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Title:

I) Basics on ~~expanders~~ spectral analysis of graphs and expanders

Let $G=(V,E)$ be a graph

V is finite, E is a "multisubset" of $V\times V$

$\forall (u,v)\in V\times V$ $E(u,v)=$ # times (u,v) appears in E

$$E(u,v)=E(v,u)$$

$[E(u,v)]_{(u,v)\in V\times V}$ integer valued symmetric matrix

$\forall u\in V$ $\deg_G(u)=\sum_v E(u,v)$

<u>G is d-regular</u> if $\forall u$ $\deg_G(u)=d$

Then the <u>normalized adjacency matrix</u> is $A_G=\frac{1}{d}[E(u,v)]_{u,v}$

A_G is a symmetric stochastic matrix.

<u>Prop 1</u> Let $A=[a_{ij}]$ $n\times n$ sym. stoch. matrix

Denote $\lambda_n\le\ldots\le\lambda_1$ its eigenvalues

Then <u>(a)</u> $-1\le\lambda_n\le\ldots\le\lambda_1\le 1$.

<u>(b)</u> $\forall x_1,\ldots,x_n\in\mathbb{R}$

$$\frac{1}{n^2}\sum_{i,j}(x_i-x_j)^2\le\frac{1}{1-\lambda_2}\times\frac{1}{n}\sum_{i,j}a_{ij}(x_i-x_j)^2$$

<u>(c)</u> $\forall x_1,\ldots,x_n,y_1,\ldots,y_n\in\mathbb{R}$

$$\frac{1}{n^2}\sum_{i,j}(x_i-y_j)^2\le\frac{1}{1-\lambda}\frac{1}{n}\sum_{i,j}a_{ij}(x_i-y_j)^2$$

where $\lambda=\max(|\lambda_2|,|\lambda_n|)$.

<u>Pf</u> (a) Let $x,y\in\mathbb{R}^n$ $\|x\|_2,\|y\|_2\le 1$

$$|\langle Ax,y\rangle|=\left|\sum_i\left(\sum_j a_{ij}x_j\right)y_i\right|=\left|\sum_i\sum_j a_{ij}^{1/2}y_i\,a_{ij}^{1/2}x_j\right|$$

$$\le\sum_i\sum_j\frac{1}{2}a_{ij}y_i^2+\frac{1}{2}a_{ij}x_j^2\le\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2\le$$

$+$ $A\mathbb{1}=\mathbb{1}$

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(b) easier than (c)

(c). $\frac{1}{n}\sum_{i,j}(x_i-y_j)^2 = \sum_i x_i^2 + \sum_j y_j^2 - \frac{2}{n}\sum_i x_i(\sum_j y_j)$

$= \|x\|^2 + \|y\|^2 - 2\langle By, x\rangle \qquad B = \begin{bmatrix} \frac1n & \cdots & \frac1n \\ \frac1n & \cdots & \frac1n \end{bmatrix}$

$\sum_{i,j} a_{ij}(x_i-y_j)^2 = \sum_i(\sum_j a_{ij})x_i^2 + \sum_j(\sum_i a_{ij})y_j^2 - 2\sum_i(\sum_j a_{ij}y_j)x_i$

$= \|x\|^2 + \|y\|^2 - 2\langle Ay, x\rangle$

We want to show that

$(1-\lambda)(\|x\|^2+\|y\|^2 - 2\langle By,x\rangle) \le \|x\|^2+\|y\|^2 - 2\langle Ay,x\rangle$

$\Leftrightarrow \quad 2\langle (A-B)y, x\rangle \le \lambda[\|x\|^2 + \|y\|^2 \cancel{-}\, 2\langle By,x\rangle]$

$\exists$ o.n.b. $A \sim \begin{bmatrix} 1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots\ \lambda_n \end{bmatrix} \qquad B \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$x = x_1 + x_2 \quad y = y_1 + y_2$ in the decomposition $\mathbb{R}^n = [\mathbb{1}] \oplus \mathbb{1}^\perp$

$|2\langle (A-B)y, x\rangle| = |2\langle (A-B)y_2, x_2\rangle| \le 2\lambda\|x_2\|\|y_2\|$

$\le \lambda(\|x_2\|^2+\|y_2\|^2) \le \lambda(\|x_2\|^2 + \|y_2\|^2 + x_1^2 + y_1^2 \cancel{-} 2x_1y_1)$

$= \lambda(\|x\|^2+\|y\|^2 \pm 2\langle By, x\rangle)$. $\square$

Rem 1) λ is optimal in (c) (λ_2 optimal in (b)).

