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6.— Unitary coactions and corepresentations

<u>Def</u>: • A <u>corepresentation</u> $c = (c_{jk}) \in M_n(A)$ for a $*$-Hopf algebra is called <u>nondegenerate</u> if c is invertible in $M_n(A)$; it is <u>unitary</u> if $cc^* = c^*c = I$, where $c^* = (c^*_{kj})_{1\le j,k\le n}$; it is <u>unitarisable</u> if it is equivalent to a unitary corep: there is $A \in M_n(\mathbb{C})$ s.t. AcA^{-1} is unitary

• A coaction $\gamma: V \longrightarrow V \otimes A$ is called unitary (wrt an inner product $\langle\,,\,\rangle$ on V) if $\sum \langle v_{(1)}, w_{(1)}\rangle v_{(2)}^* w_{(2)} = \langle v, w\rangle 1$ (i.e., is constant)
$\langle\langle \gamma(v), \gamma(w)\rangle\rangle$, where $\langle\langle v\otimes a, w\otimes b\rangle\rangle = \langle v, w\rangle a^* b$

<u>Proposition</u>: γ a corepresentation on V, $e_1, \dots, e_n$ an on basis of V, $\gamma(e_j) = e_k \otimes c_{kj}$

Then TFAE:
(a) γ is unitary
(b) $\sum c_{\ell j}^* c_{\ell k} = \delta_{jk} 1$ [i.e., $c^*c = I$]
(c) $s(c_{jk}) = c^*_{kj}$
(d) $\sum c_{j\ell} c_{k\ell}^* = \delta_{jk} 1$ [i.e, $cc^* = I$]
(e) c is unitary

<u>Proof</u>: <u>(a) ⇒ (b)</u>

$$\delta_{jk} 1 = \langle\langle \gamma(e_j), \gamma(e_{j'})\rangle\rangle = \sum \sum \langle\langle e_k \otimes c_{kj}, e_{k'} \otimes c_{k'j'}\rangle\rangle$$
$$= \sum \sum \langle e_k, e_{k'}\rangle c_{kj}^* c_{k'j'}$$
$$= \sum c_{kj}^* c_{kj'}$$

<u>(b) ⇒ (c)</u>

$$s(c_{jk}) = \sum \delta_{j\ell}\, s(c_{\ell k})$$
$$= \sum c_{pj}^* \underbrace{c_{p\ell}\, s(c_{\ell k})}_{\substack{m\circ(id\otimes s)\circ\Delta c_{pk} \\ = \varepsilon(c_{pk}) = \delta_{pk}}} = c_{kj}^*$$

<u>(c) ⇒ (d)</u>

$$\delta_{jk} 1 = \varepsilon(c_{jk}) 1$$
$$= \sum c_{j\ell}\, s(c_{\ell k}) = \sum c_{j\ell} c_{k\ell}^* = \varepsilon(c_{pk}) = \delta_{pk}.$$

We needed in fact: <u>Def</u>: Two coactions $\gamma: V \longrightarrow V \otimes A$, $\delta: W \longrightarrow W \otimes A$. $L: V \longrightarrow W$ is called an intertwiner between γ and δ if $\delta \circ L = (L \otimes id) \circ \gamma$.

$$\begin{array}{ccc} V & \xrightarrow{\gamma} & V \otimes A \\ L\downarrow & & \downarrow L\otimes id \\ W & \xrightarrow{\delta} & W \otimes A \end{array}$$

If $c, d \in M_n(A)$ corepresentations associated to γ, δ, $n = \dim V$, $m = \dim W$:

$$\underset{M_n(A)}{d} \, L = \underset{M_{m\times n}(\mathbb{C})}{L} \, c \quad \in M_{m\times n}(A).$$

We call γ and δ (or c and d) equivalent if there is a bijective intertwine between them.

<u>Proposition</u>: $\gamma : V \longrightarrow V \otimes A$, $W \subset V$ an invariant subspace. Then $W^\perp$ also is.

