GDT

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$C^1(\mathbb{T}^n)^*$ is w.s.c. (Bourgain, 83, 84)
(see Wojtaszczyk's book)

1 Intro

def: Y is wsc if $\forall (y_n)$ w-Cauchy
$\exists y \in Y$ s.t. $y_n \xrightarrow{w} y$

(w-Cauchy $\iff \langle y_n, y^* \rangle$ CV $\forall y^* \in Y^*$)

we will stuy this for $Y = X^*$, $X \subset C(\Omega, E)$
where Ω compact, E finite dim

Example: $\Phi : C^1(\mathbb{T}^s) \longrightarrow C(\mathbb{T}^s, \ell_\infty^{s+1})$

$$f \longmapsto \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$$

def: $X \subset C(\Omega, E)$ is <u>rich</u> if $\exists \sigma$ proba measure on Ω
s.t. $\forall \varphi \in C(\Omega)$, $\forall (x_n) \subset X$ bdd

$$\int_\Omega |x_n| \, d\sigma \longrightarrow 0 \Rightarrow \operatorname{dist}(\varphi x_n, X) \longrightarrow 0$$

<u>prop 1:</u> $\Phi(C^1(\mathbb{T}^s))$ is rich in $C(\mathbb{T}^s, \ell_\infty^{s+1})$

<u>thm 2:</u> If $X \subset C(\Omega, E)$ is rich and $K \subset X^*$, K bdd

TFAE: (i) K non-relatively w-compact

(ii) $\exists \sum \varphi_n$ w.u. CV. s.t.

$$\overline{\lim} \sup_{k \in K} |k(\varphi_n)| > 0$$

(iii) $\exists (k_n) \subset K \sim$ u.v.b. of ℓ_1

<u>Today</u>: (i) $\Rightarrow$ (ii)

def: X has Pełczyński property (P) if $\forall K$ not rel. w-compact bdd

$\exists \sum x_n$ w.u.c. s.t. $\inf_n \sup_{k \in K} k(x_n) > 0$

proposition 3: $\left(\text{if } X \text{ is a B space } (i) \Rightarrow (ii) \right) \Rightarrow X$ has (P)

$\Rightarrow X^*$ w.s.c.

~~Proof~~

Cor: $C^1(\mathbb{T})^*$ is w.s.c.

$|\cdot| = \|\cdot\|_{\ell_\infty^{s+1}}$

(2) Proof of Prop 1:

$$\Phi : C^1(\mathbb{T}^s) \longrightarrow C(\mathbb{T}^s, \ell_\infty^{s+1})$$

$$f \longmapsto \left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_s} \right)$$

let $\sigma = \lambda$ the normalized Lebesgue measure

Let $(f_n) \subset C^1(\mathbb{T}^s)$ bdd s.t. $\int_{\mathbb{T}^s} |\Phi(f_n)| d\lambda \xrightarrow[n\to\infty]{} 0$

we want $\forall \varphi \in C(\mathbb{T}^s)$: $\text{dist}(\varphi \Phi(f_n), X) \longrightarrow 0$

We take φ a trigonometric polynomial

$$\Phi(f \cdot \varphi) = \left(\varphi f, \frac{\partial f \cdot \varphi}{\partial x_1}, \dots \right)$$

$$= \left(\varphi f, \frac{\partial f}{\partial x_1} \varphi, \dots \right) + \left(0, \frac{\partial \varphi}{\partial x_1} \cdot f, \dots \right)$$

$$= \varphi \Phi(f) + f \left(0, \frac{\partial \varphi}{\partial x_1}, \dots \right) \text{ so}$$

$$\| \Phi(f\varphi) - \varphi \Phi(f) \|_\infty \leq C_\varphi \|f\|_\infty$$

We will prove that $\underbrace{\|\Phi(f_n\varphi) - \varphi\Phi(f_n)\|_\infty}_{\leq C_\varphi \|f_n\|_\infty} \to 0$

$\int_{\mathbb{T}^s} |\Phi(f_n)| \, d\lambda \to 0$ implies $\|f\|_1 \to 0$, $\|\frac{\partial f_n}{\partial x_i}\|_1 \to 0$

Let $x_0 \in \mathbb{T}^s$, if $f_n(x_0) \not\to 0 \Rightarrow \exists$ subsequence (f_{n_k})

s.t. $f_{n_k}(0) \to \alpha \neq 0$

By Ascoli $\exists (f_{n_k'}) \subset (f_{n_k})$ s.t.

