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Construction of base graphs in Mendel-Naor.

<u>Reminder 1</u>: If $K : X\times X \longrightarrow [0,+\infty[$ symmetric kernel and

$A = [a_{ij}]$ is a symmetric stochastic matrix, ~~and~~ then

$$\gamma^+(A,K) = \inf\{\gamma^+>0 : \forall x_1,\dots,x_n\in X\ \forall y_1,\dots,y_n\in X$$
$$\frac{1}{n^2}\sum_{i,j} K(x_i,y_j) \le \frac{\gamma^+}{n}\sum_{i,j} a_{ij}K(x_i,y_j)\}$$

If $G=(X,E)$ is a d-regular graph and $A_G = \frac{1}{d}[E(u,v)]_{u,v\in V}$, the normalized adjacency matrix, then

$$\gamma^+(G,K) = \gamma^+(A_G,K) = \inf\{\gamma^+>0 : \forall f,g : V\longrightarrow X\ \frac{1}{|V|^2}\sum_{u,v}K(f(u),g(v))$$
$$\le \frac{\gamma^+}{|V|\times d}\sum_{(u,v)\in E}K(f(u),g(v))\}$$

$\boxed{\text{Goal}}$: to build $G_n=(V_n,E_n)$ 3-regular graphs so that $|V_n|\to+\infty$ and $\forall\ (X,\|\cdot\|_X)$ superreflexive $\forall p\ge 1$ $\sup_n \gamma^+(G_n,\|\cdot\|_X^p)<\infty$

<u>Preliminaries</u>: $(X,\|\cdot\|_X)$ normed space and $f:\{1,\dots,n\}\longrightarrow X$, $\|f\|_{L_p^n(X)} = \left(\frac{1}{n}\sum_{i=1}^n\|f(i)\|_X^p\right)^{1/p}$

and $A\otimes \mathrm{Id}_X : L_p^n(X)\longrightarrow L_p^n(X)$ and $L_p^n(X)_0 = \{f\in L_p^n(X) : \sum_{i=1}^n f(i)=0\}$

$$f \longmapsto \left(i\longmapsto \sum_j a_{ij}f(j)\right)$$

Remark: $L_p^n(X)_0$ stable under $A\otimes\mathrm{Id}_X$ because A is stochastic:

$\sum_i g(i) = \sum_i\left(\sum_j a_{ij}f(j)\right) = \sum_j\left(\sum_i a_{ij}\right)f(j) = \sum_j f(j) = 0$ and

$\lambda_X^{(p)}(A) := \|A\otimes\mathrm{Id}_X\|_{B(L_p^n(X)_0)}$

N.B.: If $X=\mathbb{R}$, $p=2$, $\lambda_{\mathbb{R}}^{(2)}(A) = \max_{i\ge 2}|\lambda_i(A)| = \lambda(A)$ and $\gamma^+(A,|\cdot|^2) = \frac{1}{1-\lambda(A)}$

(i.e., $K(x,y) = |x-y|^2$) $= \gamma^+(A,\|\cdot\|_H^2)$

Lemma 6.1.: $\gamma_+(A,\|\cdot\|_X^p) \le \left(1+\frac{4}{1-\lambda_X^p(A)}\right)^{1/p}$

Remark: there is a converse estimate when X is K-convex: see Lemma 6.6 of the paper.

Proof: $\boxed{\left(\frac{1}{n^2}\sum\sum \|f(i)-g(j)\|_X^p\right)^{\frac1p}} \le \|\bar f-\bar g\|_X + \left(\frac{1}{n^2}\sum_i\sum_j \|f_0(i)-g_0(j)\|_X^p\right)^{\frac1p}$

(1) where $\bar f = \frac1n\sum f(i)$ and $f_0 = f-\bar f \in L_p^n(X)_0$ (This is just the triangular $\le$)

$$\le \|\bar f-\bar g\|_X + \left(\frac{1}{n^2}\sum\sum 2^{p/q}\left(\|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)\right)^{\frac1p}$$

$\left|\frac{a+b}{2}\right|^p \le \frac{a^p+b^p}{2}$
$(a+b)^p \le 2^{p-1}(a^p+b^p)$

$$= \|\bar f-\bar g\|_X + 2^{\frac{p-1}{p}}\boxed{\left(\frac1n\sum_i \|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)^{\frac1p}}\ (1')$$

and $\|\bar f-\bar g\| = \|\frac1n\sum_i f(i)-g(i)\|_X = \|\frac1n\sum_i\sum_j a_{ij} f(i)-g(i)\|_X$

$$= \|\frac1n\sum_i\sum_j a_{ij}(f(i)-g(j))\|_X \le \left(\frac1n\sum\sum a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}$$