<u>Proof</u>: Consider $v \in W^\perp$, $\gamma(v) = \overset{\in V}{v_{(1)}} \otimes \overset{\in A}{v_{(2)}}$. If $w \in W$,

$$\langle v_{(1)}, w\rangle v_{(2)}^* = \langle v_{(1)}, w_{(1)}\rangle v_{(2)}^* w_{(2)} s(w_{(3)}) \quad \text{by the antipode axiom}$$
$$= \langle v, \underset{W}{w_{(1)}}\rangle s(w_{(2)}) = 0$$

<u>Lemma</u>: ⓐ If $L : V \longrightarrow W$ is an intertwiner between two coactions $\gamma : V \longrightarrow V\otimes A$, $\delta : W \longrightarrow W \otimes A$,

then $L(V)$ and $L^{-1}(\{a\}) = \ker L$ are invariant subspace.

ⓑ (first Schur lemma) If γ and δ are irreducible then $L = 0$ or L is bijective.

ⓒ (second Schur lemma) If $\gamma = \delta$ are irreducible, then $L = \lambda I$ for some $\lambda \in \mathbb{C}$.

<u>Proof</u>. $L(V) \subset W$ being invariant follows directly from $\delta \circ L = (L \otimes id)\circ \gamma$.

If $v \in \ker L$, then $0 = \delta(L(v)) = (L \otimes id)\,\gamma(v) = L(v_{(1)}) \otimes v_{(2)}$,

so $\gamma(v) \in (\ker L) \otimes A$

<u>Remark</u>: ① If $\gamma : V \longrightarrow V \otimes A$ is a coaction, then $\gamma^* : A^* \otimes V \longrightarrow V$;

If $f \in A^*$, $\gamma^*(f \otimes v) = f(v_{(2)}) v_{(1)} = (id \otimes f)\circ \gamma(v)$.

We also write $\gamma^*(f) \in \mathrm{Lin}(V)$, with $\gamma^*(f)v = \gamma^*(f \otimes v)$. This defines a corepresentation of A^* acting on V.

In fact: $\gamma^*(\varepsilon)v = v_{(1)}\, \varepsilon(v_{(2)}) = v$, so $\gamma^*(\varepsilon) = id_V$;

$$\gamma^*(g)(\gamma^*(f)v) = \gamma^*(g)\, f(v_{(2)})\, v_{(1)} = f(v_{(3)}) g(v_{(2)}) v_{(1)}$$
$$= (g \otimes f)\Delta(v_{(2)}) v_{(1)}$$
$$= (g \star f)(v_{(2)}) v_{(1)} = \gamma^*(g \star f)v.$$

so it is a corepresentation.

② If A is a bialgebra or a C^*-bialgebra, then A^* is a unital algebra.

If $\dim A < \infty$, then A^* becomes a bialgebra because $m^* : A^* \to (A \otimes A)^* \simeq A^* \otimes A^*$ defines a coproduct.

If A is a Hopf algebra, $\dim A < \infty$, then A^* is again a Hopf algebra, with $s = s^*$ and $s(f) = f \circ s$.

If A is a $*$-Hopf-algebra, then A^* is a $*$-algebra with $f^*(a) = \overline{(f(s(a))^*)}$

If $\dim A < \infty$, then A^* is even a $*$-Hopf-algebra.

If $\gamma : V \longrightarrow V \otimes A$ is unitary, then γ^* is a $*$-representation:

$$\langle v, \gamma^*(f|w\rangle = \langle \overset{v_{(1)} \underbrace{v_{(2)} s(v_{(3)})}_{\varepsilon(v_{(2)})}}{\overset{\|}{v}}, w_{(1)}\rangle f(w_{(2)}) = \langle v_{(1)}, w_{(1)}\rangle f((s(v_{(3)}))^* v_{(2)}^* w_{(2)}^*$$

$$= f((s(v_{(2)}))^* \langle v_{(1)}, w\rangle$$

$$= \overline{\langle f((s(v_{(2)}))^*)} v_{(1)}, w\rangle$$

$$\langle \gamma^*(f^*) v, w\rangle$$

<u>Prop and Def</u>: Let $c = (c_{jk}) \in M_n(\bar{A})$ be a unital matrix representation then $c' = (s(c_{kj}))_{1 \le j,k \le n}$ is also a (unital) matrix corepresentation, the <u>contragredient</u> corepresentation of C.

