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Haagerup and CBAP for groups.

1) Haagerup property

Definition 1: A von-Neumann-algebra (N, τ) has Haagerup (HP) if
$\exists\, (T_\alpha)$ net of completely positive maps that preserve τ, $T_\alpha : N \circlearrowleft$

$$\begin{cases} T_\alpha \longrightarrow id \text{ in the point-}w^*\text{-topology} \\ \tilde{T}_\alpha : L^2(N,\tau) \circlearrowleft \text{ is compact} \end{cases}$$

Q: Is there such a notion without trace, e.g. type III?
A: As far as I know, no.

Definition 2: A discrete group Γ has the (HP) if
$\exists\, \alpha$ proper isometric (affine) action of Γ on a real Hilbert space.

Remarks: 1) proper means $\#\{\gamma : \alpha_\gamma(h) \in B(h,R)\} < \infty$ for all h (or just some h) and all R.

2) $\| \alpha_\gamma(h) - \alpha_\gamma(k) \|_H = \|h-k\|_H$

3) $\alpha_\gamma(h) = \underbrace{\pi_\gamma}_{\text{orthogonal}}(h) + \underbrace{b(\gamma)}_{\text{1-cocycle (this is where properness comes from)}}$. Here 1-cocycle means, as

$$\alpha_{\gamma_1\gamma_2}(h) = \alpha_{\gamma_1} \circ \alpha_{\gamma_2}(h) = \pi_{\gamma_1} \circ \alpha_{\gamma_2}(h) + b(\gamma_1)$$
$$= \pi_{\gamma_1}(\pi_{\gamma_2}(h) + b(\gamma_2)) + b(\gamma_1)$$
$$= \pi_{\gamma_1\gamma_2}(h) + \pi_{\gamma_1} \circ b(\gamma_2) + b(\gamma_1),$$

$$\boxed{b(\gamma_1\gamma_2) = \pi_{\gamma_1} \circ b(\gamma_2) + b(\gamma_1)}$$

and the cocycle is proper if $|\{\gamma : b(\gamma) \in B(0,R)\}| < \infty$.

Definition 2': Γ has the HP if there is a proper 1-cocycle $b : \Gamma \longrightarrow H$
$\pi : \Gamma \longrightarrow \mathcal{O}(H)$

N.B. As far as I know, nobody has proved HP through Def 1.

All proofs ~~use~~ construct proper 1-cocycle.

<u>Theorem</u>: The vNA $(vN(\Gamma), \tau)$ has HP iff Γ has HP.

<u>Proof</u>: $\boxed{\Leftarrow}$ For every orthogonal rep. $\Gamma \longrightarrow \mathcal{O}(H)$, there is a measure preserving automorphism $\alpha : \Gamma \curvearrowright (\mathbb{R}^n, d\gamma^n)$ with $n = \dim H$
gaussian measure

For each $h \in H$ there is $\omega(h)$, a $\mathcal{N}(0, \|h\|^2)$ - standard Gaussian random variable

such that $\alpha_\gamma \circ \omega(h) = \omega(\pi_\gamma(h))$ with correlation $(\omega(h), \omega(k)) = \langle h, k \rangle$

$\rightarrow \omega : H \rightarrow L^2(\mathbb{R}^n, d\gamma^n)$ isometry.

$\rightarrow$ The coordinates in $\mathbb{R}^n$ form an o-n basis

and you choose another on family.

It induces an isometric action on L^∞: $\hat{\alpha} : L^\infty(\mathbb{R}^n, d\gamma^n) \longrightarrow L^\infty(\mathbb{R}^n, d\gamma^n) \otimes \ell^\infty(\Gamma)$

$$f \longrightarrow f \circ \alpha, \qquad \begin{aligned} \alpha : \mathbb{R}^n \times \Gamma &\longrightarrow \mathbb{R}^n \\ (x, \gamma) &\longrightarrow \gamma \cdot x \end{aligned}$$

$$L^\infty(\mathbb{R}^n, d\gamma^n) \rtimes_\alpha \Gamma = \{\hat{\alpha}(L^\infty(\mathbb{R}^n, d\gamma^n)) \cup \mathrm{Id} \otimes (\ell^1\Gamma, \star)\}'' \subseteq B(L^2(\mathbb{R}^n, d\gamma^n) \otimes \ell_2\Gamma)$$

$\longrightarrow$ If $\lambda_\gamma \in \ell^1\Gamma$, $\lambda_\gamma \hat{\alpha}(f(x)) \lambda_\gamma^* = f(\alpha_\gamma(x))$

