $n \geq 2^{13}$

and $\cancel{\frac{1-e^{-t}}{2}} = \tau_t \geq \left(\frac{p \log + \beta n}{n}\right)^{1/2}$. Now remember $B > 2$!

We have degree $= d_t^n \leq \frac{1}{\sigma_t^n} = \frac{1}{\tau_t^{4\tau_t n}(1-\tau_t)^{n-4\tau_t n}} \leq \frac{1}{\tau_t^{8\tau_t n}} \leq e^{(\log m_n)^{1-\delta(K,p)}}$

We can apply Lemma 7,2 : $e^{-t\Delta_\pi}$ $\quad$ for $\delta(K,p) < \frac{1}{B}$, $B = B(K,p) > 2$.

Then $e^{-t\Delta}_\pi$ is a stochastic symmetric operator on $L_p(\mathbb{F}_2^n/V_n^\perp)$ $\quad$ there is a check needed here.

there are 2 mistakes on page 58. Let $s \in \mathbb{F}_2^n/V_n^\perp$: $(e^{-t\Delta_\pi})(\delta_s) = e^{-t\Delta}\mathbb{1}_s$

Let $y \in \mathbb{F}_2^n$: $\left(e^{-t\Delta}\mathbb{1}_s\right)(y) = \sum_{u\in s}' \left(e^{-t\Delta}\delta_u\right)(y) = \sum_{u\in s}' \left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1}$

$$= \left(e^{-t\Delta}\mathbb{1}_s\right)(y') \quad \text{if } y-y' \in V_n^\perp$$

this does not depend on $y \in T$ (where $T \in \mathbb{F}_2^n/V_n^\perp$) symmetric in s and T.

$$q_{s,T} = \sum_{u\in s}' \left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1} \quad \text{for } y \in T$$

$$q_{T,s} = \sum_{w\in V^\perp}' \left(\frac{1-e^{-t}}{2}\right)^{\|y+w-x\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|y+w-x\|_1} = q_{s,T}.$$

then
$$\sum_s' q_{s,T} = \sum_s' \sum_{u\in s}' \left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1}$$
$$= \sum_{x\in\mathbb{F}_2^n}' \left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1}$$
$$= \sum_{k=0}^{n} \binom{n}{k}\left(\frac{1-e^{-t}}{2}\right)^{k}\left(\frac{1+e^{-t}}{2}\right)^{n-k} = 1$$

$Q = [q_{s,T}]$ is symmetric and stochastic

Theorem 5,1 : $\|Q \otimes Id_X\|_{B(L_p(\mathbb{F}_2^n/V_n^\perp),X)_0} = \lambda_X^{(p)}(Q) \leq \frac{1}{2}$

Lemma 6,1 $\Rightarrow$ $\gamma^+(Q, \|\cdot\|_X^p) \leq g^p$ and finally, for $f,g : \mathbb{F}_2^n/V_n^\perp \to X$.

$$\frac{1}{|\mathbb{F}_2^n/V_n^\perp|^2}\sum_{s,T}' \|f(s)-g(T)\|_X^p \leq \frac{g^p}{|\mathbb{F}_2^n/V_n^\perp|}\sum_s'\sum_T' q_{s,T}\|f(s)-g(T)\|_X^p$$

$$= \frac{g^p}{|\mathbb{F}_2^n/V_n^\perp|}\sum_T'\sum_s'\left(\sum_{u\in s}'\left(\frac{1-e^{-t}}{2}\right)^{\|u-v\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-v\|_1}\right)\|f(s)-g(T)\|_X^p$$

for all $v \in T$

$$= \frac{g^p}{|\mathbb{F}_2^n/V_n^\perp| \times |V_n^\perp|}\sum_T'\sum_{v\in T}'\sum_s'\sum_{u\in s}' \underbrace{\qquad}_{\|f(s)-g(T)\|_X^p}$$

$$= \frac{g^p}{2^n}\sum_{u\in\mathbb{F}_2^n}'\sum_{v\in\mathbb{F}_2^n}'\left(\frac{1-e^{-t}}{2}\right)^{\|u-v\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-v\|_1}\|\pi f(u)-\pi g(v)\|$$

$$\overset{\text{by 7.2}}{\leq} \frac{3\cdot g^p}{|E_t^n|} \sum_{(u,v)\in E_t^n} \|\pi f(u) - \pi g(v)\|$$

But in my new graph $H_{n,}$

$$= \frac{3\cdot g^p}{|E_t^n|} \sum_S \sum_T \left(\sum_{u\in S} \sum_{v\in T} e_t^n(\|u-v\|_1) \right) \|f(S) - g(T)\|_X^p$$

$$= \frac{3\cdot g^p}{|E_t^n|} \sum_S \sum_T |V_n^\perp| \, E_n(S,T) \, \|f(S) - g(T)\|_X^p$$

$$= \frac{3\cdot g^p}{|E_t^n|} \sum_{(S,T)\in E_n} \|f(S) - g(T)\|_X^p \quad \left(\text{here } |E_n| = \frac{|E_t^n|}{|V_n^\perp|}\right)$$

"Chernoff bounds/estimates"

Tony
21 6 2013

Lemma 7.2

"For $t \in]0, \frac{1}{4}[$, $p \in [1, \infty[$, $n \geq 2^{13}$ s.t. $\tau_t \geq \sqrt{\frac{p \ln(18n)}{18n}}$ (condition [216]).

Let $G_t^n := (\mathbb{F}_2^n, E_t^n)$ be a graph s.t. $\forall x,y \in \mathbb{F}_2^n$ $\# E_{(x,y)} = e_t^n(\|x-y\|_1)$.

