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I) The Gorelik principle ('96 Lindenstrauss, Schechtmann, Johnson) ① 2/3/16

<u>Lemma 0:</u> X B. space $X_0 \subseteq X$, $\dim X/X_0 < \infty$

Let $0 < c < d$. Then $\exists A$ compact in dB_X s.t.

$\forall \phi : A \longrightarrow X$ continuous s.t.

$\forall a \in A \quad \|\phi(a) - a\| \leq c$

Then $\phi(A) \cap X_0 \neq \emptyset$.

<u>Proof:</u> $\exists f : cB_{X/X_0} \longrightarrow dB_X$ continuous selection

$\forall z \in cB_{X/X_0} : Q(f(z)) = z$

$Q : X \longrightarrow X/X_0$ quotient map

Let $A = f(cB_{X/X_0})$ compact in dB_X

Consider $\phi : A \to X$ continuous with $\|\phi(a) - a\| \leq c$

Set $g : B_{X/X_0} \longrightarrow X/X_0$

$z \longmapsto z - (Q \circ \phi \circ f)(z) = Q(f(z) - \phi(f(z)))$

$\|f(z) - Q(f(z))\| \leq c + \|Q\| \leq 1 \Rightarrow g(cB_{X/X_0}) \subset cB_{X/X_0}$

Brouwer $\Rightarrow \exists z \in c B_{X/X_0}$ s.t. $g(z) = z$

i.e. $(Q \circ \Phi)(f(z)) = 0$

$\Rightarrow \exists a \in A \quad Q(\Phi(a)) = 0$ i.e. $\Phi(a) = X_0$ $\square$

Gorelik (version 1): X, Y B sp.

$f : X \longrightarrow Y$ homeo s.t. f^{-1} is M-Lip.

Let $d, b > 0$ s.t. $Mb = c < d$

Then $\forall X_0 \leq X$, $\dim X/X_0 < \infty$ $\exists K$ compact in Y s.t. $b B_Y \subset K + f(2d B_{X_0})$

Proof: Take A given by Lemma for $c < d$ and X_0 $(A \subset d B_X)$

Fix $y \in b B_Y$ define $\Phi : A \to X$
$a \mapsto f^{-1}(y + f(a))$

continuous

$$\|\Phi(a) - a\| = \|f^{-1}(y + f(a)) - f^{-1}(f(a))\| \leq$$

$$\leq M \|y\| \leqslant c$$

Lemma $\Rightarrow \exists a \in A$ s.t. $\Phi(a) \in X_0$

$\|\Phi(a)\| \leq c + \|a\| \leq 2d \Rightarrow \Phi(a) \in 2d B_{X_0}$

i.e. $f^{-1}(y + f(a)) \in 2d B_{X_0}$

i.e. $y \in K + f(2d B_{X_0})$ where $K = -f(A) \subset Y$ cpt

<u>Application</u>: $(y_k^*) \subset Y^*$, $y_k^* \xrightarrow{w^*} 0$, $\|y_k^*\| \geq \varepsilon$

fix $X_0 \subseteq X$, $\dim X/X_0 < \infty$

$\forall k\ \exists y_k \in bB_Y$ $\langle y_k^*, y_k \rangle \geq b\varepsilon$

But $y_k^* \to 0$ uniformly on K

$\exists (x_k) \subset 2dB_{X_0}$ $\underline{\lim} \langle y_k^*, x_k \rangle \geq b\varepsilon$

* Can be done so that (x_k) is w-null

"w^*-null sequences in Y are normed by f(w^*- null sequences) in X"

<u>Gorelik (version 2)</u>: X, Y B spaces, $f: X \to Y$ homeomorphism, f^{-1} uniformly continuous

Let $d, b > 0$ $\omega(f^{-1}, b) = c < d$

Then $\forall X_0 \subseteq X$ s.t. $\dim X/X_0 < +\infty$, $\exists K$ compact in Y s.t. $bB_Y \subset K + f(2dB_{X_0})$

