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cf Bader Monod Gelander : pb de pt fixe sur L^1. (Nardzewski)(Ryll)

Strengthened property (T) and explicit construction of superexpanders.

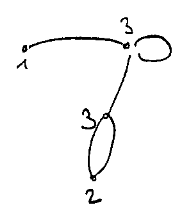
<u>Def</u>: • (X,d_X), (Y,d_Y) espaces métriques. $i : X \to X$ is a <u>coarse</u> embedding if $\exists\, \varphi_-, \varphi_+ : \mathbb{R}_+ \circlearrowleft$ increasing, $\underbrace{\varphi_- \xrightarrow[+\infty]{} +\infty}_{\text{proper}}$, $\varphi_- \le \varphi_+$ s.t. $\varphi_- \circ d_X(x,x') \le d_Y(i(x),i(x'))$
$\le \varphi_+ \circ d_X(x,x')$.

• (X_n, d_n) suite d'espaces métriques <u>coarsely embeds</u> in (Y, d_Y) if
$\forall n\ \exists i_n : X_n \to Y$ coarse st. φ_-, φ_+ ne dépendent pas de n.

<u>But</u>: Build graphs, X_n, find conditions on Y Banach space s.t. $(X_n) \xhookrightarrow{\text{coarse}} Y$

<u>Remark</u>: Conditions on Y are needed: $\forall X$ $X \xhookrightarrow{\text{isometric}} \ell^\infty(X)$.
Hence $\forall X_n$ graphs (finite) $X_n \hookrightarrow Y$ if Y has no cotype.

<u>Expanders</u>: Un graphe (fini) est $G=(V,E)$.



The degree of $v \in V$ = # of edges having v as endpoint.
G is d-regular if $\deg v = d$ for all v.
The adjacency matrix of G is $A_G \in M_V(\mathbb{C})$, $(A_G)_{xy} = \frac{1}{d}$ # edges between x,y.
Then $A_G^* = A_G$ and $A_G \underset{(1,1,\dots,1)}{\mathbb{1}} = \mathbb{1}$. Let $\ell_2^0(V) = \mathbb{1}^\perp \subseteq \ell_2(V) = \{(x_v)_{v\in V} : \sum x_v = 0\}$
$A_G : \ell_2^0(V) \to \ell_2^0(V)$. Define $\lambda_2(G) = \|A_G\|_{B(\ell_2^0(V))} \le 1$ $(=1 \Leftrightarrow G$ bipartite$)$

<u>Def</u>: A family $G_n = (V_n, E_n)$ of d-regular finite connected graph is a family of expanders if $\sup_n \lambda_2(G_n) < 1$ and $|V_n| \to \infty$.
[If $\{x,x\} \in E$ for all x, then $A_G = \frac{1}{d} I_d + \frac{d-1}{d} A_{G'}$; the smallest eigenvalue of A_G is $\ge \frac{1}{d} - \frac{d-1}{d}$].

<u>Prop</u>: [Gromov] If (G_n) is an expander, then $G_n \not\hookrightarrow H$ Hilbert space.

<u>Proof</u>: By contradiction: if we had $i_n : V_n \to H$, φ_-, φ_+; we can assume $\varphi_+(1)=1$; thus the i_n are 1-Lipschitz.

$$\|A_{G_n}|_{\ell_2^0} \otimes id_H\|_{B(\ell_2^0 \otimes_2 H)} = \lambda_2(G_n) \leq 1-\varepsilon.$$

Wlog we can assume $\forall n$ $\sum_{v \in V_n} i_n(v) = 0$ if $\xi_n = (i_n(v))_{v\in V_n} \in \ell_2^0 \otimes H$

$$\|\xi_n\| \leq \|A_{G_n} \otimes id_H \xi_n\| + \|(A_{G_n} - Id)\otimes id_H \xi_n\|$$
$$\leq (1-\varepsilon)\|\xi_n\| + \Big\|\underbrace{\Big(\sum_{\substack{y\in V_n\\ y\sim x}} \frac{\xi_n(y)}{d} - \xi_n(x)\Big)}_{\sum_{\substack{y\in V_n\\ y\sim x}} \frac{(\xi_n(y)-\xi_n(x))}{d}}\Big\| \leq (1-\varepsilon)\|\xi_n\| + \sqrt{|V_n|}.$$

average of elements which have norm ≤ 1 because of 1-Lipschitzness of i_n.

Thus $\|\xi_n\| \leq \frac{1}{\varepsilon}\sqrt{|V_n|} \Leftrightarrow \frac{1}{|V_n|}\sum_{v\in V_n} \|i_n(v)\|^2 \leq \frac{1}{\varepsilon^2}$.

