Quantum stochastic integration

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16 4 2012

as a gen$^{\text{ion}}$ of classical stochastic integration.

- $(X_t)_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\geq 0}$ if
  - $X_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ for all $t \geq 0$
  - $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$ for all $t \geq s \geq 0$.
  - by convention, if $X_0 = 0$ and $(X_t^2 - t)_{t\geq 0}$ is also a $(\mathcal{F}_t)$-martingale then X is a normal martingale.

Examples: 1 Brownian motion = Wiener process
2 Compensated Poisson process.

1) Wiener process:

A stochastic process $(X_t)_{t\geq 0}$ is a std Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$ if

- $X_0 = 0$
- $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega$
- $\mathbb{P}[X_t - X_s \in A] = \frac{1}{\sqrt{2\pi(t-s)}} \int_A e^{-\frac{x^2}{2(t-s)}} dx$ for all $t > s \geq 0$.
- $\{X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}\}$ are independent if $t_n \geq t_{n-1} \geq \dots \geq t_0 \geq 0$

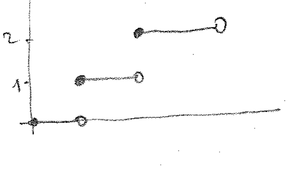
If X is a std Brown motion and $\mathcal{F}_t = \sigma(X_s - X_r : t \geq s > r \geq 0)$, then X is a normal martingale.

Proposition: TFAE: • X is a normal martingale for $(\mathcal{F}_t)_{t\geq 0}$
• $X_t \in L^2(\Omega, \mathcal{F}_t, \mathbb{P})$ and such that $\mathbb{E}[X_t - X_s | \mathcal{F}_s] = 0$ cond. expectation
$\mathbb{E}[(X_t - X_s)^2 | \mathcal{F}_s] = t - s$ cond. variance

Theorem (P. Levy) If X is a normal martingale with $t \mapsto X_t(\omega)$ continuous a.s., then X is a std Brown motion.

2) Poisson process: Let (N_t) be a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with

- $N_0 = 0$
- $t \mapsto N_t(\omega)$ are nondecreasing, integer-valued, right-continuous
- $\mathbb{P}[N_t - N_s = n] = \frac{(t-s)^n}{n!} e^{-(t-s)}$ $(n \geq 0)$



- $\{N_{t_1} - N_{t_0}, N_{t_2} - N_{t_1}, \dots, N_{t_n} - N_{t_{n-1}}\}$ is independent for all $0 \leq t_0 \leq t_1 \leq \dots \leq t_n$.
- N is the standard Poisson process and $(\underbrace{N_t - t}_{\hat{N}_t})_{t\geq 0}$ is the compensated Poisson process.

If $\mathcal{F}_t = \sigma(\hat N_t - \hat N_r : 0 \le r < s \le t)$, then $\mathbb{E}[\hat N_t - \hat N_s \mid \mathcal{F}_s] = \mathbb{E}[\hat N_t - \hat N_s]$

$= \mathbb{E}[N_t - N_s - (t-s)] = 0$

and $\mathbb{E}[(\hat N_t - \hat N_s)^2 \mid \mathcal{F}_s] = \mathbb{E}[((N_t - N_s) - (t-s))^2] = \mathbb{E}[(N_t - N_s)^2] - (t-s)^2 = t-s$

→ Wiener Paley Zygmund

↳ Applied maths → you want to differentiate Brownian motion, but you cannot do that!

With probability 1, $t \mapsto B_t(\omega)$ is not differentiable anywhere.

In fact, $\mathbb{E}[(B_t - B_s)^2] = t-s$ → hölder 1/2

→ But you can use Brown motion as an integrator.

Stochastic integration for normal martingale:

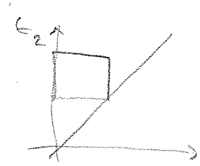
→ Make sense of $\int_0^t f(s)\, dX_s$, where $f \in L^2(\mathbb{R}_+)$ and X is a normal martingale.

→ well defined for a step function $t_1 < \dots < t_n$.

→ More generally, we might look at $I_n(f) = \int_0^{t_n} \cdots \int_0^{t_1} f(t_1, \dots, t_n)\, dX_{t_1} \cdots dX_{t_n} = \int_{\Delta_n(t)} f(\underline{t})\, dX_{\underline{t}}$, $\underline{t} = (t_1, \dots, t_n)$.

Suppose $A = (r_1, s_1] \times \dots \times (r_n, s_n]$ (*) where $0 \le r_1 < s_1 \le r_2 < s_2 \le \dots \le r_n < s_n$.

Let $I_n(1_A) = (X_{s_1} - X_{r_1}) \cdots (X_{s_n} - X_{r_n})$



1st surprising fact: this will be integrable!

Lemma: $I_n(1_A) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$; if B is another set of the form (*),

then $\mathbb{E}[I_n(1_A) I_n(1_B)] = \int_{A \cap B} dt_1 \dots dt_n = \langle 1_A, 1_B \rangle_{L^2(\Delta_n)}$

where $\Delta_n = \{(t_1, \dots, t_n) \in \mathbb{R}_+^n : t_1 < t_2 < \dots < t_n\}$ equipped w. Lebesgue measure.

