① 5/12/13
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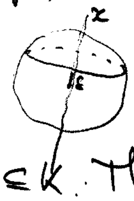
<u>Th</u> [Lafforgue] $SL_3(\mathbb{Q}_p)$ has (T) wrt to the class of spaces with type >1, which is the class of K-convex spaces.

<u>Remark:</u> G cannot have (T) wrt L^1 unless G is compact.

<u>Proof:</u> If you take the trivial rep. π, then $\pi(f_n) = \int_G f_n(g)\pi(g)\,dg = \int f_n(g)\,dg$, so that the net $(f_n) \in C_c(G)$ in the definition of (T) satisfies $\int f_n(g) \to 1$, and $G \curvearrowright L^1(G)$ by left translations, transitively: $L^1(G)^G = \{0\}$, while $\|\lambda(f)\|_{B(L^1(G))} \geq |\int f(g)\,dg|$ (exercise)

<u>ThA</u> [Lafforgue] $SL_3(\mathbb{R})$ has (T) wrt $\mathcal{E}_{mypt}$, where $\mathcal{E}_{mypt} = \{E : \exists C, \exists \varepsilon > 0,\ \|T_0 - T_\delta\|_{L^2(S^2;E)} \leq C|\delta|^\varepsilon\}$

Here, for $\delta \in [-1,1]$, $T_\delta : L^2(S^2) \circlearrowleft$ and if f is complex continuous on S^2, $T_\delta f(x) =$ average of $f(y)$ on $\{y : \langle x, y\rangle = \delta\}$



Second def of T_δ: Let $K = SO_3(\mathbb{R})$, $U = \begin{bmatrix} 1 & 0 \\ 0 & SO_2 \end{bmatrix} \subseteq K$. Then

$U\backslash K \simeq S^2$
$k \mapsto k^{-1}e_1$
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^3$

Consider $\lambda : K \to \mathcal{U}(L^2(K))$

Claim: If $k \in K$, $\iint \lambda(u_1 k u_2)\,du_1\,du_2 \in B(L^2(K))$ (Haar prob. measures on $\mathcal{U}$)

$$= \begin{cases} 0 & \text{on } L^2(U\backslash K)^\perp \\ T_\delta & \text{on } L^2(U\backslash K) \end{cases} \text{ if } k_{11} = \delta.$$

<u>Fact</u> (description of $\mathcal{U}\backslash K/\mathcal{U}$) For $k, k' \in K$, $UkU = Uk'U \Leftrightarrow k_{11} = k'_{11}$.

<u>Proof:</u> $\Rightarrow$ is clear. (i.e. $\exists u_1, u_2 \in U$ $u_1 k u_2 = k'$)

$\Leftarrow$ take $k \in K$ with $k_{11} = \delta$: $k = \begin{pmatrix} \delta & \ell \\ c & A \end{pmatrix}$: $\|c\| = \sqrt{1-\delta^2}$, so that

There is $u_1 \in SO_2$ s.t. $u_1 c = \begin{pmatrix} \sqrt{1-c^2} \\ 0 \end{pmatrix}$. Similarly, $u_2 \in SO_2$ s.t. $\ell u = (-\sqrt{1-\delta^2}\ \ 0)$

Then $\begin{pmatrix} 1 & 0 \\ 0 & u_1 \end{pmatrix} k \begin{pmatrix} 1 & 0 \\ 0 & u_2 \end{pmatrix} = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \begin{bmatrix} \delta & 0 \\ 0 & \end{bmatrix} \\ 0 & \end{pmatrix}$

and you take $x_\delta = \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

you have no choice here
you take δ here.

Remark: the representation may be taken to grow slightly.

We prove $E \in \mathcal{E}_{mypt}$ (strictly convex), $\pi : G \to B(E)$ isometric representation, then

$\pi(KgK') = \int_{K\times K} \pi(kgk')\,dk\,dk' \xrightarrow[g\to\infty]{\|\cdot\|_{B(E)}} P$ projection on E^G

1^{st} step: Cauchy criterion for $g \longmapsto \pi(KgK)$

Fact: $K\backslash G/K \simeq \Lambda = \{(r,s,t) \in \mathbb{R}^3 : r+s+t=0,\ r \geq s \geq t\}$

$K\begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t\end{pmatrix}K \longleftarrow (r,s,t)$ (polar decomposition)

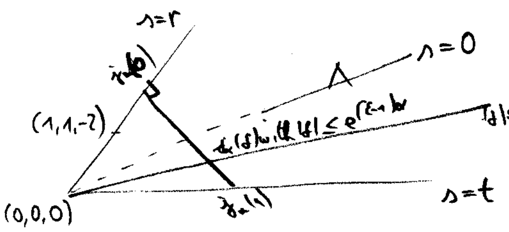
$K\ g\ K \longmapsto (r = \log\|g\|_{B(\mathbb{R}^3)},\ s,\ t = -\log\|g^{-1}\|)$