$(f_{n_k'})$ cv in $\|\cdot\|_\infty$ to f, $f(x_0) \neq 0$

contradicts $\|f_{n_k'}\|_1 \to 0$ So $f_n(x_0) \to 0$

$|f_n(x) - f_n(x_0)| \leq \|f_n'\|_1 \xrightarrow[n\to\infty]{} 0$

$\|f_n\|_\infty \to 0$

(3) <u>proof of prop 3:</u>

- $((i) \Rightarrow (ii)) \Rightarrow (P)$ obvious
- X has $(P) \Rightarrow X^*$ is wsc

Assume $(x_n^*) \subset X^*$ is w-Cauchy but not w-cv

Take $K = \{x_k^*, k \in \mathbb{N}\}$ K non-relatively w-cpt

Otherwise by Eberlein-Smulyan $\exists (x_{k_j}^*) \subset K$

s.t. $x^*_{k_j} \xrightarrow{w} x^*$ + (x_k^*) w-Cauchy $\Rightarrow$

$\Rightarrow (x^*_k)$ w-convergent ↯

- Since X has (P) $\exists \sum_i x_n$ wuc, $\forall n$ $\sup_k x_k^*(x_n) \geq 2\delta$

Goal: find k_1, k_2 s.t. $x^*_{k_j}(x_{n_j}) \geq \delta$

Take k_1 s.t. $x_{k_1}^*(x_1) \geq \delta$, $n_1 = 1$

for all k, $x_k^*(x_n) \xrightarrow[n\to\infty]{} 0$ $\left(\sum x_n \text{ wuc}\right)$

thus $\exists n_2$ s.t. $\forall k \leq k_1, \forall n \geq n_2$ $|x_k^*(x_n)| < \frac{\delta}{2}$

So $\exists k_2 ~~s~~ > k_1$ s.t. $|x_{k_2}^*(x_{n_2})| \geq \delta$

<u>Define</u> $T : c_0 \longrightarrow X$ write (x_j) for (x_{n_j})

$e_j \longmapsto x_j$.

Since $\sum x_j$ wuc

<u>Note:</u> $\sum x_j$ WUC $\forall x^*$ $\sum |x^*(x_j)| < \infty$

$$\Longrightarrow \left\|\sum_{j=1}^n \lambda_j x_j\right\| \leq C \|(\lambda_j)\|_\infty$$

so T is well defined and bdd by Banach-Steinhaus

$T^* : X^* \longrightarrow \ell_1$

$$\langle e_j, \underbrace{T^* x_{k_j}^*}_{y_j} \rangle = \langle x_j, x_{k_j}^* \rangle \geq \delta$$

ℓ_1 Schur

so (y_j) cannot be $\|\cdot\|_1$ CV $\Longrightarrow$ so not w-Cauchy

$\Longrightarrow$ $(x_{k_j}^*)$ not w-Cauchy in X^* $\lightning$

(4) Proof of (i) $\Longrightarrow$ (ii)

Let K be non-rel. w-cpt. bdd in X^*

$\Rightarrow \exists (y_n^*) \subset K$ without any w-CV subsequence

By Alaoglu it has a w^*-CV subsequence

We will assume that $y_n^* \xrightarrow{w^*} 0$ and $\exists x^{**} \in X^{**}, \|x^{**}\| = 1$ 3

s.t. $\forall n \ |x^{**}(y_n^*)| \geq 2\delta$

Step 1: Construct $(x_1, \dots, x_n) \subset X$, $(k_1^*, \dots, k_n^*) \subset K$

s.t. $\|k_j^*\| \leq C$, $\|x_j\| = 1$

$$|k_j^*(x_k)| \leq \delta/3 \quad \text{if } 1 \leq j < k \leq n$$
$$\geq \delta \quad \text{if } 1 \leq k < j \leq n$$

Proof: Take $n_1 = 1$, $|x^{**}(y_1^*)| \geq 2\delta$

Goldstein: $\exists x_1 \in S_X$ s.t. $|y_1^*(x_1)| \geq \delta$

Take $n_2 > n_1$ s.t. $|y_{n_2}^*(x_1)| < \frac{\delta}{3}$ (as $y_n^* \xrightarrow{w^*} 0$)

Choose $x_2 \in S_2$ s.t. $|y_{n_2}^*(x_2)| \geq \delta$ and $|y_{n_2}^*(x_1)| \geq \delta$

Take $n_3 > n_2$ s.t. $|y_{n_3}^*(x_i)| \leq \frac{\delta}{3}$, $i = 1, 2$, ...

etc.

Rename $y_{n_j}^* = k_j^*$

Step 2: Lemma: H Hilbert space, $x_1, \dots, x_n$, $\|x_i\| \leq 1$

$\Rightarrow \exists A, B \subset \{1, \dots, n\}$ non-empty with $\max A < \min B$

s.t. $$\left\| \frac{1}{|A|} \sum_{i \in A} x_i - \frac{1}{|B|} \sum_{i \in B} x_i \right\| \leq \frac{2}{\sqrt{\ln n}}$$

Proof: Denote $\alpha_k = \max \left\{ \left\| \frac{1}{|A|} \sum_{i \in A} x_i \right\|, |A| = 2^k \right\}$

$$\beta_k = \inf \left\{ \left\| \frac{1}{|A|} \sum_A x_i - \frac{1}{|B|} \sum_B x_i \right\|, \ |A| = |B| = 2^k, \ A < B \right\}$$

$k = 0 \dots \log_2 n - 1$

$$\left\|\frac{x+y}{2}\right\|^2 + \left\|\frac{x-y}{2}\right\|^2 = \frac{1}{2}\left(\|x\|^2 + \|y\|^2\right)$$