Take $R>0$. Let $\mathcal{E}_R = \{v \in V_n : \|i_n(v)\| \leq R\}$.

$$\frac{|\mathcal{E}_R^c|}{|V_n|} R^2 \leq \frac{1}{|V_n|}\sum_{v\in V_n}\|i_n(v)\|^2 \leq \frac{1}{\varepsilon^2}. \text{ Thus } 1 - \frac{|\mathcal{E}_R|}{|V_n|} \leq \frac{1}{\varepsilon^2 R^2}.$$

If $R = \frac{\sqrt{2}}{\varepsilon}$, we have $|\mathcal{E}_R| \geq \frac{1}{2}|V_n|$. But let us give a lower bound:

Pick $v_0 \in \mathcal{E}_R$: $\|i_n(v) - i_n(v_0)\| \leq 2R$ for all $v \in \mathcal{E}_R$.

Therefore $d(v, v_0) \leq C$ where C is such that $\varphi_-(C) > 2R$.

$$\mathcal{E}_R \subseteq \{v \in V_n : d(v,v_0) \leq C\} \leq \underbrace{1 + d^2 + \dots + d^C}$$

by induction: v_0 has $\leq d$ neighbours, etc. ..

Remark: the proof is very flexible; more generally, if the G_n are graphs, with $|V_n| \to \infty$ d-regular, there is K st. $A_n \in M_{V_n}(\mathbb{C})$ with $\begin{matrix} A_n = A_n^* \\ A_n \mathbb{1} = \mathbb{1}\end{matrix}$, a Banach space Y st. $\sup_n \|A_n\|_{\ell_2^0(Y)} \leq\, <1$ and the $(A_n)_{x,y} = 0$ for x,y with $d(x,y) > K$.

n.b.: $\ell_2^0(Y) = \{(y_v)_{v\in V_n} : \Sigma y_v = 0\}$

Th (Pisier) If Y is θ-Hilbertian, $\theta < 1$ [i.e., $Y = (Y_0, Y_1)_\theta$ with $\begin{matrix} Y_0 \text{ Hilbert space} \\ Y_1 \text{ compatible B-sp.}\end{matrix}$]

then for any sequence of expanders G_n, $G_n \underset{\text{coarsely}}{\not\hookrightarrow} Y$.

Proof: i) Observation: If $\sup \lambda_2(G_n) \lneq 1$, let $P_n: \ell_2(V_n) \to \ell_2^0(V_n)$ orthogonal proj.
$\|P_n \otimes id\|_{\ell_2(V_n, Y)} \leq 2$ for all B-spaces Y. Consider $A_{G_n} P_n \otimes id$: it has norm $\begin{matrix} \leq \lambda_2(G_n) \text{ on } Y_0 \\ \leq 2 \text{ on } Y_1 \end{matrix}$

Therefore it has norm $\leq \underbrace{\lambda_2(G_n)^{1-\theta} 2^\theta}_{<1}$ on $Y_\theta = Y$

[In fact $A_{G_n} P_n \otimes id$ also has norm $\leq \begin{cases} \lambda_2(G_n) & \text{on } \ell_2(V_n; Y_0) \\ 2 & \text{on } \ell_2(V_n; Y_1) \end{cases}$.

General case: If $\theta < 1$ fixed, there is $K \in \mathbb{N}$ s.t. $\|A_{G_n}^K\|_{B(\ell_2^0(V_n))}^{1-\theta} 2^\theta < 1$.

$$\| A_{G_n}^K \|_{B(\ell_2^0(V_n))}^{1-\theta} = \|A_{G_n}\|^{K(1-\theta)}_{B(\ell_2^0(V_n))}$$

and we use the first step.

this implies $G_n \not\hookrightarrow Y$ by the Remark.

[In his Memoir AMS, Pisier proves Y is θ-Hilbertian iff $\forall T: \begin{matrix} L^\infty(\Omega,\mu) \to L^\infty & \text{norm } 1 \\ L^2 \to L^2 & \varepsilon \\ L^1 \to L^1 & \text{norm } 1, \end{matrix}$

then $\|T \otimes id_X\|_{L^2(\Omega, X)} \leq C\varepsilon^\theta$.

Goal: to construct an explicit family of expanders st $G_n \overset{\text{coarsely}}{\not\to} Y_\theta$ with $\theta < 1$, where $Y_\theta = (Y_0, Y_1)_\theta$, Y_0 has type p, cotype q, $\frac{1}{p} - \frac{1}{q} < \frac{1}{4}$.

Q: What are these spaces? All such spaces have nontrivial type. Converse?

II Property (T) and construction of expanders (Margulis)

Convention: G l.c.g. A representation of G on X is a homomorphism $\pi: G \to B_{inv}(X)$ such that $\forall x$ $g \mapsto \pi(g).x$ is norm continuous.