If S is invertible, then C irreducible $\Leftrightarrow$ C' irreducible.

<u>Proof</u>: Consider $c'_{jk} = s(c_{kj})$ and $\Delta c'_{jk} = \Delta s(c_{kj}) = (s \otimes s) \circ \tau \circ \Delta c_{kj}$

But $\varepsilon \circ s = \varepsilon$. This implies c unital $\Leftrightarrow$ c' unital.
$$= (s \otimes s) \circ \tau \sum (c_{kl} \otimes c_{lj})$$
$$= \sum_l \underset{c'_{jl}}{s(c_{lj})} \otimes \underset{c'_{lk}}{s(c_{kl})}$$

[We have $Ac' = c'A \Rightarrow CA^T = A^T C$ by applying s^{-1} componentwise and taking transpose]

<u>Def</u>: Let A be a Hopf algebra. Then $\gamma_R = \Delta : A \to A \otimes A$ is called the <u>right regular coaction</u>.
The <u>left regular coaction</u> is $\gamma_L = \tau \circ (s \otimes id) \circ \Delta : A \to A \otimes A$
$$a \longmapsto a_{(2)} \otimes s(a_{(1)})$$

They decompose into an infinite direct sum of f.d. irreducible coactions

<u>Remark</u>: By the fundamental theorem of coalgebras, each element of A is contained in a f.d subcoalgebra ($=$ f.d. γ_R-invariant subspace)
thus γ_R (and γ_L) are "highly" reducible.

Lemma: Let $\gamma : \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} V_{ij} = V \longrightarrow V \otimes A$ be a coaction of A on a f.d. vectorspace V s.t.
$\bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m_i} \gamma_{ij}$ the γ_{ij} are equivalent to some irreducible coaction γ_i,
where $\gamma_1, \dots, \gamma_n$ are mutually inequivalent " $\gamma = \bigoplus_{i=1}^{n} m_i \gamma_i$ "
Let $U \subset V$ be an invariant subspace of V s.t. $\delta = \gamma|_U$ is irreducible
Then there is a unique $i \in \{1, \dots, n\}$ s.t. $\underset{\text{equivalent}}{\delta \sim \gamma_i}$ and $U \subseteq \bigoplus_{j=1}^{m_i} V_{ij}$

Proof: • Denote by $P_{ij} : V \longrightarrow V_{ij}$ the projection onto P_{ij}. Then $(P_{ij} \otimes id) \circ \gamma = \gamma_{ij} \circ P_{ij}$
and $Q_{ij} = P_{ij}|_U$ $(Q_{ij} \otimes id) \circ \delta = \gamma_{ij} \circ Q_{ij}$

By Schur's lemma, $Q_{ij} = 0$ or Q_{ij} is bijective.
If we had $Q_{i_1 j_1}, Q_{i_2 j_2} \neq 0$, then $\gamma_{i_1 j_1} \sim \delta \sim \gamma_{i_2 j_2}$, so $i_1 = i_2$.
Therefore there exists a unique $i \in \{1, \dots, n\}$ st $Q_{kj} = 0$ if $k \neq i$. Then
$U \subset \bigoplus_{k=1}^{n} \bigoplus_{j=1}^{m_k} Q_{kj} U = \bigoplus_{j=1}^{m_i} Q_{ij} U = \bigoplus_{j=1}^{m_i} V_{ij}$.