(this is Jensen's inequality with $\mu(\{(i,j)\}) = \frac1n a_{ij}$

Define $B = \left[\begin{array}{c|c}0&A\\\hline A&0\end{array}\right]$, $h = f_0\oplus g_0 \in L_p^{2n}(X)_0$.

$$(1-\lambda)\|h\|_{L_p^{2n}(X)} = \|h\|_{L_p^{2n}(X)} - \left(\frac{\lambda^p\|f_0\|^p_{L_p^n(X)}+\lambda^p\|g_0\|^p_{L_p^n(X)}}{2}\right)^{\frac1p}$$

$$\le \|h\|_{L_p^{2n}(X)} \left(\frac12\|(A\otimes Id_X)(f_0)\|^p_{L_p^n(X)} + \frac12\|A\otimes Id_X(g_0)\|_X^p\right)^{\frac1p}$$

$$= \|h\|_{L_p^{2n}(X)} - \|(B\otimes Id_X)(h)\|_{L_p^{2n}} \le \|(Id - B\otimes Id_X)(h)\|_{L_p^{2n}(X)}$$

$$= \left(\frac{1}{2n}\sum_i\|f_0(i)-\sum_j a_{ij}g(j)\|_X^p + \frac{1}{2n}\sum_i\|g_0(i)-\sum_j a_{ij}f(j)\|_X^p\right)^{1/p}$$

as A is stochastic and symmetric, $\le \left(\frac1n\sum_i\sum_j a_{ij}\|f_0(i)-g_0(j)\|_X^p\right)^{1/p}$

$$\le \|\bar f-\bar g\|_X + \frac1n\left(\sum_i\sum_j a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}.$$

Let us ~~sum~~ summarize: we had to bound (1) and $\|h\| = \left(\frac{1}{2n}\sum_i\|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)^{\frac1p}$

So $\left(\frac1n\sum_i\|f_0(i)\|_X^p+\|g_0(i)\|_X^p\right)^{\frac1p} \le \frac{2^{1/p}}{1-\lambda}\cdot 2\cdot\left(\frac1n\sum\sum a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}$

Finally, (1) $\le \left(1+2^{\frac{p-1}{p}}\cdot\frac{2^{\frac1p}\cdot 2}{1-\lambda}\right)\left(\frac1n\sum\sum a_{ij}\|f(i)-g(j)\|_X^p\right)^{\frac1p}$.

thus $\gamma^+ \le \left(1+\frac{4}{1-\lambda_X^p(A)}\right)^{\frac1p}$.

Theorem (Corollary 7.4) $\forall\delta\in]0,1[\ \forall p\in]1,\infty[\ \exists n_0(p,\delta)\ \exists \left(H_n^p(\delta)\right)_{n\ge n_0(p,\delta)}$

$d_n^p(\delta)$-regular such that (i) $|V(H_n(\delta))| = m_n = 2^{D_n}$, $D_n \ge \frac{n}{10}$

(ii) $d_n^p(\delta) \le e^{(\log m_n)^{1-\delta}}$

and $\forall (X,\|\cdot\|_X)$ K-convex Banach space $\exists \delta_0^p(X)\ \forall\delta < \delta_0^p(X)\ \sup \gamma^+(H_n^p(\delta),\|\cdot\|_X^p) \le 9^{p+1}$

Lemma 7.3 $\forall K, p > 1 \ \exists n(K,p) \ \exists \delta(K,p) \in]0,1[\ \exists (H_n(K,p))_{n \geq n(K,p)}$:

$d_n^p(K,p)$-regular $\quad d_n(K,p) \leq e^{(\log m_n)^{1-\delta(K,p)}}$ with $m_n = |V(H_n(K,p))| = 2^{D_n}$ for $D_n \geq \frac{n}{10}$

and for any X K-convex with $K(X) \leq K$, for all $n \geq n(K,p)$, $\gamma^+(H_n(K,p), \|\cdot\|_X^p) \leq g^{p+1}$

For $7.3 \Rightarrow 7.4$, see the paper.