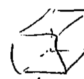
Then G_t^n is a d_t^n-regular graph, $\frac{1}{3\sigma_t^n} \leq d_t^n \leq \frac{1}{\sigma_t^n}$ (condition [217]). Moreover, for every metric space (X, d_X) and every $f, g : \mathbb{F}_2^n \longrightarrow X$ we have

$$\frac{1}{3|E_t^n|} \sum_{(x,y) \in E_t^n} d_X(f(x), g(y))^p \leq \frac{1}{2^n} \sum_{x,y \in \mathbb{F}_2^n} (e^{-t\Delta}\delta_x)(y)\, d_X(f(x), g(y))^p \leq \frac{3}{|E_t^n|} \sum \ldots \text{"} \quad [218]$$

Recall that $\tau_t = \frac{1-e^{-t}}{2}$ and $(e^{-t\Delta}\delta_x)(y) = \tau_t^{\|x-y\|_1}(1-\tau_t)^{n-\|x-y\|_1}$ [96]

$\sigma_t^n = \tau_t^{4\tau_t n}(1-\tau_t)^{n-4\tau_t n}$ and $e_t^n(k) = \frac{\tau_t^k (1-\tau_t)^{n-k}}{\sigma_t^n}$ [207]

Proof: Let $\tau = \tau_t$, $\sigma = \sigma_t^n$, $d = d_t^n$.

The space is very symmetric, like a cube  : $\forall x,y\ \forall k\ \# S_{\|\cdot\|_1}(x,k) = \# S_{\|\cdot\|_1}(y,k) = \binom{n}{k}$

Thus G_t^n is regular and $d = \sum_{k=0}^{n} \binom{n}{k} e_t^n(k) \leq$ "$\frac{e_t^n(k)}{e^{-t\Delta}\delta_x(y)}$" $= \frac{1}{\sigma}$

and $\sum_{k=0}^{n} \binom{n}{k} e_t^n(k) \geq \sum_{k \in [0, 4\tau n]} \binom{n}{k} e_t^n(k)$.

N.B.: $f(k) = \tau^k(1-\tau)^{n-k}$ decreases iff $\tau < \frac{1}{2}$ (which is the case!)

We have $\sigma = f(4\tau n)$, so that $f(k) \geq \sigma$ if $k \leq 4\tau n$

$$\geq \frac{1}{2} \sum_{k \in [0, 4\tau n]} \binom{n}{k} \frac{\tau^k(1-\tau)^{n-k}}{\sigma}$$

$$= \frac{1}{2\sigma}\left(1 - \sum_{k > 4\tau n} \binom{n}{k} \tau^k (1-\tau)^{n-k}\right).$$

N.B.: Chernoff $\leq$: $\sum_{k \geq a} \binom{n}{k} \tau^k (1-\tau)^{n-k} < e^{-2(a-\tau n)^2/n}$

$$\geq \frac{1}{2\sigma}\left(1 - e^{-18\tau^2 n}\right) \underset{[216]}{\geq} \frac{1}{2\sigma}\left(1 - \frac{1}{3}\right) = \frac{1}{3\sigma}$$

this is [217] ∎

Now the "moreover" part: it is also a doubling $\leq$

$$\frac{1}{2^n}\sum_{x,y\in\mathbb{F}_2^n}\left(e^{-t\Delta}\delta_x\right)(y)\,d_X(f(x),g(y))^p \overset{[96]}{=} \frac{1}{2^n}\sum_{x,y\in\mathbb{F}_2^n} \tau^{\|x-y\|_1}(1-\tau)^{n-\|x-y\|_1} d_X(f(x),g(y))^p \quad (2)$$

$$\geq \frac{\sigma}{2^n}\frac{d}{2}\sum_{x,y\in\mathbb{F}_2^n} e_t^n(\|x-y\|_1)\,d_X(\ ,\ )^p \geq \frac{1}{3|E_t^n|}\sum_{(x,y)\in E_t^n} d_X(\ ,\ )^p \qquad [f\cosh\leq \text{in } [218]]$$

The other $\leq$: $\frac{1}{2^n}\sum_{x,y\in\mathbb{F}_2^n}(e^{-t\Delta}\delta_x)(y)\, d_X(f(x),g(y))^p = \frac{1}{2^n}\sum_{s=0}^{n}\tau^s(1-\tau)^{n-s}\sum_{\|x-y\|_1=s} d_X(f(x),g(y))^p$

Let us split $\sum_{s=0}^{n}$ in $\sum_{s\leq 4\tau n}+\sum_{s>4\tau n}$:

$$\leq \frac{1}{2^n}\Bigg[\underbrace{\sum_{s\leq 4\tau n} 2\sigma\, e_t^n(s)\sum_{\|x-y\|_1=s} d_X(\ ,\ )^p}_{\leq 2\sigma\sum_{(x,y)\in E_t^n} d_X(f(x),g(y))^p} + \sum_{s>4\tau n}\tau^s(1-\tau)^{n-s}\sum_{\|x-y\|_1=s} d_X(\ ,\ )^p\Bigg]$$

<u>Claim</u>: $\forall s\in\,]4\tau n, n]\ \sum_{\|x-y\|_1=s} d_X(f(x),g(y))^p \leq \binom{n}{s}18\sigma n^p\sum_{(x,y)\in E_t^n} d_X(f(x),g(y))^p.$

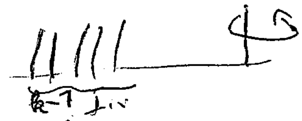
$$\leq \frac{\sigma}{2^n}\sum_{(x,y)\in E_t^n} d_X(f(x),g(y))^p\left(2+\sum_{s>4\tau n}18n^p\binom{n}{s}\tau^s(1-\tau)^{n-s}\right)$$

$$\leq \frac{\sigma}{2^n}\sum \text{———}^p\left(2+18n^p e^{-18\tau^2 n}\right)$$

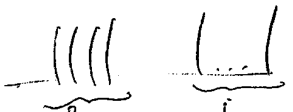
$$\underset{\text{choice of }\tau}{\leq} \frac{3}{|E_t^n|}\sum_{(x,y)\in E_t^n} d_X(f(x),g(y))^p.$$

<u>Proof of the claim</u>: For $k\in[\![0, 4\tau n]\!]$, $m\geq 0$ s.t. $k+2m\leq n$.