For each t, there is $\pi_t : vN(\Gamma) \longrightarrow L^\infty(\mathbb{R}^n, d\gamma^n) \rtimes \Gamma$

$$\lambda_\gamma \longmapsto e^{it\omega(b(\gamma))}\lambda_\gamma$$

and $\pi_t : \lambda_{\gamma_1\gamma_2} \longrightarrow e^{it\omega(b(\gamma_1\gamma_2))}\lambda_{\gamma_1\gamma_2} = e^{it\omega(b(\gamma_1))}\lambda_{\gamma_1} e^{it\omega(b(\gamma_2))}\lambda_{\gamma_2}$

$$= e^{it\omega(b(\gamma_1))} e^{it\hat{\alpha}_{\gamma_1}(\omega(b(\gamma_2)))}\lambda_{\gamma_1\gamma_2}$$
$$= e^{it\omega(b(\gamma_1))} e^{it\omega(\pi_{\gamma_1}(b(\gamma_2)))}\lambda_{\gamma_1\gamma_2}$$
$$= e^{it\omega(b(\gamma_1\gamma_2))}\lambda_{\gamma_1\gamma_2}.$$

Verification for involution is similar.

Consider $\pi_0 : vN(\Gamma) \rightarrow L^\infty(\mathbb{R}^n, d\gamma^n) \rtimes \Gamma$
$$\lambda_\gamma \longmapsto \lambda_\gamma$$

$vN(\Gamma) \subseteq L^\infty(\mathbb{R}^n, d\gamma^n) \rtimes \Gamma$: there is a unique conditional expectation

$$E : L^\infty(\mathbb{R}^n, d\gamma^n) \rtimes \Gamma \rightarrow vN(\Gamma), \quad E(f\lambda_\gamma) = \int_{\mathbb{R}^n} f \, d\gamma^n \, \lambda_\gamma$$

N.B.: conditional expectations are completely positive.

Consider $\mathbb{T}_t = E \circ \pi_t : VN(\Gamma) \circlearrowleft$

$$\lambda_\gamma \longrightarrow E(e^{it b(\gamma)} \lambda_\gamma) = e^{-t \|b(\gamma)\|_H^2} \lambda_\gamma .$$

Let $\varphi : \gamma \longmapsto e^{-t\|b(\gamma)\|^2}$: $\varphi \in c_0\Gamma$ and $T_t \; L^2(VN(\Gamma), \tau) \circlearrowleft$ is compact.

$\boxed{\Rightarrow}$ (even though this direction is never used) Suppose (T_α) witness the HP for $VN(\Gamma)$. Define $\varphi_\alpha(\gamma) = \tau(\lambda_\gamma^* T_\alpha(\lambda_\gamma))$. This $\in \mathbb{R}$ by assumption (I cheated a little bit). Let $\varepsilon_\alpha = 1 - \varphi_\alpha(\gamma_0)$ for some fixed γ_0.
Then $\varepsilon_\alpha \xrightarrow[\alpha]{} 0$; let $\psi(\gamma) = \lim_\alpha \frac{1-\varphi_\alpha(\gamma)}{\varepsilon_\alpha}$ [Here I cheat a little bit. Perhaps one has to pass to a subsequence]
In fact, the φ_α are positive definite. Recall:

$\varphi : \Gamma \to \mathbb{C}$ is <u>pos. def.</u> if $\forall \gamma_1, \dots, \gamma_n \in \Gamma \; (\varphi(\gamma_i^{-1}\gamma_j))_{i,j}$ is positive definite.

This implies that ψ is conditionally negative definite:

$\forall \gamma_1, \dots, \gamma_n \in \Gamma \; \forall c_1, \dots, c_n \in \mathbb{R} \; \sum_j c_j = 0 \quad (c_1, \dots, c_n) \left[\psi(\gamma_i^{-1}\gamma_j)\right]_{i,j} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \leq 0$

Let $V \subseteq \mathbb{R}\Gamma$ be the space of mean 0 elements:

$[\ ,\ ]_\psi : V \times V \longrightarrow \mathbb{R}$

$[\gamma_1, \gamma_2] = \frac{1}{2}\left(\psi(\gamma_1) + \psi(\gamma_2) - \psi(\gamma_1^{-1}\gamma_2)\right)$ and $V/_{\{x : [x,x]_\psi = 0\}} \simeq H$.