Reminder 2: If $A \subset \{1,\dots,n\}$ and $W_A : \mathbb{F}_2^n \longrightarrow X$, $x \longmapsto (-1)^{\sum_A x_j}$ and $f \in L^p(\mathbb{F}_2^n, X)$,

$$e^{z\Delta} f = \sum_{A \subset \{1,\dots,n\}} e^{z|A|} \hat{f}(A) w_A$$

$$R_z(x) = \prod_{j=1}^{n} (1 + (-1)^{x_j} e^z) = (1-e^z)^{\sum_j x_j} (1+e^z)^{n - \sum x_j}$$

$$= \sum_{A \subset \{1,\dots,n\}} e^{z|A|} (-1)^{\sum_{j \in A} x_j} \quad : e^{z\Delta} f = R_z * f.$$

Then $$(e^{z\Delta} f)(y) = \sum_{w \in \mathbb{F}_2^n}' \left(\frac{1-e^z}{2}\right)^{\sum_j y_j - w_j} \left(\frac{1+e^z}{2}\right)^{n - \sum_j y_j - w_j} \cdot f(w)$$

$$= \sum_{w \in \mathbb{F}_2^n}' \left(\frac{1-e^z}{2}\right)^{\|y-w\|_1} \left(\frac{1+e^z}{2}\right)^{n - \|y-w\|_1} f(w)$$

because we just work modulo 2.

In particular, $$(e^{-t\Delta} \delta_x)(y) = \left(\frac{1-e^{-t}}{2}\right)^{\|x-y\|_1} \left(\frac{1+e^{-t}}{2}\right)^{n - \|x-y\|_1}.$$

$\tau_t = \frac{1-e^{-t}}{2}$; $\sigma_t^n = \tau_t^{4n\tau_t} (1-\tau_t)^{(1-4\tau_t)n}$ and $e_t^n : \{0,\dots,n\} \longrightarrow \mathbb{N}$, $k \longmapsto \left\lfloor \frac{\tau_t^k (1-\tau_t)^{n-k}}{\sigma_t^n} \right\rfloor$

Lemma 7.2: Action of $e^{-t\Delta}$ and Poincaré inequalities: let $t \in]0, \frac{1}{4}[$, $p \geq 1$, $n \geq 2^{13}$ so that $\tau_t \geq \left(\frac{n \log 18n}{18n}\right)^{1/2}$; $G_t^n = (\mathbb{F}_2^n, E_t^n)$, $V(G_t^n) = \mathbb{F}_2^n$: for all $x, y \in \mathbb{F}_2^n$ the edge (x,y) appears $e_t^n(\|x-y\|_1)$ times : G_t^n is d_t^n-regular, $\frac{1}{3\sigma_t^n} \leq d_t^n \leq \frac{1}{\sigma_t^n}$.

For all metric space (X, d_X), $$\frac{1}{3|E_t^n|} \sum_{E_t^n} d_X(f(x), g(y))^p \leq \frac{1}{2^n} \sum_{x,y \in \mathbb{F}_2^n} (e^{-t\Delta}\delta_x)(y)\, d_X(f(x), g(y))^p$$
$$\leq \frac{1}{|E_t^n|} \sum_{E_t^n} d_X(f(x), g(y))^p$$

End of talk: $7.2 \Rightarrow 7.3$.

Claim: there is a linear subspace V_n of $\mathbb{F}_2^n$ so that $D_n = \dim(V_n) \geq \frac{n}{40}$, $k_n = \min_{x \in V_n \setminus \{0\}} \|x\|_1 \geq \frac{n}{10}$

It is a "good linear code"

Proof: Let V be the k-dimensional linear subspace of $\mathbb{F}_2^n$ ~~of dim~~ such that

(*) $= \left|\{x\in\mathbb{F}_2^n : \exists v\in V\ \|x-v\|_1 < \frac{n}{10}\}\right| \le 2^k \times \sum_{\ell<\frac{n}{10}} \binom{n}{\ell} = 2^k A_n$

We have $A_n = 2^n P\left[\sum_i X_i < \frac{n}{10}\right]$ X_i independent Bernoulli "facts of life"

$= 2^n P\left[\sum_i \varepsilon_i > \frac{4n}{5}\right]$ ε_i ——— Rademacher.

Then $P\left[\sum_i \varepsilon_i > a\right] \le e^{-\frac{a^2}{2n}}$ and $A_n \le 2^n e^{-\frac{8n}{25}}$ and (*) $\le 2^n e^{k-\frac{8n}{25}}$

Pick $k\in \left]\frac{n}{10}, \frac{8n}{25}\right[$ and V k-dimensional and note that (*) $< 2^n \Rightarrow \exists x_0\in\mathbb{F}_2^n\ \forall v\in V\ \|x_0-v\|_1 \ge \frac{n}{10}$.

$V_n = \mathrm{span}(V\cup\{x_0\})$, $\min_{V_n\setminus\{0\}} \|x\|_1 \ge \frac{n}{10}$

Define: $H_n = H_n(k,p)$, $V(H_n) = \mathbb{F}_2^n / V_n^\perp$, $V_n^\perp = \{x\in\mathbb{F}_2^n : \forall y\in V_n\ \sum_i x_i y_i = 0 \bmod 2\}$

and $|V(H_n)| = 2^{D_n}$ and $D_n = \dim V_n \ge \frac{n}{10}$.