2) We can replace $(\ )^2$ by $\|\ \|_H^2$ in (a) and (b).

Proposition 2 Let G d-regular
(a) $\lambda_2 < 1 \iff G$ connected
(b) $\lambda_n = -1 \iff G$ has a bipartite connected component.

Pf (a). Assume G is disconnected and U is a connected component. Then $A\mathbb{1}_U = \mathbb{1}_U$ $\left[A1_U(v) = \sum_{u\in U} A_{vu} = \begin{matrix} 1 & \text{if } v \\ 0 & \text{if } v \end{matrix}\right.$

- Assume G is connected. Let f so that $A_G . f = f$

Pick v so that $f(v)$ is maximal

Then $\sum_{u, (u,v)\in E} \frac1d f(u) = f(v)$

$\Rightarrow \forall u$ adjacent to v $\quad f(u) = f(v)$ + G connected $\Rightarrow f = \lambda\mathbb{1}$

connected component

(b) • $H \subset G$ is bipartite if $H = X \dot\cup Y$ and every edge starting in X ends in Y.

Consider $f = 1_X - 1_Y$ we claim that $Af = -f$

$\forall v \in X \quad Af(v) = \sum_{u \in Y, (u,v) \in E} \frac{1}{d} f(u) = -1 = -f(v)$

also $\forall\ v \in Y \quad Af(v) = -f(v)$.

$\forall v \in G \setminus H \quad f(v) = Af(v) = 0$

• Assume $\exists f$ so that $Af = -f \qquad f \neq 0$

Pick v so that $|f(v)|$ is maximal

Then $-f(v) = Af(v) = \sum_{u, (u,v) \in E} \frac{1}{d} f(u)$

$\Rightarrow \forall u$ connected to $v \quad f(u) = -f(v)$

Same argument works for those u's.

So $\forall u$ in the connected component of v, we have $f(u) = \pm f(v)$ and for adjacent vertices the signes are opposite $\to$ this connected component is bipartite.

<u>Edge expansion</u> $\quad G = (V, E) \quad F \subset V$

∂F = set of edges connecting F to $V \setminus F$

$$h(G) = \inf_{|F| \leq n/2} \left\{ \frac{|\partial F|}{|F|} \right\}.$$

<u>Examples</u> 1) If G is a n-cycle and F connected with $|F| \sim \frac{n}{2}$ then $\frac{|\partial F|}{|F|} \sim \frac{4}{n} \qquad h(G) \lesssim \frac{4}{n} \to 0$

2) G is the full graph with $|V| = n$.

$|F| = \ell \Rightarrow |\partial F| = \ell(n-\ell) \qquad \frac{|\partial F|}{|F|} = n - \ell$

$|F| \leq n/2 \longrightarrow h(G) = \frac{n}{2} \to +\infty$.

<u>Def</u> Let (G_n) be a sequence of finite connected d-regular graphs. (G_n) is a family of expanders if $|V_n| \to +\infty$ and $\exists\, \varepsilon > 0 \ \forall n \ \ h(G_n) \geq \varepsilon$

④

Theorem 3 Let G be a finite connected d-regular graph (without loops?)

Then $\frac{1}{2} d(1-\lambda_2) \leq h(G) \leq d\sqrt{2(1-\lambda_2)}$.

Corollary 4 (G_n) d-regular graphs is a sequence of expanders $\iff$ $\lambda_2(A_{G_n})$ is uniformly bounded away from 1

$\iff$ $\gamma_{G_n} = \frac{1}{1-\lambda_2(A_{G_n})}$ uniformly bounded.

~~enough to show that $\frac{1}{1-\lambda_2^2} \leq C \implies G_n$ expanders~~

Pf Th 3 ① $h(G) \geq \frac{1}{2} d(1-\lambda_2)$

Rem: Let $F, H \subset V$

$$d\langle A 1_F, 1_H\rangle = \sum_v \Big(\sum_{u\sim v} 1_F(u)\Big) 1_H(v)$$

$= \#$ of edges connecting F and H

So if $d\langle A1_F, 1_F\rangle \leq (d-s)|F|$

then $|\partial F| \geq s|F|$.