Denote $f : \Lambda \longrightarrow B(E)$

$\lambda = (r,s,t) \longmapsto f(\lambda) = \pi\left(K\begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t\end{pmatrix}K\right)$

N.B.: We are looking for a net of f's with compact support to prove Th A. Here we get Radon measures KgK. Take $f \in C_c(SL_3(\mathbb{R}))$ with $\int f = 1$, $f \geq 0$. Take $g_n \in G$, $g_n \to \infty$, $f_n \in C_c(G)$.

$f_n(g) = \iint f(k g g_n k')\, dk\, dk'$



We have here $U \subseteq K \subseteq G$.

<u>Lemma</u>: $\|\pi(Ux_0U) - \pi(Ux_\delta U)\|_{B(E)} \leq C|\delta|^s$

$\iint \pi(u_1 x_0 u_2)\, du_1\, du_2$.

N.B.: If you have any unitary rep. for a compact group, it is weakly contained in the left regular representation: every norm $\leq$ valid for the second is valid for the first. Here, same trick for Banach space. "Full absorption principle"

<u>Proof</u>: $i : E \longrightarrow L^2(K;E)$

$i(x)(k) = \pi(k^{-1})x$.

$\lambda(k_1)i(x)(k) = i(x)(k_1^{-1}k) = \pi(k^{-1})[\pi(k_1)x] = i(\pi(k_1)x)(k)$

$\lambda(k_1)i(x) = i(\pi(k_1)x)$

$i\left((\pi(Ux_0U) - \pi(Ux_\delta U))x\right) = [\lambda(Ux_0U) - \lambda(Ux_\delta U)]\, i(x)$

$\|(\pi(Ux_0U) - \pi(Ux_\delta U))x\|_E = \|i(\ \ )\|_{L^2(K;E)} = (T_0 - T_\delta)i(x)$

$\leq \|T_0 - T_\delta\|_{B(L^2(K;E))}$

"Hölder continuity of exponents"

Embed $(K,U) \hookrightarrow (G,K)$: consider, for $\alpha > 0$, $D_\alpha = \begin{pmatrix} e^\alpha & 0 & 0 \\ 0 & e^{-\frac{\alpha}{2}} & 0 \\ 0 & 0 & e^{-\frac{\alpha}{2}}\end{pmatrix} \in G$,

$i_\alpha : K \longrightarrow G$

$k \longmapsto D_\alpha k D_\alpha$

D_α commutes with U, so that

$K\, i_\alpha(u_1 k u_2)K = K\, i_\alpha(k)K$ and

$\underset{U\backslash K/U}{\overset{\shortparallel}{f_\alpha : [-1,1]}} \longrightarrow \underset{\Lambda}{K\backslash G/K}$

i_α factors : gives $U^K/U \to U^G/U$

Lemma: $\|f(j_\alpha(0)) - f(j_\alpha(\delta))\|_{B(E)} \leq C|\delta|^s$

$$f(j_\alpha(0)) = \pi(KD_\alpha x_0 D_\alpha K) = \pi(KD_\alpha)[\pi(Ux_0U)]\pi(D_\alpha K)$$

$$\|\pi(KD_\alpha)\| \leq \int \underbrace{\|\pi(kD_\alpha)\|}_{\leq 1} dk$$

Let us compute: $j_\alpha(1)$: $i_\alpha(n_1) = \begin{pmatrix} e^{2\alpha} & & \\ & e^{-\alpha} & \\ & & e^{-\alpha}\end{pmatrix}$, so that $j_\alpha(1) = (2\alpha, -\alpha, -\alpha)$

$i_\alpha(n_0) = \begin{pmatrix} 0 & e^{-\frac{\alpha}{2}} & 0 \\ e^{\frac{\alpha}{2}} & 0 & 0 \\ 0 & 0 & 1\end{pmatrix}$, so that $j_\alpha(0) = (\frac{\alpha}{2}, \frac{\alpha}{2}, -\alpha)$

Proof: $i_\alpha(n_\delta) = \begin{pmatrix} e^{2\alpha}\delta & -e^{\frac{\alpha}{2}}\sqrt{1-\delta^2} & 0 \\ e^{\frac{\alpha}{2}}\sqrt{1-\delta^2} & \delta e^{-\alpha} & 0 \\ 0 & 0 & e^{-\alpha}\end{pmatrix}$. By definition,