For each $\pi\in S_n$ and $i\in\{1,\dots,2m+1\}$ we put $z_i^\pi=\sum_{j=1}^{k-1} e_{\pi(j)}+e_{\pi(k+i-i)}\in\mathbb{F}_2^n$



$y_i^\pi=\sum_{j=1}^{i} z_j^\pi\in\mathbb{F}_2^n$



$y_0^\pi=z_0^\pi=0$.

For every $x\in\mathbb{F}_2^n$ and $\pi\in S_n$, $d_X(f(x),g(x+y_{2m+1}^\pi))\leq \sum_{i=0}^{m} d_X(f(x+y_{2i}^\pi),g(x+y_{2i+1}^\pi)) + \sum_{i=0}^{m-1} d(g(x+y_{2i+1}^\pi), f(x+y_{2i+2}^\pi))$

By Hölder's $\leq$, $d_X(f(x),g(x+y_{2m+1}^\pi))^p\leq(2m+1)^{p-1}\left(\sum_i d_X(\ \ )^p+\sum_i d_X(\ \ )^p\right)$ [221]

We have $\|y_{2m+1}^\pi\|_1=k+2m$, $y_{2m+1}: S_n\longrightarrow S_{\|\cdot\|_1}(k+2m)$, $\pi\longmapsto y_{2m+1}^\pi$

and $\#\{\pi: y_{2m+1}^\pi=w\}=\frac{n!}{\binom{n}{k+2m}}$ for all $w\in S_{\|\cdot\|_1}(k+2m)$.

$$\frac{1}{2^n\binom{n}{k+2m}}\sum_{\|x-y\|=k+2m} d_X(f(x),g(y))^p=\frac{1}{2^n n!}\sum_{x\in\mathbb{F}_2^n}\sum_{\pi\in S_n} d_X(f(x),g(x+y_{2m+1}^\pi))^p \qquad [222].$$

This is just a change of variables!

Also, for all $j=0,\dots,2m$, $\displaystyle\sum_{x\in\mathbb{F}_2^n}\sum_{\pi\in S_n} d_X\left(f(x+y_j^\pi),g(x+y_{j+1}^\pi)\right)^p = \sum_{\pi\in S_n}\sum_{x\in\mathbb{F}_2^n}\left[d_X\left(f(x+y_j^\pi),g(x+y_j^\pi+z_{j+1}^\pi)\right)\right]^p$ (3)

N.B.: $x\in\mathbb{F}_2^n \longmapsto x+y_j^\pi \in\mathbb{F}_2^n$ is a bijection

$$\dots=\sum_{\pi\in S_n}\sum_{u\in\mathbb{F}_2^n} d_X\left(f(u),g(u+z_j^\pi)\right)^p = \frac{n!}{\binom{n}{k}}\sum_{\|u-v\|=k} d_X(f(u),g(v))^p \quad [223]$$

here we used $z_j: S_n \longrightarrow S_{\|\cdot\|_1}(k)$, $\pi\longmapsto z_j^\pi$ and $\#\{\pi: z_j^\pi=w\}=\frac{n!}{\binom{n}{k}}$ for all w.

Now let us use [221], [222], [223]: $\displaystyle\frac{1}{2^n\binom{n}{k+2m}}\sum_{\|u-y\|=k+2m} d_X(f(u),g(y))^p \le \frac{(2m+1)^p}{2^n \cancel{n!}}\frac{\cancel{n!}}{\binom{n}{k}}\sum_{\|u-y\|_1=k} d(f(u),g(y))^p$ [224]

This argument of averaging and getting still, better appears also in Enflo

Let us make again a change of variable:

Let's fix $s\in\,]4\tau n,n]$ and $m\in[(s-4\tau n)/2,s/2]$ and if we put $k=s-2m$,

we will get $0\le k\le 4\tau n$ and $k+2m\le n$.

[224] yields $\displaystyle\frac{\binom{n}{s-2m}}{n^p\binom{n}{s}}\sum_{\|u-y\|=s} d_X(f(u),g(y))^p\le\sum_{\|u-y\|_1=s-2m} d_X(\quad)^p$ multiply by $e_\varepsilon^n(s-2m)$ and sum $\displaystyle\sum_{m\in I}$:

We get $\displaystyle\frac{\sum_{m\in I}\binom{n}{s-2m}e_\varepsilon^n(s-2m)}{n^p\binom{n}{s}}\sum_{\|u-y\|_1=s} d_X(\quad)^p\le\sum_{(u,y)\in E_\varepsilon^n} d_X(f(x),g(y))^p$

We need concentration $\le$ ies for binomial sums:

Claim II: $\displaystyle\sum_{m\in I}\binom{n}{s-2m}e_\varepsilon^n(s-2m)\ge\frac{1}{180}$ for all $s\in\,]4\tau n,n]$

then $\displaystyle\sum_{\|u-y\|_1=s} d_X(\ ,\ )^p\le 180\, n^p\binom{n}{s}\sum_{(u,y)\in E_\varepsilon^n} d_X(\ ,\ )^p$ by the claim

Proof of claim II: like Chernoff's $\le$: "std binomial concentration bound"

Let the $(Y_i)_{i=1}^n$ be independent r.v. s.t. $\mathbb{P}[Y_i=1]=\tau$, $\mathbb{P}[Y_i=0]=1-\tau$ and let $Y=Y_1+\dots+Y_n$.