- Let $F \subseteq V$ $\quad 1_F = \frac{|F|}{|V|} 1_V + f \qquad f \in 1_V^{\perp}$

$\|1_F\|_2^2 = |F|$, $\left\|\frac{|F|}{|V|} 1_V\right\|_2^2 = \frac{|F|^2}{|V|} \implies \|f\|_2^2 = |F| - \frac{|F|^2}{|V|}$

$$\langle A1_F, 1_F\rangle = \left\langle A\frac{|F|}{|V|}1_V, \frac{|F|}{|V|}1_V\right\rangle + \langle Af, f\rangle$$
$$\leq \frac{|F|^2}{|V|} + \lambda_2\left(|F| - \frac{|F|^2}{|V|}\right)$$
$$= (1-\lambda_2)\frac{|F|^2}{|V|} + \lambda_2|F|$$

If $|F| \leq \frac{|V|}{2}$ $\quad \langle A1_F, 1_F\rangle \leq (1-\lambda_2)\frac{|F|}{2} + \lambda_2|F| = \frac{1+\lambda_2}{2}|F|$

$$= \left(1 - \frac{(1-\lambda_2)}{2}\right)|F|$$

So $|F| \leq \frac{|V|}{2} \implies |\partial F| \geq d\left(\frac{1-\lambda_2}{2}\right)|F|$

Thus $h(G) \geq \frac{1}{2} d(1-\lambda_2)$. □

② $h(G) \leq d\sqrt{2(1-\lambda_2)}$

Let $g \in 1_V^{\perp}$ st $Ag = \lambda_2 g$ $\qquad g \neq 0$

$V^+ = \{v \; g(v) > 0\}$ wma $|V^+| \leq \frac{|V|}{2}$

Note: $g \neq 0$ + $g \perp 1_V \implies g^+, g^- \neq 0$

Lemma 1 $\langle (Id - A) g^+, g^+ \rangle \leq (1-\lambda_2) \|g^+\|_2^2$

Pf $Ag^+ + Ag^- = \lambda_2 g^+ + \lambda_2 g^-$ $\quad g^+ \perp g^-$

So $\langle Ag^+, g^+ \rangle + \langle Ag^-, g^+ \rangle = \lambda_2 \langle g^+, g^+ \rangle$

Also $\langle Ag^-, g^+ \rangle \leq 0 \Rightarrow \langle Ag^+, g^+ \rangle \geq \lambda_2 \langle g^+, g^+ \rangle$.

Fix an orientation of the graphe. For each edge e choose an ~~et~~ endpoint e^+ and denote the other e^-

For $f \in l_2(V)$ define $\nabla f \in l_2(E)$ by $\nabla f(e) = f(e^+) - f(e^-)$

Define $b_{eu} = \begin{cases} 1 & \text{if } u = e^+ \\ -1 & \text{if } u = e^- \\ 0 & \text{otherwise} \end{cases}$ $\quad B = [b_{eu}]_{E \times V}$

$f = [\;] \in l_2(V)$ $\quad \forall e \in E \quad \sum_{u \in V} b_{eu} f(u) = f(e^+) - f(e^-)$

$\nabla f = B . f$

Lemma 2 $\|\nabla f\|_2^2 = \langle d(I - A) f, f \rangle$

(So $\|\nabla g^+\|_2^2 \leq d(1-\lambda_2) \|g^+\|_2^2$

Pf $\|\nabla f\|_2^2 = \langle Bf, Bf \rangle = \langle {}^tB B f, f \rangle$

$L = {}^tB B$ $\quad L_{uv} = \sum_{e \in E} b_{eu} b_{ev} = -\#$ appearances of uv $\quad u \neq v$

$L_{uu} = d$

$L = d\,Id - dA$. $\square$

Lemma 3 G connected $f : V \to \mathbb{R}$ $\quad m$ median of f

Then $\sum_{v \in V} |f(v) - m| \leq \frac{1}{h(G)} \sum_e |f(e^+) - f(e^-)|$

Pf Assume $m = 0$ and $|V| = 2k+1$ is odd ~~and f=0~~

$f_1 \leq \dots \leq f_k \leq f_{k+1} = 0 \leq f_{k+2} \leq \dots \leq f_{2k+1}$ values of f

$L_i^- = \{v, f(v) \leq f_i\}$ $i \leq k$ $\quad L_i^+ = \{v, f(v) \geq f_i\}$ $i \geq k+2$