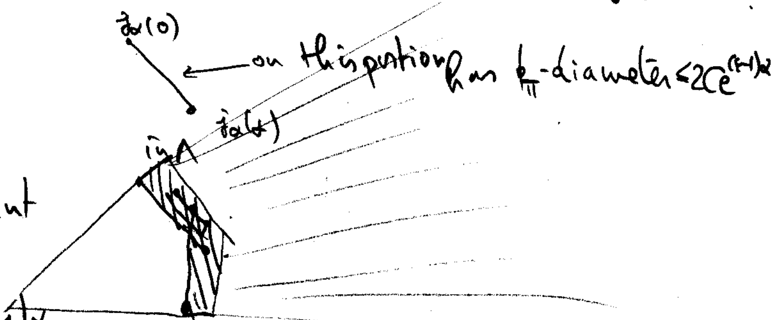
$j_\alpha(\delta) = (r, s, t)$, $r+s+t=0$. $r = \log\|i_\alpha(n_\delta)\|$

$t = -\log\|i_\alpha(n_\delta)^{-1}\|$

$t = -2\alpha$.

If $s = \varepsilon t$, $j_\alpha(\delta) = ((1+\varepsilon)\alpha, -\varepsilon\alpha, -\alpha)$, $e^{2\alpha}|\delta| \leq \|i_\alpha(n_\delta)\| = e^{(1+\varepsilon)\alpha}$.

If $d_\pi(\lambda, \lambda') = \|f(\lambda) - f(\lambda')\|_{B(E)}$



Take its symmetric segment and take the region:
this d_π-diameter $\leq 6Ce^{(\varepsilon-1)\alpha}$

Then the whole region has d_π-diameter $\leq 6C\sum e^{(\varepsilon-1)(1+\varepsilon)^n\alpha}$

Thus the Cauchy criterion is satisfied and $P = \lim \pi(KgK)$ exists

<u>Claim</u>: $\pi(Kg)P = P$ for all $g \in G$.

$$\lim \pi(Kg)\pi(KhK) = \int_K \pi(Kg\,k\,hK)\,dk \xrightarrow[\infty]{\text{unif for } k\in K}$$

Hence $P^2 = P$.

Indeed, "$\lim\limits_{g\to\infty} \pi(KgK)P = \lim\limits_{g\to\infty} P$

$\pi(Kg)P$

If $x \in E^G$, $Px = x$. If $x \in \operatorname{Im} P$, let us show that $x \in E^G$.

By the claim, for all $g \in G$, $\int_K \underbrace{\pi(kg)}_{\text{strongly continuous}} x\,dk = \underbrace{x}_{\text{average}}$. By strict convexity, $\pi(kg)x = x$.

If the space was not strictly convex, it would become much more difficult to conclude.

Study of the mysterious class $\mathcal{E}_{myst}$.

Lemma: (Lafforgue) $\|T_0 - T_\delta\|_{B(L^2(K))} \leq 4\sqrt{\delta}$.

Thus $\mathbb{C} \in \mathcal{E}_{myst}$ and every Hilbert space belongs to it.

[Recall if $E = H$ Hilbert space, $L^2(K;H) \simeq L^2(K) \otimes_2 H$: $\langle f\otimes x, g\otimes y\rangle = \langle f,g\rangle_{L^2}\langle x,y\rangle$
$\|T_1 \otimes T_2\|_{B(H_1 \otimes H_2)} = \|T_1\|\,\|T_2\|$.

Recall $T_\delta = \lambda(U x_\delta U) \in B(L^2(K))$: $\lambda : K \to \mathcal{U}(L^2(K))$
$\lambda(k_1) f(k_2) = f(k_1^{-1} k_2)$.

Prop: (Pisier) $\mathcal{E}_{myst} \supseteq \vartheta$-Hilbertian spaces with $\vartheta \in]0,1[$
$= \{[X_0, X_1]_\vartheta : X_0 \text{ Hilbert space}, X_1 \text{ arbitrary}\}$

Proof: $\|T_\delta\|_{B(L^2(K;E))} \leq 1$ for all E: if $f \in L^2(K;E)$, $T_\delta f(k) = \iint f(u_1 x_\delta^{-1} u_2 k)\,du_1 du_2$
$T_\delta = \iint \underbrace{\lambda(u_1 x_\delta u_2)}_{\text{isometry!}} du_1 du_2$

If $X = [X_0, X_1]_\vartheta$, $L^2(K;X) = [L^2(K;X_0), L^2(K;X_1)]_\vartheta$, so that
$\|T_0 - T_\delta\|_{B(L^2(K;X))} \leq \|T_0 - T_\delta\|_{X_0}^{1-\vartheta} \|T_0 - T_\delta\|_{X_1}^{\vartheta} \leq (4\sqrt{|\delta|})^{1-\vartheta} 2^{\vartheta}$

This implies the results by Baptiste Olivier in his talk!