The number of edges joining $x+V_n^\perp$ and $y+V_n^\perp$

= ——— with one end in $x+V_n^\perp$ and the other end in $y+V_n^\perp$, divided by $|V_n^\perp|$, makes

$\frac{1}{|V_n^\perp|}\sum_{(u,v)\in V_n^\perp\times V_n^\perp} e_\epsilon^n(\|x-y+u-v\|_1) = \sum_{u\in V_n^\perp} e_\epsilon^n(\|x-y+u\|_1)$

and the degree at $\dot{x} = x+V_n^\perp$ is $\sum_{y\in V_n^\perp}\left(\sum_{u\in V_n^\perp} \|x-y+u\|_1\right) = \sum_{z\in\mathbb{F}_2^n} e_\epsilon^n(\|x-z\|_1) = d_\epsilon^n$ (by 7.2!)

Let $\pi: \mathbb{F}_2^n \longrightarrow \mathbb{F}_2^n/V_n^\perp$ be the quotient map and X be a B-space:

If $f\in L^p(\mathbb{F}_2^n/V_n^\perp, X)$, we get $\pi f: \mathbb{F}_2^n \longrightarrow X$, $x \longmapsto f\circ\pi(x)$ $\in L_p(\mathbb{F}_2^n,X)$ constant on cosets.

Claim: let $f\in L^p(\mathbb{F}_2^n/V_n^\perp, X)_0$ (i.e. $\sum_{\dot x} f(\dot x)=0$ so that $\widehat{\pi f}(\emptyset)=0$). Then $\pi f\in L_p^{\ge k_n}(\mathbb{F}_2^n,X)$ with $k_n = \min_{x\in V_n\setminus\{0\}} \|x\|_1 \ge \frac{n}{10}$.

Proof: $V_n^{\perp\perp} = V_n$ (exercise). Let $\emptyset\ne A\subset\{1,\dots,n\}$ and $|A|<k_n$. Then $1_A \notin V_n = V_n^{\perp\perp}$ so that $\exists v\in V_n^\perp$ $\langle 1_A, v\rangle \ne 0$ $\sum_A v_i = 1 \pmod 2$

$\widehat{\pi f}(A) = \int_{\mathbb{F}_2^n} \pi f(u)\, w_A(u)\,du = \int_{\mathbb{F}_2^n} \pi f(u+v)\, w_A(u)\,du = \int_{\mathbb{F}_2^n}\pi f(u)\, w_A(u-v)\, du$

$= (-1)^{\sum_A v_i}\int_{\mathbb{F}_2^n} \pi f(u)\, w_A(u)\,du = -\widehat{\pi f}(A)$. Therefore $\widehat{\pi f}(A)=0$.

Remark: Theorem 5.1 $\Rightarrow$ | If f is K-convex with $K(X)\le K$, then there is $A=A(K,p)$, $B=B(K,p)>2$, $C=C(K,p)$, so that ~~$\forall f\in L_p(\mathbb{F}_2^n$~~ $\|e^{-t\Delta}\|_{B(L_p^{\ge k}(\mathbb{F}_2^n,X))} \le Ce^{-Ak\min(t,t^B)}$

Here $\boxed{\forall f\in L^p(\mathbb{F}_2^n/V_n^\perp,X)_0 \quad \frac{\|e^{-t\Delta}\pi f\|_{L_p(\mathbb{F}_2^n,X)}}{\|\pi f\|_{L^p(\mathbb{F}_2^n,X)}} \le Ce^{-Ak_n\min(t,t^B)}}$

Pick $t=t(n,p,k) = \left(\frac{\log 2C}{k_n A}\right)^{1/B}$; then $Ce^{-Ak_n\min(t,t^B)} \le \frac{1}{2}$.

There is $n(K,p)$ such that for all $n \geq n(K,p)$ $\quad t < \frac{1}{4}$ and $n \geq 2^{13}$

and $\frac{1-e^{-t}}{2} = \tau_t \geq \left(\frac{p \log 18n}{n}\right)^{1/2}$. Now remember, $B>2$!

We have degree $= d_t^n \leq \frac{1}{\sigma_t^n} = \frac{1}{\tau_t^{4\xi_t n}(1-\tau_t)^{n-4\xi_t n}} \leq \frac{1}{\tau_t^{8\xi_t n}} \leq e^{(\log m_n)^{1-\delta(K,p)}}$

We can apply Lemma 7,2 : $e^{-t\Delta_\pi}$

for $\delta(K,p) < \frac{1}{B}$, $B = B(K,p) > 2$. There is a check needed here.