Then $\mathbb{E}[Y]=\tau n$ and $\mathbb{P}\left[|Y-\mathbb{E}(Y)|\ge\varepsilon\mathbb{E}Y\right]\le 2e^{-c_\varepsilon\mathbb{E}Y}$ where $c_\varepsilon=\min\left(-\ln\left(e^\varepsilon(1+\varepsilon)^{-(1+\varepsilon)}\right),\frac{\varepsilon^2}{2}\right)$

Let $\varepsilon=\frac{1}{2}$: $c_\varepsilon=\min\left\{\frac{3}{2}\ln\left(\frac{3}{2}\right)-\frac{1}{2},\frac{1}{8}\right\}$; and $\ln\frac{3}{2}\ge 0{,}41$, so that $c_{\frac{1}{2}}\ge\frac{1}{10}$.

then $\displaystyle\dots=\sum_{k\notin[(1-\varepsilon)\mathbb{E}Y,(1+\varepsilon)\mathbb{E}Y]}\binom{n}{k}\tau^k(1-\tau)^{n-k}$

All together, we get $\displaystyle\underbrace{\sum_{k\in\left[\frac{\tau n}{2},\frac{3\tau n}{2}\right]}\binom{n}{k}\tau^k(1-\tau)^{n-k}}_{a_k}\ge\underbrace{1-2e^{-\frac{\tau n}{10}}}_{\ge 0{,}89\text{ as }\tau\ge\frac{1}{3\sqrt{n}}\text{ by [216] and }n\ge 8000}\ge\frac{8}{9}$

Let us use the hypothesis that $f \in]0, \frac{1}{4}[$: then $\tau \in]0, \frac{1}{8}[$.

For each $k \in [\frac{\tau n}{2}, \frac{3\tau n}{2}]$, $\frac{a_{k+1}}{a_k} = \frac{\tau}{1-\tau} \frac{n-k}{k+1} \in \left[\frac{1-\frac{3\tau}{2}}{3(1-\tau)}, \frac{2-\tau}{1-\tau}\right] \subseteq [\frac{1}{4}, 4]$

Then $$\sum'_{\substack{k\in[\frac{\tau n}{2}, \frac{3\tau n}{2}] \\ k \text{ even}}} a_k \geqslant \frac{1}{8} \sum'_{k\in[\frac{\tau n}{2}, \frac{3\tau n}{2}]} a_k$$

$$\sum'_{\substack{k\in[\frac{\tau n}{2}, \frac{3\tau n}{2}] \\ k \text{ odd}}} a_k \geqslant \frac{1}{8} \sum'_{k\in[\frac{\tau n}{2}, \frac{3\tau n}{2}]} a_k .$$

Here many situations can happen! $- \mid \mid \mid \mid \mid \underset{\text{last even}}{+}$: $2\sum_{\text{even}} \geqslant (\frac{1}{4}\sum_{\text{odd}}) + \sum_{\text{even}}$

would give $\sum_{\text{even}} \geqslant \frac{1}{5}\sum_{\text{all}} \ldots$

So putting $k = s - 2m$

$$\sum'_{m\in\left[\frac{s-3\tau n}{2}, \frac{s-\frac{\tau n}{2}}{2}\right]} \binom{n}{s-2m} \tau^{s-2m}(1-\tau)^{n-(s-2m)} \geqslant \frac{1}{9}$$

$$\leqslant 2\sigma \sum'_{m\in\left[\frac{(s-4\tau n)}{2}, \frac{s}{2}\right]} \binom{n}{s-2m} e^n_\epsilon(s-2m) .$$ This proves Claim II.

Luis

Sup-partitions of unity and smooth approximation

Problem: Let X be a B-space. Do we have that for every $\varepsilon>0$ and $f: X \xrightarrow{Lip} \mathbb{R}$ there is $g: X \underset{Lip}{\xrightarrow{C^k}} \mathbb{R}$ with $|g(x)-f(x)|<\varepsilon$ and $\operatorname{Lip} g \leq \underset{C_0(X)}{\underset{\shortparallel}{C_0}} \operatorname{Lip} f$?

① Smooth approximation.

Problem: Given $\varepsilon>0$ and $f: X \xrightarrow{cont} \mathbb{R}$, is there $g: X \xrightarrow{C^k} \mathbb{R}$ with $|g(x)-f(x)|<\varepsilon$

Main tool: partitions of unity.

Def: X has C^k-partitions of unity if, for every open cover $\mathcal{U}$ of X, there is a family of C^k-smooth functions $\{\varphi_\alpha : X \to [0,1]\}_{\alpha\in\Gamma}$ s.t.

① $\forall \alpha \in \Gamma\ \exists \mathcal{U}_\alpha \in \mathcal{U}$. $\operatorname{supp}\varphi_\alpha = \overline{\{x\in X : \varphi_\alpha(x)\neq 0\}} \subset \mathcal{U}_\alpha$

② $\{\operatorname{supp}\varphi_\alpha\}_{\alpha\in\Gamma}$ is locally finite: $\forall x\in X\ \exists x\in V\subset X$ open $\#\{\alpha\in\Gamma : \operatorname{supp}\varphi_\alpha \cap V \neq \emptyset\} < \infty$

③ $\forall x \in X\ \sum_{\alpha\in\Gamma}' \varphi_\alpha(x) = 1$.