Lemma: A a Hopf algebra with invertible antipode and γ, δ two irred. coactions on f.d. vector spaces. Then $\gamma \sim \delta$ $\quad \underset{(c_{jk})}{\gamma}, \underset{(d_{jk})}{\delta}$

$$\Updownarrow$$

$$\mathrm{span}\{c_{jk}\} = \mathrm{span}\{d_{jk}\}$$

Proof: "$\Rightarrow$" is immediate: $d = A c A^{-1}$
"$\Leftarrow$" $V = \mathrm{span}\{c_{jk}\} = \mathrm{span}\{d_{jk}\}$, $(\dim V \overset{?}{=} n^2)$
$\gamma_R|_V = \Delta|_V$. Fix k: $v_i = c_{ki}$, $\Delta v_i = c_{kj} \otimes c_{ji}$, $U_k = \mathrm{span}\{c_{k1}, \dots, c_{kn}\}$
Then $\gamma_R|_V \sim \bigoplus_n \gamma \sim \bigoplus_n \delta$: $V = \bigoplus_{k=1}^{n} U_k$

Lemma: $\gamma \cong c$ an irred f.d. corep, A Hopf algebra with invertible antipode.
Then $\{c_{jk} : 1 \leq j, k \leq n\}$ is linearly independent.

Lemma $\boxtimes$ irr: $\gamma : V \longrightarrow V \otimes A$, $\delta : W \longrightarrow W \otimes A$ irred f.d. coaction.
(to be proved later) Then $\gamma \boxtimes \delta : V \otimes W \longrightarrow V \otimes W \otimes A \otimes A$ is irred.

We consider $\underset{\gamma_R \boxtimes \gamma_L}{\gamma} : \underset{V}{\mathrm{span}\{c_{jk}\}} \longrightarrow V \otimes A \otimes A$
$a \longmapsto a_{(2)} \otimes S(a_{(1)}) \otimes a_{(3)}$
$c_{jk} \longmapsto c_{pq} \otimes S(c_{jp}) \otimes c_{qk}$

$e_k \longmapsto e_\ell \otimes c_{\ell k}$ are irred.
$e_k \longmapsto c_\ell \otimes S(c_{k\ell})$

γ is equivalent to the coaction on $\mathbb{C}^n \otimes \mathbb{C}^n$ defined by
$e_j \otimes e_k \longmapsto e_p \otimes e_q \otimes S(c_{jp}) \otimes c_{qk}$

By lemma $^{\boxtimes\text{irr}}$, $\delta = \gamma \boxtimes \gamma'$ is irreducible. Clearly $\delta \sim \tilde{\gamma}$, so $\dim V = n^2$

Then $\gamma_R|_V \sim \bigoplus_n \gamma \sim \bigoplus_n \delta \overset{\text{Lemma}}{\Longrightarrow} \gamma \sim \delta$ ▪ But there should be a direct proof with involution and antipode.

Let us improve the penultimate lemma:

Theorem: $\mathcal{A}$ Hopf algebra with involutive antipode. Let C^α, $\alpha \in I$, be a family of mutually inequivalent irreducible matrix corep^s, $\dim C^\alpha = d_\alpha$. Then $\{c^\alpha_{jk} : \alpha \in I, 1 \le j, k \le d_\alpha\}$ is linearly independent.

This follows from the dichotomy: either $\operatorname{span}\{c_{jk}\} = \operatorname{span}\{d_{jk}\}$ and $\gamma \sim \delta$,
or $\operatorname{span}\{c_{jk}\} \cap \operatorname{span}\{d_{jk}\} = \{0\}$ and $\gamma \not\sim \delta$.

Def: A linear functional $f : A \to \mathbb{C}$ is called an integral if $(\mathrm{id} \otimes f) \circ \Delta(a) = (f \otimes \mathrm{id}) \circ \Delta(a) = f(a) 1$
If $f : A \to \mathbb{C}$ is furthermore a faithful state, i.e. $f(1) = 1$, and $f(a^* a) > 0$ for $a \in A \setminus \{0\}$, then we call it a Haar state.