Then $\pi_\gamma([\gamma']) = [\gamma \cdot \gamma' - \gamma]$
$b(\gamma) = [\gamma]$

Let us check: $[\pi_x(\gamma_1), \pi_x(\gamma_2)] = [x \cdot \gamma_1 - x, x \cdot \gamma_2 - x]$

$= [x\cdot\gamma_1, x\cdot\gamma_2] - [x\cdot\gamma_1, x] - [x, x\cdot\gamma_2] + [x,x]$

$= \frac{1}{2}\big[\psi(x\cdot\gamma_1) + \psi(x\cdot\gamma_2) - \psi(\gamma_1^{-1}\gamma_2) - \psi(x\cdot\gamma_1) - \psi(x)$
$+ \psi(\gamma_1^{-1}) - \psi(x) - \psi(x\cdot\gamma_2) + \psi(\gamma_2)$
$+ \psi(x) + \psi(x) + \psi(e)\big)$

$= \frac{1}{2}\left(\psi(\gamma_1^{-1}) + \psi(\gamma_2) + \psi(\gamma_1^{-1}\gamma_2)\right)$

$= [\gamma_1, \gamma_2]$

What we did here is to define

$\pi_\gamma(\gamma' + \gamma'')$: you define on the whole space and observe that the definition is much simpler on H.

<u>Theorem</u>: If Γ acts properly on a (simplicial) tree, then Γ has HP

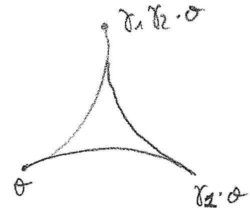
<u>N.B.</u>: proper means here that the tree is considered as a metric space

Proof: fix a point $\theta \in V$ and define $b_\theta(x)(e) = \begin{cases} 1 & \text{if } e \in [0, x] \\ -1 & \text{if } \bar{e} \in [0, x] \\ 0 & \text{otherwise} \end{cases}$
Here $\bar{e}$ means e backwards: if $e = (x, y)$, $\bar{e} = (y, x)$.

Here $b_o(x) \in \ell^2_{\mathbb{R}} E$. $\|b_o(x)-b_o(y)\|^2_{\ell^2 E} = 2d(x,y)$

We will need later that $e^{-rd(x,y)}$ is a p.d kernel.

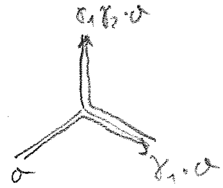
Define $b(\gamma)=b_o(\gamma\cdot o)$: then $b(\gamma_1\gamma_2)=b_o(\gamma_1\gamma_2\cdot o)=$



$= b_o(\gamma_1\cdot o)+b_{\gamma_1\cdot o}(\gamma_1\gamma_2\cdot o)$

$= b_o(\gamma_1)+\pi(\gamma_1)\,b_o(\gamma_2\cdot o)$

$= b(\gamma_1)+\pi(\gamma_1)\,b(\gamma_2)$



Examples:
- free groups acting on their Cayley graphs
- $\Gamma_1 *_\Lambda \Gamma_2$, $\Gamma_1,\Gamma_2,\Lambda$ finitely act on their Bass-Serre tree
- $\mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4 \simeq SL_2(\mathbb{Z})$
- Γ acting on a space ~~of~~ with walls. (generalisation of trees)
  $\Gamma_1 \wr \Gamma_2$ with Γ_1 finite, Γ_2 acting on a tree
- Amenable groups (proof with Følner sets to define the net of c.p. maps).

② CBAP

Definition: A C^*-algebra A has the completely bounded approximation property (CBAP) if there is a net (φ_α) of c.b. finite rank maps s.t.

① $\varphi_\alpha : A \circlearrowleft$ converges in the point-norm topology

② $\lim_\alpha \|\varphi_\alpha\|_{cb} < \infty$.

Remark: Denote by $\Lambda_{cb}(A)$ the minimum constant s.t. $\lim_\alpha \|\varphi_\alpha\|_{cb} = \Lambda_{cb}(A)$

All known values are odd integers

Definition for vN algebras: similar, normal maps that converge in the point-w^* topology.

Grothendieck's theorem:

Let S, T be sets; $\varphi : S\times T \longrightarrow \mathbb{C}$ is a Schur multiplier if, given $A=(a_{s,t})$ in $B(\ell^2 T, \ell^2 S)$, $M_\varphi A = (\varphi(s,t)\, a_{s,t})$ is also bounded: $M_\varphi : B(\ell_2 T, \ell_2 S) \circlearrowleft$ is bounded.