Toruńczyk (1973): Theorem: If X is a B-space, ① X admits C^k-smooth partitions of unity iff ② $\forall \varepsilon>0\ \forall f: X\xrightarrow{cont}\mathbb{R}\ \exists g: X \xrightarrow{C^k\text{-smooth}} \mathbb{R}$ $|g(x)-f(x)|<\varepsilon$
iff ③ $\exists \Gamma\neq\emptyset\ \exists \varphi: X \underset{\text{homeomorphic}}{\hookrightarrow} c_0(\Gamma)\ \forall \gamma\in\Gamma\ e_\gamma^*\circ\varphi : X \xrightarrow{C^k} \mathbb{R}$

Examples:
1. X separable with C^k-norm: it even has C^k-bumps: Bonic & Frampton 1966.
2. X reflexive with C^k-norm: it even has C^k-bumps: Toruńczyk 1973
3. X weakly compactly generated with C^k-norm: C^k-bump.
($X=\overline{\operatorname{span}}\,K$, K weakly compact)

Proof of Toruńczyk: ① ⇒ ②: $f: X\to\mathbb{R}$, $\varepsilon>0$: $\forall x\in X\ \exists r_x>0\ \forall y\in B(x,r_x)$ $|f(x)-f(y)|<\varepsilon$

Let $\mathcal{U}=\{B(x,r_x) : x\in X\}$. It is an open cover and ① yields a C^k-partition of unity $\{\varphi_\alpha\}$ subordinated to $\mathcal{U}$. Let, for each $\alpha\in\Gamma$, $x_\alpha\in X$ with $\operatorname{supp}\varphi_\alpha\subset B(x_\alpha, r_{x_\alpha})$.
Then $g(y) := \sum_{\alpha\in\Gamma}' \varphi_\alpha(y) f(x_\alpha)$ is C^k-smooth and $|g(y)-f(y)|<\varepsilon$ on X.

One should not try the following approach: $f : X \xrightarrow[\varepsilon>0]{\mathrm{Lip}} \mathbb{R}$, $\varepsilon>0$, $\{\varphi_\alpha\}_{\alpha\in\Gamma}$ a C^k-partition of unity: even if $\mathrm{Lip}\,\varphi_\alpha \le L$, $g(y) = \sum_{\alpha\in\Gamma} \varphi_\alpha(y) f(x_\alpha)$ has no controlled Lip constant because the finite # of α with $\varphi_\alpha(y)\neq 0$ is not uniformly bounded. It is however locally Lip.

② Sup-partitions of unity

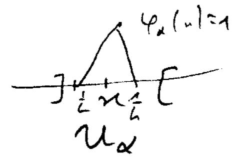
Let us return to the first problem? We might even look for a universal constant C_0, or prove that $C_0 \le 1+\varepsilon$.

Def [Fry 2003] X admits C^k-sup-partitions of unity subordinated to an open cover $\mathcal{U}$ if there is a family of functions $\{\varphi_\alpha : X \xrightarrow[\text{smooth}]{C^k} [0,1]\}_{\alpha\in\Gamma}$ with $\mathrm{Lip}\,\varphi_\alpha \le L$, ① $\forall \alpha\in\Gamma\ \exists\, \mathcal{U}_\alpha \in \mathcal{U}$ $\operatorname{supp}\varphi_\alpha \subseteq \mathcal{U}_\alpha$

② $\{\operatorname{supp}\varphi_\alpha\}_{\alpha\in\Gamma}$ locally finite

③ $\forall x\in X\ \sup_{\alpha\in\Gamma} \{\varphi_\alpha(x)\} = 1$, i.e. $\forall x\in X\ \exists \alpha\in\Gamma\ \varphi_\alpha(x)=1$.

Remark: the Lip constant L depends on the cover $\mathcal{U}$; furthermore $\mathcal{U}$ must be "uniform": $\forall x\in X\ \exists\alpha\in\Gamma\ \varphi_\alpha(x)=1$, so that $\exists\, \mathcal{U}_\alpha\in\mathcal{U}$ $\operatorname{supp}\varphi_\alpha \subset \mathcal{U}_\alpha$.
As $\mathrm{Lip}\,\varphi_\alpha \le L$, $B(x, \frac{1}{L}) \subset \operatorname{supp}\varphi_\alpha \subset \mathcal{U}_\alpha$.



Theorem [Hájek–Johanis 2010] Let X be a B-space. Then

① X admits C^k-sup-partitions of 1 subordinated to $\{B(x,1) : x\in X\}$

$\Updownarrow$

② $\exists\,\Gamma\neq\emptyset\ \exists\ \varphi : X \hookrightarrow c_0(\Gamma)$ bilipschitz embedding $\forall \gamma$ $e_\gamma^* \circ \varphi : X \xrightarrow[\text{smooth}]{C^k} \mathbb{R}$

$\Downarrow$

③ $\forall \varepsilon>0\ \forall f : X \xrightarrow{\mathrm{Lip}} \mathbb{R}\ \exists g : X \xrightarrow[C^k\text{-smooth}]{\mathrm{Lip}} \mathbb{R}$ $|g(x)-f(x)|<\varepsilon$ on X with $\mathrm{Lip}\, g \le \operatorname{dist}\varphi \cdot \mathrm{Lip} f$.

Furthermore ③ $\Rightarrow$ ① for X separable, no counterexample is known.