$f_i^\Delta = f_{i+1} - f_i$ $\quad f_i^\nabla = f_i - f_{i-1}$

$\sum |f_i| = (f_2 - f_1) + (2f_3 - 2f_2) + \dots + k(0 - f_k) + k(f_{k+1} - 0) + \dots + (f_{2k+1} - f_{2k}$

$\sum_v |f(v)| = \sum_{i=1}^{k} |L_i^-| f_i^\Delta + \sum_{k+2}^{2k+1} |L_i^+| f_i^\nabla$

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$|L_i^-|, |L_i^+| \leq k \leq \frac{|V|}{2} \Rightarrow |\partial L_i^-| \geq h|L_i^-| \quad |\partial L_i^+| \geq h|L_i^+|$

So $\sum_{v \in V} |f(v)| \leq \frac{1}{h}\left(\sum_i^k |\partial L_i^-| f_i^\Delta + \sum_{k+2}^{2k+1} |\partial L_i^+| f_i^\Delta\right)$

For a fixed edge uv the contribution of uv to

$(\quad)$ is $|f(u) - f(v)|$ ......

End of proof $\quad f(v) = (g^+(v))^2$

$|V^+| \leq \frac{|V|}{2} \Rightarrow 0$ is a median of f

So $\sum_v |f(v)| = \|g^+\|_2^2 \leq \frac{1}{h}\sum_{e \in E} |g^+(e^+) - g^+(e^-)||g^+(e^+) + g^+(e^-)|$

So $\cancel{\ } h\|g^+\|_2^2 \leq \left(\sum_e |g^+(e^+) - g^+(e^-)|^2\right)^{1/2}\left(\sum_e |g^+(e^+) + g^+(e^-)|^2\right)^{1/2}$

$\leq \|\nabla g^+\|_2 \left(2\sum_e |g^+(e^+)|^2 + |g^+(e^-)|^2\right)^{1/2}$

$\leq \sqrt{2d}\, \|\nabla g^+\|_2 \, \|g^+\|_2$

Lemma 2 $\Rightarrow \quad h \leq d\sqrt{2(1-\lambda_2)}$. $\square$

(Note that $g \neq 0 \quad g \perp 1_V \Rightarrow g^+, g^- \neq 0$).

II) Generalized Poincaré constants

- Let X be an arbitrary set $K : X \times X \to [0,\infty)$ symmetric

<u>$\gamma(A,K)$</u> is the infimum of all $\gamma > 0$ st

$$\forall x_1, \dots, x_n \in X \quad \frac{1}{n^2} \sum_{i,j} K(x_i, x_j) \leq \frac{\gamma}{n} \sum_{i,j} a_{ij} K(x_i; x_j) \quad (1)$$

(where A is a $n \times n$ sym stoch. matrix).

<u>$\gamma^+(A,K)$</u> is the infimum of all $\gamma_+ > 0$ st

$$\forall x_1, \dots, x_n \in X \quad \forall y_1, \dots, y_n \in X \quad \frac{1}{n^2} \sum_{i,j} K(x_i; y_j) \leq \frac{\gamma_+}{n} \sum_{i,j} a_{ij} K(x_i; y_j)$$

If $G = (V, E)$ is a d-regular graph we denote

$\gamma(G,K) = \gamma(A_G, K) \qquad \gamma^+(G,K) = \gamma^+(A_G, K)$

$$(1) \iff \frac{1}{|V|^2} \sum_{V \times V} K(f(u), f(v)) \leq \frac{\gamma}{|V| \times d} \sum_{E} K(f(u), f(v))$$

$$(\forall f : V \to X)$$

$$(2) \iff \forall f : V \to X \qquad \forall g : V \to X$$

$$\frac{1}{|V|^2} \sum_{V \times V} K(f(u), g(v)) \leq \frac{\gamma_+}{|V| \times d} \sum_{E} K(f(u), g(v)).$$

<u>Clearly</u> $\gamma \leq \gamma^+$.