(same proof)

Net equivalence of B. spaces

Let $0<a<b$, an (a,b)-net in a metric space (M,d) is $\mathcal{M}\subset M$ s.t. $\forall z\neq z'\in\mathcal{M} : d(z,z')\geq a$

$\forall x\in M\ \exists z\in\mathcal{M}\ d(z,x)<b$

Th.: (Lindenstrauss, Matoušková – Preiss)

X B. space, s.t. $\dim X=\infty$

Then any two nets in X are Lip-equivalent.

Th (BL book) $\exists$ net $\mathcal{M}$ in $\mathbb{R}^2$: $\mathcal{M} \underset{L}{\not\sim} \mathbb{Z}^2$

Def.: X, Y infinite dim Banach spaces

$X \underset{N}{\sim} Y$ if $\exists(\forall)$ nets $\mathcal{M}\subset X$, $\mathcal{N}\subset Y$

s.t. $\mathcal{M}\underset{L}{\sim}\mathcal{N}$.

Rem: * $X\underset{UH}{\sim}Y \Rightarrow X\underset{N}{\sim}Y$

* $X\underset{N}{\sim}Y \not\Rightarrow X\underset{UH}{\sim}Y$ (Kalton)

* $X\underset{L}{\sim}Y \Rightarrow X_{\mathcal U}\underset{L}{\sim}X_{\mathcal U}$ so net-equivalence preserves local properties of Banach spaces

Gorelik (version 3): X, Y B. sp., $X \underset{N}{\sim} Y$

Then $\exists\, c>0\ \exists f: X \longrightarrow Y\ \exists g: Y \longrightarrow X$ continuous

s.t. $\forall x \in X\quad \|(g\circ f)(x) - x\| \leq c$

$\forall y \in Y\quad \|(f\circ g)(y) - y\| \leq c$

And $M \geq 1\ \exists \alpha_0 > 0$ s.t. $\forall X_0 \subseteq X$, $\dim X/X_0 < \infty$

$\exists K \underset{cpt}{\subset} Y\ \forall \alpha > \alpha_0 \qquad \frac{\alpha}{16M} B_Y \subset K + c B_Y + f(\alpha B_{X_0})$

(Note: interesting only if $\frac{\alpha}{16M} \gg c$)

Moreover f and g are coarse Lipschitz

i.e. $\|f(x) - f(x')\| \leq D + M\|x - x'\|$

Proof: wma $\exists (x_i)_{i\in I}$, $(y_i)_{i\in I}$ $(1,\lambda)$ nets

in X and Y resp. s.t.

$\forall i,j.\quad \frac{1}{M}\|x_i - x_j\| \leq \quad \|y_i - y_j\| \leq M\|x_i - x_j\|$

Let $(\varphi_i)_{i\in I}$ be a partition of unity subordinate

to $(B(x_i,\lambda))_{i\in I}$

– $(\psi_i)_{i\in I}$ ———— $(B(y_i,\lambda))_{i\in I}$

$$f(x) = \sum_{i \in I} f_i(x)\, y_i \quad , \quad g(y) = \sum_{i \in I} g_i(y)\, x_i$$

Lemma: $\forall x \in X \quad \|x - x_i\| \leq r \Rightarrow \|f(x) - y_i\| \leq M(\lambda + r)$

$\forall y \in Y \quad \|y - y_i\| \leq r \Rightarrow \|g(y) - x_i\| \leq M(\lambda + r)$

Pf.: $f_j(x) \neq 0 \Rightarrow \|x - x_j\| \leq \lambda \Rightarrow \|x_i - x_j\| \leq \lambda + r$

$$\Downarrow$$

$$\|y_i - y_j\| \leq M(\lambda + r)$$

$$\|f(x) - y_i\| = \Big\| \sum_{j \mid f_j(x) \neq 0} f_j(x)(y_j - y_i) \Big\| \leq M(\lambda + r) \quad \square$$