Theorem: A $*$-Hopf-algebra. Tfae:

(a) $\mathcal{A} = \operatorname{span}\{u^\alpha_{jk}, u^\alpha = (u^\alpha_{jk})$ are f.d. unitary corepresentations$\}$
$= \operatorname{span}\{u^\alpha_{jk} : \alpha \in I, 1 \le j, k \le d_\alpha\}$, where $\{u^\alpha : \alpha \in I\}$ is a maximal collection of mutually inequivalent irred. unitary corep^s

(b) $\mathcal{A}$ admits a Haar state

(c) $\mathcal{A}$ is the dense $*$-Hopf-algebra of a compact quantum group $C(G)$ (cf. Moes, van Daele, Woronowicz)

Def: in this case, we call $\mathcal{A}$ a Woronowicz $*$-algebra or CQG $*$-algebra

Proof: (b) $\to$ (a). Assume that $\mathcal{A} = \bigoplus \operatorname{span}\{c^\alpha_{jk} : 1 \le j, k \le d_\alpha\}$, where the c^α are mutually inequivalent irred. f.d. corep^s [proof that $\mathcal{A}$ decomposes in this way]

Lemma: If A has a Haar state, then any f.d. irred. corep. can be unitarised:
$(v, w) = \langle v_{(1)}, w_{(1)} \rangle \, h(v_{(2)}^* w_{(2)}) = \langle \gamma(v), \gamma(w) \rangle_\otimes$, where

$$\langle v\otimes a, w\otimes b\rangle_\otimes = \langle v,w\rangle \underbrace{\langle a,b\rangle_h}_{h(a^*b)}.$$

We look at $(v_{(1)}, w_{(1)}) v_{(2)}^* w_{(2)} = \langle v_{(1)}, w_{(1)}\rangle \underbrace{h(v_{(2)}^* w_{(2)})\, v_{(3)}^* w_{(3)}}_{(h\otimes id)\Delta(v_{(2)}^* w_{(2)}) = h(v_{(2)}^* w_{(2)})1}$

$= (v,w)1$

This is just like Hopf-algebra translation of the group case.

<u>Converse</u>: $(a) \Rightarrow (b)$.

<u>Lemma</u>: We define $h: A \longrightarrow \mathbb{C}$ by $h(u^\alpha_{jk}) = \begin{cases} 1 & \text{if } u^\alpha = 1 \\ 0 & \text{else}\end{cases}$.

This is an integral on A. Furthermore $h(S(a)) = h(a)$ for all $a \in A$.

<u>Proof</u>: $(h\otimes id)\Delta(u^\alpha_{jk}) = \sum_l h(u^\alpha_{jl}) u^\alpha_{lk} = h(u^\alpha_{jk})1 = \dots = (id\otimes h)\Delta(u^\alpha_{jk})$ for $\alpha \in I$ and all j,k.

If $c \sim 1$, then $c' \sim 1$. Since $S(1)=1$, $1'=1$.
This implies that $h(S(u^\alpha_{jk})) = \left.\begin{cases}1 & \text{if } u^\alpha = 1\\ 0 & \text{else}\end{cases}\right\} = h(u^\alpha_{jk})$

<u>Lemma</u>: If c,d are two matrix corep^s of A, then $A^{(jk)} = (A^{(jk)}_{il}) = h(c_{ij} S(d_{kl}))$
$B^{(jk)} = (B^{jk}_{il}) = h(S(d_{ij}) c_{kl})$

intertwine between c and d, $A^{(jk)} d = c A^{(jk)}$

<u>Proof</u> $\sum_l h(c_{ij}\; s(d_{kl})) d_{lp} \overset{?}{=} c_i\, h(c_j s(d_{kp}))$ $c = d\, B^{(jk)}$ for all j,k

$\| $

$(id\otimes h)\Delta(c_{ij} s(d_{kl})) = \sum_{r,s} c_{ir}\, s(d_{sl})\, h(c_{rj}\, s(d_{ks}))$

$= \sum_{l,r,s} h(c_{rj}\, s(d_{ks}))\, c_{ir} \underbrace{s(d_{sl}) d_{lp}}_{\varepsilon(d_{sp}) = \delta_{sp}} = \sum_r h(c_{rj}\, s(d_{kp}))\, c_{ir}$ ✓

Consider c,d irreducible. <u>Case 1</u>: $c \not\sim d$. Then $A^{(jk)} = 0 = B^{(jk)}$ for all j,k.
Then $h(c_{ij}\, s(d_{kl})) = 0$, $h(s(d_{ij}) c_{kl}) = 0$ for all i,j,k,l.