But this is only interesting for fixed point properties, not for coarse embeddings.

For coarse embeddings, we need other results.

Observation (Lafforgue - de la Salle) $\forall p > 4\ \exists C_p$ $\|T_0 - T_\delta\|_{S^p(L^2(K))} \leq C_p |\delta|^{\frac{1}{2}-\frac{2}{p}}$.

Recall: If H is a Hilbert space, $S^p(H) = \{T \in B(H) : \underbrace{\mathrm{Tr}(T^*T)^{p/2}}_{\|T\|_{S^p}^p} < \infty\}$

If $p > q$, $S^p \supseteq S^q$

Consequence: $\mathcal{E}_{myst} \supseteq \{E : \text{type } p, \text{ cotype } q, \frac{1}{p} - \frac{1}{q} < \frac{1}{4}\}$.

Recall: E has type p if $\exists T_p(E)$ $\|\sum g_i x_i\|_{L^2(\Omega;E)} \leq T_p(E) (\sum \|x_i\|^p)^{\frac{1}{p}}$
g_i i.i.d. $\mathcal{N}(0,1)$

cotype q $\exists C_q(E)$ $\geq \frac{1}{C_q(E)} (\sum \|x_i\|^q)^{\frac{1}{q}}$.

Pisier - Xu prove that there are nonreflexive examples

Theorem (König, Retherford, Tomczak-Jägermann): If $u: \ell^2(E) \circlearrowleft$ has norm 1 and rank $\leq n$, then $\|u \otimes id_E\|_{B(\ell^2(E))} \leq K_{pq} T_p(E) C_q(E)\, n^{\frac{1}{p}-\frac{1}{q}}$.

there is a conjecture: $\frac{1}{p}-\frac{1}{q}$ may be replaced by $\max\left(\frac{1}{p}-\frac{1}{2}, \frac{1}{2}-\frac{1}{q}\right)$.

Corollary: if $u: \ell^2 \to \ell^2$, $\|u\|_{S^r} < \infty$, $\frac{1}{r} > \frac{1}{p}-\frac{1}{q}$, then $\|u \otimes id_E\|_{B(\ell^2(E))} \leq K_{r,p,q} T_p(E) C_q(E) \|u\|_{S^r}$.

Proof: write $u = \sum_k' \alpha_k u_k$, with u_k of norm 1, rank $\leq 2^k$, with $\sum' |\alpha_k|^r 2^k \leq 2\|u\|_{S^r}^r$.

$\to$ look at the singular values of u: $(\underbrace{s_1}_{1}, \underbrace{s_2, s_3}_{2}, \underbrace{s_4 \ldots}_{4}, \ldots) \in \ell_r$

take $\alpha_k = s_{2^k}$: u_k has singular values $\frac{1}{s_{2^k}}(s_{2^k}, \ldots, s_{2^{k+1}-1})$

$\longrightarrow$ comparison of weak ℓ_r and ℓ_r.

Then $\|u\|_{B(\ell^2(E))} \leq \sum_k' \alpha_k \|U_k\|_{B(\ell^2(E))} \overset{KRT}{\leq} C \sum' \alpha_k 2^{k\left(\frac{1}{p}-\frac{1}{q}\right)} \leq C\left(\sum_k' |\alpha_k|^r 2^k\right)^{\frac{1}{r}} \underbrace{\left(\sum_k' 2^{k\left(\frac{1}{p}-\frac{1}{q}-\frac{1}{r}\right)r'}\right)^{\frac{1}{r'}}}_{<\infty}$

$\frac{1}{p}-\frac{1}{q}-\frac{1}{r} < 0$.

We have now a result on ℓ_2: we pass to L^2 with martingale techniques.

$L^2(S^2) \simeq \oplus_n H_n$ with $H_n = \{$restrictions to S^2 of homogeneous polynomials on $\mathbb{R}^3$ of $\partial^\circ n$, harmonic$\}$

$K \curvearrowright L^2(S^2)$. The H_n are irreducible representations and $\dim H_n = 2n+1$.

The T_δ and π_k commute for all $k \in K$ and $\delta \in]-1,1]$: $T_\delta = \oplus_n \underbrace{P_n(\delta)}_{\text{Legendre polynomial}} 1_{H_n}$.

This is exactly how Legendre introduced his polynomials.

Thus $\|T_0 - T_\delta\|_{S^p}^p = \sum_n' (2n+1)|P_n(\delta) - P_n(0)|^p$