Def A unitary rep. of G on a Hilbert space H almost has invariant vectors (a.i.v.) if $\exists (\xi_\alpha)$ net, $\|\xi_\alpha\|_H = 1$, $\forall Q \subset G$ compact $\sup_{g \in Q} \|\pi(g)\xi_\alpha - \xi_\alpha\| \underset{\alpha}{\to} 0$

If G is discrete, $\forall g \in G$ $\|\pi(g)\xi_\alpha - \xi_\alpha\| \to 0$

Def: G has (T) if for all unitary rep^s of G with a.i.v., $H^G \neq \{0\}$, where $H^G = \{\xi \in H : \pi(g)\xi = \xi$ for all $g \in G\}$

Trivial example: G finite or compact.

Difficult example: $SL(3,\mathbb{Z})$; $SL(3,\mathbb{R})$ [Kazhdan]
$SL(3,\mathbb{Q}_p)$

Th [Margulis] If Γ is a countable discrete group with (T), and S is a finite generating subset in Γ (with $1 \in S$, $s^{-1} \in S$ for $s \in S$)

If there is a sequence (X_n) of finite sets with $|X_n| \to \infty$ and an action $\Gamma \curvearrowright V_n$ that is transitive, define $G_n = (V_n, E_n)$ the graph with edges (x, gx) for $g \in S$: G_n is a $d = |S|$-regular graph, connected; G_n is an expander.

N.B. For $x \in V_n$, the neighbours of x are the gx, $g \in S$.

<u>Example</u>: $\Gamma = SL(3,\mathbb{Z})$. Take $V_n = SL(3, \mathbb{Z}/n\mathbb{Z})$. Then $\Gamma \curvearrowright V_n : \gamma \cdot a = \left(\sum_k \gamma_{ik} a_{kj}\right)$

Here $S = \left(\text{diag. matrices w. coeff } \pm 1;\ A_{ij}\ \begin{pmatrix} 1 & \pm 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$

<u>Proof</u>: 1st observation: for each unitary rep. π of G on H, $H^G = \{0\}$,

$\| \underbrace{\frac{1}{|S|} \sum_{s \in S}' \pi(s)}_{A} \|_{B(H)} < 1$. In fact, otherwise $\exists\ \xi_n \in H$, $\|\xi_n\| = 1$, $\langle A\xi_n, \xi_n\rangle \to \pm 1$

since $s \in S$ $\sum_{s \in S}' \langle \pi(s)\xi_n, \xi_n\rangle \longrightarrow \pm |S|$ and one of the terms is $= 1$;

the others are ± 1, so that $\sum_{s \in S}' \langle \pi(s)\xi_n, \xi_n\rangle \to |S|$ and $\langle \pi(s)\xi_n, \xi_n\rangle \to 1$

but $\|\pi(s)\xi_n - \xi_n\| \xrightarrow{n \to \infty} 0$ $\lightning$

2nd observation: $\exists \varepsilon > 0$ $\| \frac{1}{|S|} \sum_{s \in S}' \pi(s) \| < 1 - \varepsilon$ for π unitary rep. on H with $H^\Gamma = \{0\}$.

In fact, otherwise, if π_n is a sequence like above,

$\oplus \pi_n : G \to \mathcal{U}(\oplus H_n)$ s.t. $\| \frac{1}{|S|} \sum_s' \pi_n(s) \| \to 1$, $\oplus \pi_n$ has no fixed vector and $\| \frac{1}{|S|} \sum_{s \in S}' \oplus \pi_n(s) \| = \sup_n \| \frac{1}{|S|} \sum_{s \in S}' \pi_n(s) \| = 1$

3rd observation: consider for each n the action of Γ on $\ell^2(V_n)$: since it is transitive, there is no Γ-fixed vector. Therefore $\underbrace{\| \frac{1}{|S|} \sum_{s \in S} \pi_n(s) \|_{B(\ell^2(V_n))}}_{= A_{G_n}} \leq 1 - \varepsilon$.

$\boxed{\text{II} \quad \text{Banach space case}}$

<u>Theorem</u>: (Vincent Lafforgue) If $G = SL(3, \mathbb{Q}_p)$ [a topological group], there is $f_n \in C_c(G)$, $f_n \geq 0$, such that for all uniformly bounded representations π of G on a Banach space X with nontrivial type, $\pi(f_n) = \int \pi(g) f_n(g)\, dg \in B(X)$ [Haar m. on G] $\xrightarrow{n \to \infty}$ in $\|\ \|_{B(X)}$-norm P a projection on X^G;

Ex: If $X^G = \{0\}$, $\|\pi(f_n)\| \to 0$;

Ex: If $X = \mathbb{C}$ with trivial action, $\int f(g)\, dg \to 1$.

$K = SL(3, \mathbb{Z}_p)$ compact (and open) subgroup of G: If $g_n \in G$, $g_n \to \infty$, $f_n = \mathbb{1}_{K g_n K}$.