Then $e^{-t\Delta_\pi}$ is a stochastic symmetric operator on $L_p(\mathbb{F}_2^n/V_n^\perp)$

there are 2 mistakes on page 58. Let $s \in \mathbb{F}_2^n/V_n^\perp$: $(e^{-t\Delta_\pi})(\delta_s) = e^{-t\Delta}\mathbb{1}_s$

Let $y \in \mathbb{F}_2^n$: $\left(e^{-t\Delta}\mathbb{1}_s\right)(y) = \sum_{u\in s}\left(e^{-t\Delta}\delta_u\right)(y) = \sum_{u\in s}\left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1}$

$$= \left(e^{-t\Delta}\mathbb{1}_s\right)(y') \quad \text{if } y-y' \in V_n^\perp$$

this does not depend on $y \in T$ (where $T \in \mathbb{F}_2^n/V_n^\perp$) symmetric in s and T.

$$q_{s,T} = \sum_{u\in s}\left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1} \quad \text{for } y \in T$$

$$q_{T,s} = \sum_{w\in V^\perp}\left(\frac{1-e^{-t}}{2}\right)^{\|y+w-x\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|y+w-x\|_1} = q_{s,T}.$$

then $\sum_s q_{s,T} = \sum_s \sum_{u\in s}\left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1}$

$$= \sum_{u\in\mathbb{F}_2^n}\left(\frac{1-e^{-t}}{2}\right)^{\|u-y\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-y\|_1}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1-e^{-t}}{2}\right)^{k}\left(\frac{1+e^{-t}}{2}\right)^{n-k} = 1$$

$Q = [q_{s,T}]$ is symmetric and stochastic

Theorem 5,1 : $\|Q \otimes Id_X\|_{B(L_p(\mathbb{F}_2^n/V_n^\perp),X)_0} = \lambda_X^{(p)}(Q) \leq \frac{1}{2}$

Lemma 6,1 $\Rightarrow \gamma^+(Q, \|\cdot\|_X^p) \leq 9^p$ and finally, for $f,g : \mathbb{F}_2^n/V_n^\perp \to X$.

$$\frac{1}{|\mathbb{F}_2^n/V_n^\perp|^2}\sum_{s,T}\|f(s)-g(T)\|_X^p \leq \frac{9^p}{|\mathbb{F}_2^n/V_n^\perp|}\sum_s\sum_T q_{s,T}\|f(s)-g(T)\|_X^p$$

$$= \frac{9^p}{|\mathbb{F}_2^n/V_n^\perp|}\sum_T\sum_s\left(\sum_{u\in s}\left(\frac{1-e^{-t}}{2}\right)^{\|u-v\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-v\|_1}\right)\|f(s)-g(T)\|_X^p$$

$$= \frac{9^p}{|\mathbb{F}_2^n/V_n^\perp| \times |V_n^\perp|}\sum_T\sum_{v\in T}\sum_s\sum_{u\in s}\underbrace{\qquad\qquad}$$

for all $v \in T$ $\|f(s)-g(T)\|_X^p$

$$= \frac{9^p}{2^n}\sum_{u\in\mathbb{F}_2^n}\sum_{v\in\mathbb{F}_2^n}\left(\frac{1-e^{-t}}{2}\right)^{\|u-v\|_1}\left(\frac{1+e^{-t}}{2}\right)^{n-\|u-v\|_1}\|\pi f(u)-\pi g(v)\|_X^p$$

$$\overset{\text{by 7.2}}{\leq} \frac{3 \cdot g^p}{|E_t^n|} \sum_{(u,v) \in E_t^n} \|\pi f(u) - \pi g(v)\|$$

But in my new graph H_{n},
$$= \frac{3 \cdot g^p}{|E_t^n|} \sum_{S} \sum_{T} \left(\sum_{u \in S} \sum_{v \in T} e_t^n(\|u-v\|_1) \right) \|f(S) - g(T)\|_X^p$$
$$= \frac{3 \cdot g^p}{|E_t^n|} \sum_{S} \sum_{T} |V_n^\perp| \, E_n(S,T) \, \|f(S) - g(T)\|_X^p$$
$$= \frac{3 \cdot g^p}{|E_t^n|} \sum_{(S,T) \in E_n} \|f(S) - g(T)\|_X^p \quad \left(\text{here } |E_n| = \frac{|E_t^n|}{|V_n^\perp|}\right)$$

"Chernoff bounds/estimate"