Proof: $1\Rightarrow 2$: We shall prove in fact: If $\{\varphi_\alpha\}$ is a sup-partition of unity with $\mathrm{Lip}\,\varphi_\alpha \le L$ for all $\alpha\in\Gamma$, $L\ge 1$ and even $L>1$, then $\forall\varepsilon>0\ \exists\varphi : X \hookrightarrow c_0^+(\Gamma)$ $\operatorname{dist}(\varphi) \le (2+\varepsilon)L$.

The idea comes probably from Assouad 1978: X sep $\xhookrightarrow{\varphi} c_0^+$ with $\operatorname{dist}\varphi < 3+\varepsilon$.

③

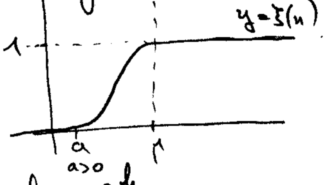
Let us take $\eta>0$ with $L > 1+\eta$ and $\frac{2(1+\eta)^2}{1-\eta} \leq 2+\frac{\varepsilon}{2}$ and $t=1+\eta$.

Exercise: There is a $M: \mathbb{R}^2 \xrightarrow[Lip]{C^\infty\text{-smooth}} \mathbb{R}$ with ① $|M(x,y)-\min\{x,y\}|<\eta$ for $x,y\in\mathbb{R}$

and ② $Lip_{\|\cdot\|_\infty} M = \sup\left\{ \frac{|M(x,y)-M(x',y')|}{\|(x,y)-(x',y')\|_\infty} : (x,y)\neq(x',y')\right\} \leq 1+\eta$.

and ③ If $\min\{x,y\}=0$, then $M(x,y)=0$.

Then, let $\xi: \mathbb{R} \longrightarrow [0,1]$ s.t.

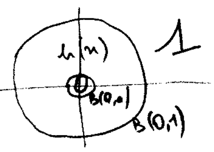


and $\xi\in C^\infty$, $Lip\,\xi \leq 1+\eta$

Let $h(x) = \xi(\|x\|)$: • If $\|\cdot\|$ is C^k, then h is C^k

• If $\|\cdot\|$ is not C^k, use starlike bodies constructed from the support of a bump function.

By hypothesis, there is $\{\varphi_\alpha\}$ C^k- sup-partition of unity subordinated to $\{B(x)\}_{x\in X}$. Let $\Phi: X \longrightarrow c_0^+(\mathbb{Z}\times\Gamma)$



$$x \longmapsto (\varphi_\alpha^n(x)) = \left(t^n M\left(h\left(\tfrac{x}{t^n}\right), \varphi_\alpha\left(\tfrac{x}{t^n}\right)\right)\right)$$

ⓘ Prove that $\Phi(x)\in c_0(\mathbb{Z}\times\Gamma)$: $\forall x\in X\ \forall\delta>0\ \exists n_0,n_1\in\mathbb{Z}$
$\begin{cases} ① t^n < \frac{\delta}{1+\eta} \text{ for } n<n_0 \\ ② \frac{\|x\|}{t^n} < a \text{ for } n>n_1 \\ ③ \forall\, n_0\leq n\leq n_1\ \exists F\subset\Gamma \text{ finite } \forall\alpha\in\Gamma \\ \varphi_\alpha\left(\frac{x}{t^n}\right)\neq 0 \end{cases}$

Then, for $n<n_0$, $\varphi_\alpha^n(x) < \frac{\delta}{1+\eta}(1+\eta) = \delta$ because $M\leq(\eta+\min)$

for $n>n_1$, $h\left(\frac{x}{t^n}\right)=0$, so that $\varphi_\alpha^n(x)=0$.

ⓘⓘ Prove that Φ is Lipschitz: $|\varphi_\alpha^n(x)-\varphi_\alpha^n(y)| \leq t^n\left|M\left(h\left(\tfrac{x}{t^n}\right),\varphi_\alpha\left(\tfrac{x}{t^n}\right)\right) - M\left(h\left(\tfrac{y}{t^n}\right),\varphi_\alpha\left(\tfrac{y}{t^n}\right)\right)\right|$

$$\leq t^n(1+\eta)\left\|\left(h\left(\tfrac{x}{t^n}\right),\varphi_\alpha\left(\tfrac{x}{t^n}\right)\right) - \left(h\left(\tfrac{y}{t^n}\right),\varphi_\alpha\left(\tfrac{y}{t^n}\right)\right)\right\|_\infty$$

$$\leq t^n(1+\eta)\max\{\underbrace{Lip\,h}_{1+\eta}, \underbrace{Lip\,\varphi_\alpha}_{L\geq 1+\eta}\}\cdot\frac{\|x-y\|}{t^n}$$

$$\leq (1+\eta)L\|x-y\|.$$

ⓘⓘⓘ On the other hand, $x\neq y \Rightarrow \exists n\in\mathbb{Z}$ $2t^n \leq \|x-y\| \leq 2t^{n+1}$

and we can assume that $\|x\|\geq t^n$: because otherwise $\|y\|\geq\|x-y\|-\|x\|\geq 2t^n-t^n\geq t^n$.

But if $\|x\|\geq t^n$, $h(x)=\xi\left(\frac{\|x\|}{t^n}\right)=1$. Since $\{\varphi_\alpha\}$ is a sup-partition,

there is $\alpha\in\Gamma$ such that $\varphi_\alpha\left(\frac{x}{t^n}\right)=1$.

$\varphi_\alpha^n(x) = t^n M(1,1) \geq t^n(\min\{1,1\}-\eta) = t^n(1-\eta)$.

If $\varphi_\alpha\left(\frac{y}{t^n}\right)\neq 0$, we would have $\frac{y}{t^n}\in \operatorname{supp}\varphi_\alpha \subset B(x_\alpha,1)$ and $2\leq\left\|\frac{x}{t^n}-\frac{y}{t^n}\right\|<2$ so that $\varphi_\alpha\left(\frac{y}{t^n}\right)=0$.