If π is a unitary representation of G on H, $H^G = \{0\}$, there is $f_n \in C_c(G)$ with $\|\pi(f_n)\| < \frac{1}{2}$ where $\int f_n = 1$. Thus, for all $\xi \in H$, $\|\xi\| = 1$, $\|\pi(f_n)\xi - \xi\| \geq \frac{1}{2}$

thus $\frac{1}{2} \leq \int f_n(g) \sup_{g \in \operatorname{Supp} f_n} \|\pi(g)\xi - \xi\| \leq \sup_{g \in \operatorname{supp} f_n} \|\pi(g)\xi - \xi\|$ $\quad \int f_n(g) \|\pi(g)\xi - \xi\| \geq \frac{1}{2}$.

Th : If $G = SL(3,\mathbb{R})$, the same holds, but only for Banach space Y_θ, Y_θ has type p, cotype q with $\frac{1}{p} - \frac{1}{q} < \frac{1}{4}$.

Corollary (2): If Γ is a lattice in $SL(3,\mathbb{Q}_p)$, $\Gamma = SL(3,\mathbb{Z})$, the associated expanders do not $\hookrightarrow$ coarsely X for all X as in the theorem.

N.B.: $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ for the absolute value $|\frac{a}{b}| = p^{v_p(b) - v_p(a)}$.

$\mathbb{Q}_p = \{ \sum'_{n \geq N} a_n p^n : N \in \mathbb{Z}, a_n \in \{0, \dots, p-1\}, a_N \neq 0 \}$: in this rep, $|\ \ | = p^{-N}$

Corollary (1): Let Γ be a lattice in $G \in \{SL(3,\mathbb{Q}_p), SL(3,\mathbb{R})\}$; then there are $g_n : \Gamma \to \mathbb{C}$ finitely supported s.t. for all isometric representations π of Γ on X as in Th A or Th B, $\pi(g_n) \xrightarrow[B(X)]{n\to\infty}$ a projection on X^G.

Def: a discrete subgroup Γ of a locally compact group G with Haar measure dg is a lattice if $\exists\, \Omega \subseteq G$ of finite measure s.t. $\Omega\Gamma = \{\omega\gamma : \omega \in \Omega, \gamma \in \Gamma\} = G$.

Ex: $\mathbb{Z} < \mathbb{R}$, $\Omega = [0,1[$; $SL(3,\mathbb{Z}) < SL(3,\mathbb{R})$

Proof of Corollary 1: Since Γ is a lattice in G, there is $f \in L^1(G)^+$ s.t. $\sum'_{\gamma\in\Gamma} f(g\gamma) = 1$ a.s. in g.

In fact, we can assume $\forall g \in G\, \exists!\, (\omega,\gamma)\ \omega\gamma = g$ and take $f = 1_\Omega$.

Define $g_n(\gamma) = \int_{G\times G} f_n(g_1) f(g_2\gamma) f(g_1 g_2)\, dg_1 dg_2$. Then $\sum'_{\gamma\in\Gamma} g_n(\gamma) = \int_{G\times G} f_n(g_1) f(g_1 g_2)\, dg_1 dg_2$ $g_n \geq 0$,

and put $\tilde{g_2} = g_1 g_2$: thus it $= \int f_n(g_1) \int f(\tilde{g_2}) = \int f_n(g_1) = 1$

[Here we normalised the Haar measure on Γ so that $\int f = 1$]

Take π isometric representation of Γ on X. Let $X' = \{ \xi : G \to X \text{ s.t. } \xi(g\gamma) = \pi(\gamma^{-1})\xi(g) \}$

with norm^2 $= \int_G f(g) \|\xi(g)\|^2 dg$. $X' \subseteq L^2(G, f\,dg; X)$

then let $\pi' : G \to \mathrm{Isom}(X')$ s.t. $\pi'(g)\xi(x) = \xi(g^{-1}x)$.

Consider $\alpha : X \to X'$, $x \mapsto \alpha(x) = \sum_{\gamma\in\Gamma} f(g\gamma)\pi(\gamma)x$ and $\beta : X' \to X$, $\beta(\xi) = \int_G f(g)\xi(g)\,dg$

We have $\beta\circ\alpha(x) = \int_G f(g) \sum'_{\gamma\in\Gamma} f(g\gamma)\pi(\gamma)x$ and $\beta\circ\pi'(f_n)\circ\alpha(x) = ?$

$\pi'(f_n)\circ\alpha(x) = \int f_n(g_1)\pi'(g_1)\alpha(x) : g \mapsto \int f_n(g_1)\alpha(x)(g_1^{-1}g) = \int f_n(g_1) \sum_{\gamma\in\Gamma} f(g_1^{-1}g\gamma)\pi(\gamma)x$.

The details went wrong, but the idea was to apply the theorem to π'.

As for Cor. 2, we shall check that $\pi(g_n) \to 0$