Remark. This shows that if we extend I_n by linearity to

$H_0 = \text{lin}\{1_A : A = (r_1, s_1] \times \dots \times (r_n, s_n]\}$, then

I_n is an isometry from $H_0 \subseteq L^2(\Delta_n)$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Proof. With A as above, $I_n(1_A) = (X_{s_1} - X_{r_1}) \cdots (X_{s_n} - X_{r_n})$

For $n = 1$, $\mathbb{E}[I_n(1_A)^2] = \mathbb{E}[(X_{s_1} - X_{r_1})^2] = \mathbb{E}[\mathbb{E}[(X_{s_1} - X_{r_1})^2 \mid \mathcal{F}_{r}]]$

$= \mathbb{E}[s_1 - r_1]$ ∵ X is a normal martingale

$= s_1 - r_1$.

For general n, let $k_1, \dots, k_{n-1} \in \mathbb{N}$ and note

that $\mathbb{E}[(k_1 \wedge (X_{s_1} - X_{r_1})^2) \cdots (k_{n-1} \wedge (X_{s_{n-1}} - X_{r_{n-1}})^2)(X_{s_n} - X_{r_n})^2]$

$= \mathbb{E}[\mathbb{E}[\sim \mid \mathcal{F}_{r_n}]] = \mathbb{E}[(k_1 \wedge (X_{s_1} - X_{r_1})^2) \cdots (k_{n-1} \wedge (X_{s_{n-1}} - X_{r_{n-1}})^2)\, \mathbb{E}[(X_{s_n} - X_{r_n})^2 \mid \mathcal{F}_{r_n}]]$

$= (s_n - r_n)\, \mathbb{E}[\tilde{\sim}]$ and then let $k_{n-1} \to \infty$ and use monotone convergence.

You get rid of k_{n-1} and thus successively of $k_{n-2},\dots,k_1$:

$$\mathbb{E}[I_n(A)^2] = (s_1-r_1)\cdots(s_n-r_n) = \int_A dt_1\cdots dt_n .$$

Exercise: use a similar argument to show that $\mathbb{E}[I_n(1_A)I_n(1_B)] = \int_{A\cap B} dt_1\cdots dt_n$

→ You can look at P.-A. Meyer, Un cours d'intégration stochastique, Sém. Prob. X, Chap IV, §§45-47.

↳ Trick: If A,B have the special product form, then you may suppose $]r_i,s_i] =]p_i,q_i]$ or $\cap = \emptyset$ (by taking the parent partition

<< the mark live in $\bigotimes_{t\geq 0} L^2(\mathbb{R}_+)$". One is not "really" taking the product $(X_{s_1}-X_{r_1})(X_{s_2}-X_{r_2})$.

<u>Theorem</u>: If X is a normal martingale, then for all $n\geq 1$, there is an isometry $I_n : L^2(\Delta_n) \longrightarrow L^2(\Omega,\mathcal{F},\mathbb{P})$ s.t. $I_n(1_A) = (X_{s_1}-X_{r_1})\cdots(X_{s_n}-X_{r_n})$ and $\mathbb{E}[I_m(f)I_n(g)] = 0$ if $m\neq n$.

Sketch of proof for the last part: If $f=1_A$ and $g=1_B$, with A and B as before, and $]r_i,s_i]$ and $]p_j,q_j]$ disjoint or equal for all i and j,

then $I_m(1_A)I_n(1_B) = \prod_{j=1}^{k}(X_{b_j}-X_{a_j})^{m_j}$ and $m_j\in\{1,2\}$

and at least one $m_j = 1$! [Split the product at the highest a_j with $m_j=1$]

<u>Definition</u>: Let $\mathcal{H}_n = I_n(L^2(\Delta_n))$ for all $n\geq 1$ and let $\mathcal{H}_0 = \mathbb{R}1_\Omega \in L^2(\Omega,\mathcal{F},\mathbb{P})$.

These are closed, mutually orthogonal subspaces of $L^2(\Omega,\mathcal{F},\mathbb{P})$ and their sum

$\mathcal{H} = \bigoplus_{n=0}^{\infty}\mathcal{H}_n$ is the chaos space of $L^2(\Omega,\mathcal{F},\mathbb{P})$. If $\mathcal{H} = L^2(\Omega,\mathcal{F},\mathbb{P})$, then X has <u>CRP</u>

CRP = chaotic representation property

cf. Wiener, Differential chaos, → Brown motion

<u>Theorem</u> [Wiener] Brown motion has the CRP.

[Throughout, $\mathcal{F} = \sigma(\bigcup_{t\geq 0}\mathcal{F}_t)$]

<u>Theorem</u>: The compensated Poisson process has the CRP

Theorem: [Émery 1989] There exists a normal martingale without the CRP.

Question: Which normal martingales have the CRP ?

<u>More general stochastic integration:</u> What about $\int_0^t F_s\,dX_s$ if F is a stochastic process on $(\Omega,\mathcal{F},\mathbb{P})$

Let $E = C\times]r,s]$, where $C\in\mathcal{F}_r$ and $s>r\geq 0$.

Then $\int_0^\infty 1_E(t)\,dX_t := 1_C(X_s-X_r)$ and $\mathbb{E}[1_C(X_s-X_r)^2] = \mathbb{E}[\mathbb{E}[1_C(X_s-X_r)^2|\mathcal{F}_r]]$

$= \mathbb{E}[1_C](s-r)$

By linearity, we get an isometry for $\mathcal{H}_0 = \mathrm{lin}\{1_{C \times]r,s]} : s > r \geq 0,\ C \in \mathcal{F}_r\}$ equipped with the inner product $\langle f, g \rangle = \mathbb{E}\left[\int_0^\infty f(s) g(s)\, ds\right]$ into $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem: $\mathcal{H}_0$ is dense in $\left\{F : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R} \;\middle|\; \mathbb{E}\left[\int_0^\infty F(s)^2 ds\right] < \infty : F \text{ is previsible}\right\}$

more than being adapted.