Then $\varphi_\alpha^n(y)=0$ and $\|\Phi(x)-\Phi(y)\|_\infty \geq |\varphi_\alpha^n(x)-\varphi_\alpha^n(y)| = \varphi_\alpha^n(x) \geq t^n(1-\eta) \geq \frac{1-\eta}{2t}\|x-y\|$

thus $\operatorname{dist}\Phi \leq (1+\eta)L\frac{2t}{1-\eta} = \frac{2(1+\eta)^2}{1-\eta}L \leq (2+\varepsilon)L$.

N.B.: One could try to adapt Pelant's proof …

Or to embed into c_0 instead of c_0^+ ? Get $1+\varepsilon$?

N.B.: in ℓ_2, $L > \frac{\sqrt{5}}{2}$ as of sup-partitions of unity; in c_0, $L = 1+\varepsilon$ …

<u>Remark</u> Th (Kalton + Lancien, 2007). If $\ell_p \overset{\varphi}{\hookrightarrow} c_0^+$ bilip. embedding, then $\operatorname{dist}\varphi \geq (2^p+1)^{\frac{1}{p}}$.

<u>Corollary</u>: If ℓ_p has C^k-sup-partitions with Lipschitz constant L, $L \geq \frac{(2^p+1)^{\frac{1}{p}}}{2}$.

In particular, $p=2$ gives $L \geq \frac{\sqrt{5}}{2}$.

<u>Proof (2) ⇒ (3)</u>: Theorem [Hájek & Johanis, 2009] $\forall \Gamma \neq \emptyset$ $\forall \varepsilon > 0$ $\forall A \subseteq c_0(\Gamma)$ $\forall f : A \xrightarrow{\text{Lip}} \mathbb{R}$ $\exists g : c_0(\Gamma) \xrightarrow[\text{Lip}]{C^\infty\text{smooth}} \mathbb{R}$ "depending locally on a finite # of coord^s" [LFC]

$|f(x) - g(x)| < \varepsilon$ on A and $\operatorname{Lip} g = \operatorname{Lip} f$.

<u>Def</u>: [Pechanec, Whitfield, Zizler 1981] A f^n $h : c_0(\Gamma) \longrightarrow \mathbb{R}$ depends locally on finitely many coordinates (LFC) if $\forall x \in c_0(\Gamma)$ $\exists \mathcal{U} \subset X$ neighbourhood of x $\exists \gamma_1, \dots, \gamma_n \in \Gamma$ $\exists G : \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\begin{tikzcd} \mathcal{U} \subset c_0(\Gamma) \arrow[r, "h"] \arrow[d, "(e^*_{\gamma_1}, \dots, e^*_{\gamma_n})"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, "H"'] & \end{tikzcd}$$

$h(x) = H(e^*_{\gamma_1}(x), \dots, e^*_{\gamma_n}(x))$

Here:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & \mathbb{R} \\ c_0(\Gamma) \supset \varphi(X) \arrow[u, bend right, "\varphi^{-1}"'] & \end{tikzcd}$$

$F(x) = f \circ \varphi^{-1}(x)$ for $x \in \varphi(X)$ and $\operatorname{Lip} F \leq \operatorname{Lip} f \operatorname{Lip} \varphi^{-1}$

By Hájek-Johanis: $\exists G : c_0(\Gamma) \longrightarrow \mathbb{R}$ C^∞ Lip LFC with $|F(x) - G(x)| < \varepsilon$

for $g : X \longrightarrow \mathbb{R}$, $x \longmapsto G \circ \varphi(x)$.

and $\operatorname{Lip} G = \operatorname{Lip} F$ and $\operatorname{Lip} g \leq \operatorname{dist} \varphi . \operatorname{Lip} f$ and $|g(x) - f(x)| \leq |G(\varphi(x)) - F(\varphi(x))| < \varepsilon$

Then g is C^k-smooth and LFC implies $G(x) = H(e^*_{\gamma_1}(x), \dots, e^*_{\gamma_n}(x))$

so that $g(x) = \underset{C^n}{H}(\underbrace{e^*_{\gamma_1}(\varphi(x))}_{C^n}, \dots, \underbrace{e^*_{\gamma_n}(\varphi(x))}_{C^n})$

<u>Last part</u>: Construction of sup-partitions of unity for a separable B-space.

<u>Th</u> [Fry 2003]. Let X be a ~~separable~~ B-space with C^k-smooth norm (C^k-smooth Lip. bump). Then $\forall \eta > 0$ $\exists$ C^k-sup-partition of unity subordinated to $\{B(x,1)\}_{x \in X}$, $\{\varphi_n\}_{n=1}^{\infty}$ and $\operatorname{Lip} \varphi_n \leq \cancel{2\eta}$ $2+\eta$.

<u>Cor</u>:

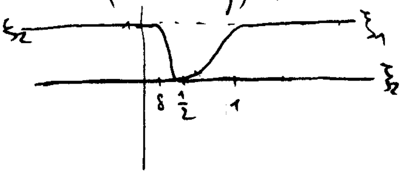
Cor: If X is sep with C^k-norm: $\forall \varepsilon > 0\ \forall f: X \xrightarrow{Lip} \mathbb{R}\ \exists g: X \xrightarrow[C^k\text{-smooth}]{Lip} \mathbb{R}$ $|g(x)-f(x)|<\varepsilon$

$\text{Lip}\, g \leq (4+\varepsilon)\, \text{Lip} f$.