Theorem: Let $\varphi : S\times T \longrightarrow \mathbb{C}$ be a function and $C\geq 0$. TFAE

(i) φ is a bounded Schur multiplier $\|M_\varphi\| \leq C$

(ii) There is a Hilbert space H and there are functions $x\in \ell^\infty(T,H)$, $y\in \ell^\infty(S,H)$ such that $\varphi(s,t) = \langle x(t), y(s)\rangle$, $\|x\|_\infty \|y\|_\infty \leq C$

(iii) $u_\varphi : \ell^1 T \longrightarrow \ell^\infty S$ is in $\Gamma_2(\ell^1 T, \ell^\infty S)$, $\gamma_2(u_\varphi) \leq C$.

(iv) M_φ is c.b. with $\|M_\varphi\|_{cb} \leq C$.

$\longrightarrow$ look up Pisier, Similarity problems and cb maps, chapter 5.

Corollary: Let χ_n be the characteristic function of the set $\{(x,y)\in T^2 : d(x,y) = n\}$, where T is a tree. Then $\|\chi_n\|_{cb} \leq 2n$.

Proof: fix $\omega : \mathbb{N} \longrightarrow T$ injective (geodesic ray). For each $x\in T$, there is a geodesic ray ω_x such that $\omega_x(0) = x$ and eventually ω_x "flows" into ω.

Define $\psi_n(x,y) = \sum_{k=0}^{n} \langle \delta_{\omega_y(k)}, \delta_{\omega_x(n-k)} \rangle_{\ell^2 V}$ for $x,y\in T$,

$$= \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \chi_{n-2m}(x,y)$$

But $\chi_n = \psi_n - \psi_{n-2}$. By the above theorem, $\|\psi_n\|_{cb} \leq n+1$, so that $\|\chi_n\|_{cb} \leq (n+1)+(n-1) = 2n$.

Corollary: Let $\Gamma \curvearrowright T$ properly. Then $\Lambda_{cb}(C^*_r\Gamma) = 1$.

Proof: Let $\vartheta_{k,t}(x,y) = \chi_{\leq k}(x,y)\, e^{-t d(x,y)}$, where $\chi_{\leq k} = \sum_{n=0}^{k} \chi_n$;

$$\|\vartheta_{k,t}\|_{cb} = \|e^{-td(\cdot,\cdot)} - \sum_{n>k} \chi_n e^{-td(\cdot,\cdot)}\|_{cb}$$

$$\leq \|e^{-td(\cdot,\cdot)}\|_{cb} + \sum_{n>k} \|\chi_n e^{-td(\cdot,\cdot)}\|_{cb} \leq 1 + \sum_{n>k} 2n e^{-tn}$$

Then, let $\varphi_{k,t}(\gamma) = \vartheta_{k,t}(o, \gamma^{-1}\cdot o)$ for $o\in T$: $\varphi_{k,t}$ is finitely supported and $\varphi_{k,t}(\gamma_1^{-1}\gamma_2) = \vartheta_{k,t}(\gamma_2\cdot o, \gamma_1\cdot o)$.

$$M_{\varphi_{k,t}} : C^*_r\Gamma \longrightarrow C^*_r\Gamma, \quad \lambda_\gamma \longmapsto \varphi_{k,t}(\gamma)\lambda_\gamma.$$ $\|\varphi_{k,t}\|_{cb} = \|\vartheta_{k,t}\|_{cb}$ s.t. $\|\varphi_{k,t}\| \leq 1+\varepsilon$.

ε is arbitrary, so we're done.

Here $\|P_n\|_\sigma \leq cn \rightsquigarrow \|D_n T_t\| \leq c_1 n e^{-\sigma t n}$.

There is a generalisation to "δ-hyperbolic ~~graphs~~" by Ozawa.

But there are hyperbolic groups that have property T: they are not Haagerup.

- $\Lambda_{cb}(\overset{\text{lattices in}}{SO(1,n)}) = \Lambda_{cb}(\overset{\text{lattices in}}{SU(1,n)}) = 1$ (or defining Λ_{cb} a little differently: finite support replaced with $\in A(G)$)

- $\Lambda_{cb}(Sp(1,n)) = 2n-1$
- $\Lambda_{cb}(F_{4(-20)}) = 21$
- If G has real rank ≥ 2, $\Lambda_{cb}(G) = \infty$