- <u>Gromov's argument for non embeddability</u>

Assume that $G_n = (V_n, E_n)$ are connected d-regular graphs and denote d_{G_n} = geodesic distance on V_n

Assume also $G_n \underset{\text{coar.}}{\hookrightarrow} (Y, d_Y)$ i.e $\exists f_n : G_n \to Y$

i.e $\exists \alpha, \beta : [0,\infty) \to [0,\infty)$ with $\lim_{t \to \infty} \alpha(t) = \infty$

and $\forall n \ \forall x, y \in V_n \quad \alpha(d_{G_n}(x,y)) \leq d_Y(f_n(x), f_n(y)) \leq \beta(d_{G_n}(x,y))$ (3)

Finally, assume that $\exists p > 0 \ \exists \gamma > 0$

$\forall n \qquad \forall f : V_n \to Y$

$$\frac{1}{|V_n|^2} \sum_{V_n \times V_n} d_Y(f(u), f(v))^p \leq \frac{\gamma}{d|V_n|} \sum_{E_n} d_Y(f(u), f(v))^p \quad (4)$$

in other words $\sup_n \gamma(G_n, d_Y^p) < \infty$

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$(3)+(4) \Rightarrow \frac{1}{|V_n|^2} \sum_{V_n \times V_n} \left[\alpha(d_{G_n}(u,v))\right]^p \leq \gamma \beta(1)^p$

Fix $k \in \mathbb{N}$ $\quad u \in V \quad |\{v, d_{G_n}(u,v) \leq k\}| \leq d^k$

$|\{(u,v), \ d_{G_n}(u,v) \leq k\}| \leq |V_n| d^k$

$|V_n| d^k \leq \frac{|V_n|^2}{2} \iff d^k \leq \frac{|V_n|}{2} \iff k \lesssim C_d \log |V_n|$

So at least half of the pairs in $V_n \times V_n$ satisfy

$d_{G_n}(u,v) \geq C_d \log |V_n|$

So $\quad \alpha(C_d \log |V_n|) \leq \gamma \beta(1)^p$

In particular $(|V_n|)_n$ is uniformly bounded.

<u>Example</u> If $G_n = (V_n, E_n)$ is a sequence of expander graphs (d-regular), then $\sup_n \gamma(G_n, \|\ \|_H^2) < \infty$ (Prop 1)

then $(V_n, d_{G_n}) \underset{c}{\not\hookrightarrow}$ Hilbert space.

III) Main result - Strategy of proof

<u>Théorem 1.1</u> There exists a sequence $G_n = (V_n, E_n)$ of 3-regular graphs such that $\lim |V_n| = +\infty$ and for every super-reflexive Banach space $(X, \|\ \|_X)$

$\sup_n \gamma_+(G_n, \|\ \|_X^2) < \infty$

A fortiori $\quad \sup_n \gamma(G_n, \|\ \|_X^2) < \infty$

They are called super-expanders

They do not embed coarsely in any superreflexive B. space

<u>Step 1</u> Construct base graphs $(H_n(\delta))_{n \in \mathbb{N}, \delta > 0}$

so that $\gamma_+(H_n(\delta), \|\ \|_X^2) \leq g^3$ ($\forall X$ K-convex)

$|V(H_n(\delta))| = m_n \nearrow$

But $H_n(\delta)$ $d_n(\delta)$-regular

$d_n(\delta) \leq e^{(\log m_n)^{1-\delta}}$

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Based on analysis of $e^{-t\Delta}$ on $L_p(\mathbb{F}_2^n)$.
(Sections 5 and 7)

- Step 2   Lower the degree but control γ^+ and γ
  - Edge completion (Lemma 2.9 easy)
  - Zigzag products | Section 8
  - Replacement products
  - $A \longrightarrow A_m = \frac{1}{m} \sum_0^{m-1} A^t$
  - If (X, d_X) has metric Markov cotype p with exponent 2 then $\gamma_+(A_m, d_X^2) \leq C \max\{1, \frac{\gamma_+(A, d_X^2)}{m^{2/p}}\}$] Section 7

Section 6 [$(X, \|\ \|_X)$ super-reflexive
$\Rightarrow \exists p \geq 2$ $(X, \|\ \|_X)$ has MMC p with exponent 2

- Step 3   Mix everything + diagonal argument] Section
  $\rightarrow$ end of proof

Possible organization

(A) Accessible stuff on graph operations
behavior of γ_+ under edge completion
zigzag products
replacement products.

(B) Césaro Means and Markov cotype

(C) Section 6 ~~Rex~~ SR $\Rightarrow$ Markov cotype

?

(D) Section 5

(E) Construction of the base graphs

(F) Section 4