Now: Let $x \in X$. Pick $i \in I$ s.t. $\|x - x_i\| \leq \lambda$

Lemma 1 $\Rightarrow \|f(x) - y_i\| \leq 2M\lambda$

$\Rightarrow \|g \circ f(x) - x_i\| \leq M(\lambda + 2M\lambda)$

$\Rightarrow \|g \circ f(x) - x\| \leq \lambda + M(\lambda + 2M\lambda) =: c$

$\square$

Lemma 2: f and g are coarse Lip.

Pf.: $x, x' \in X$, $i \in I$ $\|x - x_i\| \leq \lambda$, $j \in I$ $\|x' - x_j\| \leq \lambda$

$\|f(x) - y_i\| \leq 2M\lambda$, $\|f(x') - y_j\| \leq 2M\lambda$

$\Rightarrow \|y_i - y_j\| \leq M\|x_i - x_j\| \leq 2M\lambda + M\|x - x'\|$

Finally $\|f(x)-f(x')\| \le 6M\lambda + M\|x-x'\|$ $\square$

$\mathrm{Lip}_\infty(f) \le M$, $\mathrm{Lip}_\infty(g) \le M$ optimal

Recall: $\mathrm{Lip}_s(f) = \sup\left\{ \frac{\|f(x)-f(x')\|}{\|x-x'\|}, \|x-x'\| \ge s \right\}$

$$\mathrm{Lip}_s(f) \underset{s\to\infty}{\searrow} \mathrm{Lip}_\infty(f)$$

So $\underset{N}{\sim} \Rightarrow$ "$\underset{CL}{\sim}$" (? "$\Leftarrow$")

Proof of corollary (3): $\alpha > 0$, Let $X_0 \subset X$, $\dim X/X_0 < \infty$

~~Let A~~

$y \in \frac{\alpha}{16M} B_Y$

* Lemma 0 $\Rightarrow \exists A$ in $\frac{\alpha}{2} B_X$ s.t.

 $\forall \phi : A \to X$ satisfying $\|\phi(a)-a\| \le \frac{\alpha}{4}$, $a \in A$

 we have $\phi(A) \cap X_0 \ne \emptyset$

* $\phi : A \to X$ continuous

 $a \mapsto g(y+f(a))$

$$\|\phi(a)-a\| \le \|g(y+f(a))-a\| \le \|g(y+f(a))-g(f(a))\| + \|g(f(a))-a\|$$

$$\le C + \|g(y+f(a))-g(f(a))\| \le$$

$\leq \frac{\alpha}{8} + \|g(y+f(a)) - g(f(a))\|$ if $\alpha > 8C$

Pick $i \in I$ $\quad \|f(a) - g_i\| \leq \lambda$, $j \in I$ $\|y+f(a) - g_j\| \leq \lambda$

$\|g(f(a)) - x_i\| \leq 2M\lambda$ ....

<u>infact</u>

g is coarse Lip $\Rightarrow$

$\Rightarrow \|g(y+f(a)) - g(f(a))\| \leq 6M\lambda + M\|y\|$

$\leq 6M\lambda + \frac{\alpha}{16} \leq \frac{\alpha}{8}$ if $\alpha > 96M\lambda$

Lemma 0 $\Rightarrow \exists a \in A$ $\phi(a) \in X_0$

$\|a\| \leq \frac{\alpha}{2}$, $\|\phi(a) - a\| \leq \frac{\alpha}{4} \Rightarrow \|\phi(a)\| \leq \alpha$

$\phi(a) \in \alpha B_{X_0}$

Finally $\|f(\phi(y)) - y + f(a)\| = \|f \circ g(y+f(a)) - (y+f(a))\| \leq C$

So $y \in K + CB_Y + f(\alpha B_{X_0})$ where $K = -f(A)$ $\square$