<u>Case 2</u>: If $c = d$, then there exist constants α_{jk}, β_{jk} st $A^{(jk)} = \alpha_{jk} I$, $B^{(jk)} = \beta_{jk} I$.
So $A^{(jk)}_{il} = h(c_{ij}\, S(d_{kl})) = \alpha_{jk}\delta_{il}$, $B^{(jk)}_{il} = h(S(d_{ij}) c_{kl}) = \beta_{jk}\delta_{il}$.

Then $\sum^{(j=k,\, i=l)} \alpha_{jj} = h(c_{ij}\; S(c_{ji})) = h(\varepsilon(c_{ii})1) = 1 = \sum_{j=1}^{m} \beta_{jj}$

Case 3 $\quad c = d'' = \left(s^2(d_{jk})\right)$. $\sum h(d_{k\ell}\, s(d_{ij}))\, d_{\ell m} = \sum s^2(d_{i\ell})\, h(d_{km}\, s(d_{\ell j}))$

As $Ac = c'A$, and $Bc' = cB$, $\sum h(\underset{c''}{s^2(d_{ij})}\, s(d_{k\ell}))\, d_{\ell m} = \sum h(s^2(d_{\ell j})\, s(d_{km}))\, s^2(d_{i\ell})$

$Bd = cB$

Proposition: $\mathcal{A}$ $*$-Hopf algebra with Haar state h, c an irreducible corep. of $\mathcal{A}$

(a) d an irred. corep. that is not equivalent to c, then
$h(c_{ij}\, s(d_{k\ell})) = 0 = h(s(c_{ij})\, d_{k\ell})$ for all i,j,k,ℓ.
$d^*_{\ell k}$ if d unitary.

(b) c is equivalent to c'', and if $F \neq 0$ is an intertwiner for c and c'', then
$\operatorname{tr} F \neq 0$, $\operatorname{tr} F^{-1} \neq 0$ and $h(c_{k\ell}\, s(c_{ij})) = \delta_{kj} \dfrac{F_{i\ell}}{\operatorname{tr} F}$
$h(s(c_{k\ell})\, c_{ij}) = \delta_{kj} \dfrac{(F^{-1})_{i\ell}}{\operatorname{tr} F^{-1}}$ } Peter-Weyl-Woronowicz orthogonality relations.

Proposition: c an irreducible unitary matrix corep., $F \neq 0$ an intertwiner between c and c''. Then F is a multiple of a positive matrix and admits a unique normalisation such that $\operatorname{tr} F = \operatorname{tr} F^{-1} > 0$. This is the quantum dimension of c.

→ consider c': it is unitarisable and $AC'A^{-1} = u$ for some invertible A and a corep. u that is unitary irreducible. Then $Ac' = uA$ and $\bar{A}c = u'\bar{A}$

[I want to get $\bar{A}^T \bar{A} c = c'' \bar{A}^T \bar{A}$] $\quad \bar{A}u' = c\bar{A}^T \quad c^* A^* = A^* u^*$

$Ac' = uA \Rightarrow a_{ij}\, S c_{jk} = u_{ij}\, a_{jk}$ applying S and transposing

$\Rightarrow A^T u' = c'' A^T$

Another attempt: $\quad$ c irred. unit. $\quad c' \sim u \quad u'_{jk} = s(u_{kj}) = u^*_{jk}$
u unitary
$c'_{ij} = c^*_{ij}$ - but why?
$c'_{ij} = S(c'_{ji})$

Does this prove $\begin{cases} u'A = \bar{A}c \\ A^T u' = c'' A^T \end{cases}$? Then we would have $A^T \bar{A} c = A^T u' \bar{A} = c'' A^T \bar{A}$

and as the space of intertwiners is one-dimensional ...