So $$\left\|\frac{1}{2^{k+1}}\sum_{A\cup B} x_i\right\|^2 + \frac{1}{4}\left\|\frac{1}{2^k}\sum_A x_i - \frac{1}{2^k}\sum_B x_i\right\|^2 =$$

$$= \frac{1}{2}\left(\left\|\frac{1}{2^k}\sum_A x_i\right\|^2 + \| \cdots \|^2\right) \qquad (*)$$

Choose $C \subset \{1, \dots, n\}$, $|C| = 2^{k+1}$ s.t.

$$\alpha_{k+1} = \left\|\frac{1}{|C|}\sum_{C} x_i\right\|$$

$$C = A \cup B, \quad |A| = 2^k = |B|$$

So $(*) \Rightarrow \alpha_{k+1}^2 + \frac{\beta_k^2}{4} \leq \alpha_k^2$

$$\Rightarrow \frac{1}{4}\sum_{k=0}^{\log_2 n - 1} \beta_k^2 \leq \alpha_0^2 \leq 1$$

let $\beta := \min \beta_k$

$$\Rightarrow \beta^2 \ln n \leq \beta^2 \cdot \log_2 n \leq 4$$

$$\beta \leq \frac{2}{\sqrt{\ln n}}$$

Let $\sum \varepsilon_j < \delta/10$

<u>Step 3:</u> Construct $(z_1, \dots, z_n, \dots) \subset X$, $(x_1^*, \dots, x_n^*, \dots) \subset K$

s.t. • $|x_j^*(z_j)| > \frac{2\delta}{3}$

• $\int |z_j| \, d|\mu_k| < \varepsilon_j$ if $1 \leq j < k$

where μ_k is the extension of x_k^* to $C(-B, \in)$

• $\operatorname{dist}\left(z_k \prod_{j=1}^{k-1} (1 - |z_k|), X\right) < \varepsilon_k$

Choose n_1 s.t. $\frac{2}{\sqrt{\ln n_1}} < \varepsilon_1$, (let $n > n_1$) ($j \leq n$)

We apply the Lemma in $H = L^2\left(\max_{j=1..n} |\mu_j| + \sigma\right)$ from richness

and $(x_1, \dots, x_{n_1}) \subset X$

So there exist $A, B \subset \{1, \dots, n\}$, $\max A < \min B$

$$\forall j = 1..n \int \underbrace{\left| \frac{1}{|A|} \sum' x_i - \frac{1}{|B|} \sum' x_i \right|}_{z_1} d(|\mu_j| + \sigma) \leq \|z_1\|_{L^2} < \varepsilon$$

Let $s = \max A$ and take $x_1^* = e_s^*$

$x_1^*(z_1^A) \geq \delta$ since $i \in A : i \leq s$

$x_1^*(z_1^B) \leq \frac{\delta}{3}$ $\quad i \in B : i > s$

Thus $|x_1^*(z_1)| \geq \frac{2\delta}{3}$

Then take $n_2 > n_1$ s.t. $\frac{2}{\sqrt{\ln n_2}} < \varepsilon_2$

lemma $\rightsquigarrow$ $z_2 = \frac{1}{|A_2|} \sum'_{A_2} x_i - \frac{1}{|B_2|} \sum_{B_2} x_i$ s.t.

$A_2, B_2 \subset \{1, \dots, n_2\}$ (but $x_1, \dots, x_{n_2}$) are different

$\forall n > n_2, j \leq n$

$$\int |z_2| \, d(|\mu_j| + \sigma) < \varepsilon_2$$

and with $\int |z_2| d\sigma$ small enough to have

$\operatorname{dist}((1-|z_1|) z_2, X) < \varepsilon_2$

<u>Step 4:</u> Take $\varphi_k \in X$ s.t.

$$\left\| z_k \prod_{j=1}^{k-1} (1-|z_j|) - \varphi_k \right\| < \varepsilon_k$$

$$\sum_{k=1}^{\infty} |\varphi_k|_E \leq \sum_{k=1}^{\infty} \varepsilon_k + \sum_{k=1}^{\infty} \underbrace{|z_k|}_{1-(1-|z_k|)} \prod_{j=1}^{k-1} (1-|z_j|)$$

$$\leq \varepsilon + \sum_{k=1}^{\infty} \prod_{j=1}^{k-1} (1-|z_j|) - \prod_{j=1}^{k} (1-|z_j|) \leq C'$$

Moreover

$$|x_k^*(\varphi_k)| = \left|\int \varphi_k \, d\mu_k\right| \geq \left|\int z_k \prod_{j=1}^{k-1} (1-|z_j|) d\mu_k\right| - \varepsilon_k$$

$$\geq \underbrace{\left|\int \varepsilon_k \, d\mu_k\right|}_{\geq \frac{2\delta}{3}} - \underbrace{\left|\int \underbrace{1 - \prod_{j=1}^{k-1} (1-|z_j|)}_{\leq \sum_{j>1}^{k-1} |z_j|} \underbrace{z_k}_{|\cdot| \leq 1} \, d\mu_k\right|}_{\geq -\frac{\delta}{10}} - \varepsilon_k$$

$\square$