Proof of Th: Let us take $\eta, \delta > 0$ such that $\begin{cases} (1+\eta)^4 \leq 1 + \frac{r}{2} \\ \delta < \frac{1}{2} \\ \frac{1}{1-2\delta} \leq 1+\eta. \end{cases}$

Let us take $\|\cdot\|_1$ in $c_0(\mathbb{N})$ s.t. $\|x\|_\infty \leq \|x\|_1 \leq (1+\eta)\|x\|_\infty$ and $\|x\|_1$ is C^∞-smooth.

Let us define $\xi_1: \mathbb{R} \xrightarrow{C^\infty} [0,1]$, $\xi_2: \mathbb{R} \to [0,1]$ s.t.

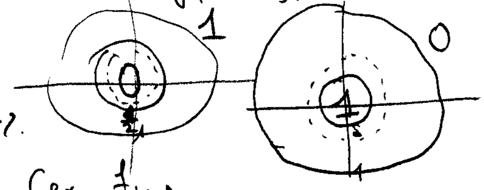


$\text{Lip}\, \xi_1 \leq 2(1+\eta)$

$\text{Lip}\, \xi_2 \leq \frac{2(1+\eta)}{1-2\delta}$

X is separable: let $(x_n)_{n=1}^\infty$ be dense in X; let us define $f_n(x) = \xi_1(\|x-x_n\|)$

$g_n(x) = \xi_2(\|x-x_n\|)$



Let $\varphi_1(x) = \|(f_1(x), 0, \dots)\|_1$

Let $\varphi_n(x) = \|(g_1(x), \dots, g_{n-1}(x), f_n(x), 0, \dots)\|_1$ for $n \geq 2$.

These are C^k-functions and $\text{Lip}\, \varphi_n \leq 2\frac{(1+\eta)^2}{1-2\delta}$ (exercise)

If $x \notin B(x_n, 1)$, $\varphi_n(x) \geq 1$: $\varphi_n(x) \underset{\|\cdot\|_1 \geq \|\cdot\|_\infty}{\geq} \|(g_1, \dots, g_{n-1}, f_n, 0, \dots)\|_\infty \geq |f_n(x)| = 1$.

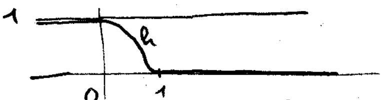
There is $n_0 \in \mathbb{N}$ s.t. $\varphi_{n_0}(x) = 0$. Thus $\overset{\text{there is } n_0 \text{ s.t.}}{x \in B(x_{n_0}, \frac{1}{2})}$ but $x \notin B(x_j, \frac{1}{2})$ for $j < n_0$.
↳ $g_j(x) = 0$.

thus $\varphi_{n_0}(x) = \|(\underset{0}{g_1(x)}, \dots, \underset{0}{g_{n_0-1}(x)}, \underset{0}{f_{n_0}(x)})\|_1 = 0$.

For $x \in X$, there is $n_0 \in \mathbb{N}$ s.t. $\varphi_n(x) \geq 1$ for $n > n_0$. Thus there is $n_0 \in \mathbb{N}$ s.t. $x \in B(x_{n_0}, \delta)$ and for $n > n_0$ $g_{n_0}(x) = 1$ and $\varphi_n(x) \geq \|(g_1(x), \dots, g_{n_0-1}(x), f_n(x), 0, \dots)\|_\infty = 1$

Thus we have sup-partitions with 1 replaced by 0;

Take $h: \mathbb{R} \to [0,1]$

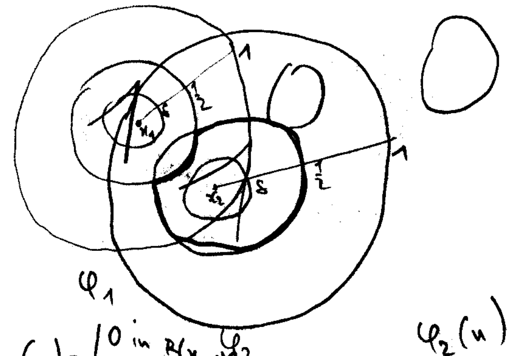


: $h \in C^\infty$, $\text{Lip}\, h \leq 1+\eta$

and $\psi_n(x) := h \circ \varphi_n(x)$ for all n: $\{\psi_n\}$ is a C^k sup-partition of 1

and $\text{Lip}\, \psi_n \leq \text{Lip}\, h \cdot \text{Lip}\, \varphi_n \leq 2\frac{(1+\eta)^3}{1-2\delta} \leq 2(1+\eta)^4 \leq 2+r$

$\uparrow \frac{1}{1-2\delta} \leq 1+\eta$ $\qquad$ $\uparrow (1+\eta)^4 \leq 1+\frac{r}{2}$.



$\varphi_1(x) = \begin{cases} 0 \text{ in } B(x_1,1)^c \\ 1 \text{ in } B(x_1, \frac{1}{2}) \end{cases}$ φ_2

$\varphi_2(x) = \begin{cases} 0 \text{ in } B(x_2,1)^c \cup B(x_1, \delta) \\ 1 \text{ in } B(x_2, \frac{1}{2}) \setminus B(x_1, \frac{1}{2}). \end{cases}$ so that there is a distance $\frac{1}{2} - \delta$ between where $\varphi_2 = 1$ and where $\varphi_2 = 0$.

Open problem: If X is reflexive with C^k-norm, then do we have (i), (ii), (iii)? (6)

N.B.: If X is Hilbert, X satisfies (iii), but we do not know about (i) and (ii).

N.B. We can also get $\mathrm{Lip}(f-g) < \varepsilon$.

Q. In Hájek–Johanis, what do we know about the extension operator? You ~~continue~~ start with a Lipschitz extension of f....