Martijn 2/10/2012

Locally compact quantum groups (lcqg)

Motivation: Pontryagin duality: G ab. grp $\longrightarrow \hat{G}$ unitary dual $\rightarrow \hat{\hat{G}} \simeq G$

A forerunner in the 70's are Kac algebras $\longrightarrow$ Kac, Vainerman | Enock, Schwartz

In the 80's, $SU_q(2)$, which is not a Kac algebra, but still has all properties of a qg.

In the 00's, Kustermans, Vaes developed C^*-algebraic and vN-algebraic qg's

Algebraic qg's are what we have done so far

$\updownarrow$ does not always work

C^*-algebraic qg's: $\left\{ \begin{array}{l} \text{reduced} \\ \text{universal} \end{array} \right.$ $\quad$ * $SU_q(1,1)$ fails.

$\updownarrow$ works well

vN-algebraic qg's . we will define them

$\boxed{\text{one should also mention Woronowicz and Baaj-Skandalis}}$

Reference: • KV, Ann Sci ENS, Math Scand.

• van Daele: . Lcqg's, a vN-algebra approach. (we will follow this one)

$\rightarrow$ it is more conceptual

• Summer school notes by Kustermans

Definition: A lcqg consists of the following data: $(M, \Delta, \varphi, \psi)$, where

- M is a vNa
- $\Delta: M \longrightarrow M \otimes M$ a normal unital $*$-homomorphism st $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$ [coassociativity]. It is the comultiplication
- $\varphi, \psi: M^+ \longrightarrow [0, \infty]$ normal, semifinite, faithful (nsf) weights st

$$\varphi((\omega \otimes \iota)\Delta(x)) = \varphi(x) \cdot \omega(1) \quad \text{for } \omega \in M_*^+, \ x \in \mathfrak{m}_\varphi^+$$
$$\psi((\iota \otimes \omega)\Delta(x)) = \psi(x) \cdot \omega(1) \quad \text{———} \quad x \in \mathfrak{m}_\psi^+$$

Recall that a weight φ is a map $M^+ \longrightarrow [0, \infty]$ preserving convex combinations.

It is faithful if $\varphi(x^*x) = 0 \Rightarrow x = 0$

normal if φ preserves suprema of increasing nets in M^+

Notation: $\mathfrak{n}_\varphi = \{x \in M : \varphi(x^*x) < \infty\}$, $\mathfrak{m}_\varphi = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi = \text{span}\{\mathfrak{m}_\varphi^+\}$ where $\mathfrak{m}_\varphi^+ = \mathfrak{m}_\varphi \cap M^+$

The weight is semifinite if $\underline{n}_\varphi$ is σ-weakly dense in M.

Let $H_0 = \underline{n}_\varphi$. Consider $H_0 \times H_0 \longrightarrow [0,\infty]$, $(x, y) \longmapsto \varphi(y^* x)$ as a scalar product and let

H be the completion of H_0 and let $\Lambda : \underline{n}_\varphi \longrightarrow H$, $x \longmapsto x$.

We have a $*$-rep. π of M on H defined by $\pi(x)\Lambda(y) = \Lambda(xy)$, $x \in M, y \in \underline{n}_\varphi$

(as $\underline{n}_\varphi$ is a left ideal: $\varphi(y^* x^* x y) \leq \|x\|^2 \varphi(y^* y)$). We call (H, Λ, π) the

GNS-representation of φ

Fact: • Λ is σ-weakly - weakly - closed. In fact, it is closed as a map

$M \to H$: the graph $G(\Lambda) \subset M \times H$ is closed.

• π is normal and injective

We shall need some facts of Tomita-Takesaki-theory "$\varphi(ab) = \varphi(b\,\sigma_{-i}(a))$".

But where is the antipode ?!

But before, some examples. ① The canonical example: $(L^\infty(G), m^*, \varphi = \int dx, \psi = \int dx^{-1})$

② $(VN(G), \Delta_{VN(G)}(\lambda_g) = \lambda_g \otimes \lambda_g, \varphi_{VN(G)}(\lambda_f) = f(e), \dots)$

$\hookrightarrow$ $\varphi(\lambda_f^* \lambda_f)$ is the L^2-norm of f

and ψ does the same job w.r.t. the right Haar m.

$\to$ Aim: construction of an antipode and left- and right- regular (co)-representations

Prop: There is a bounded operator $V \in B(H_\psi \otimes H_\psi)$ s.t. $(\iota \otimes \omega)(V)\Lambda_\psi(x) = \Lambda((\iota \otimes \omega)\Delta(x))$ for $x \in M_\psi$, $\omega \in B(H)_*$.

Moreover (i) $V^*V = 1$ (ii) $V(x \otimes 1) = \Delta(x) V$ for $x \in M$

↑ to be rigorous, one should write here the GNS of x.

I will not stress the domain issues.

Proof: look at it the other way round: $V \Lambda_\psi(x) \otimes \Lambda_\psi(y) = (\Lambda_\psi \otimes \Lambda_\psi)(\Delta(x)(1 \otimes y))$

defines an isometry V: $(\psi \otimes \psi)\left((\Delta(x)(1 \otimes y))^* (\Delta(x)(1 \otimes y))\right)$

$= (\psi \otimes \psi)\left((1 \otimes y^*)\Delta(x^* x)(1 \otimes y)\right) = \psi(x^* x)\,\psi(y^* y) < \infty$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \Lambda_\psi(x) \otimes \Lambda_\psi(y) \|^2$

(and one should repeat this for tensors of higher rank)

(ii) follows similarly: $V(u\otimes 1)\,\Lambda_\psi(a)\otimes\Lambda_\psi(b) = \Lambda_\psi\otimes\Lambda_\psi(\Delta(x)\Delta(a)(b\otimes 1))$

$= \Delta(u)V\Lambda_\psi(a)\otimes\Lambda(b)$.

Remark: In fact, V is unitary, but this is hard to prove.

Definition: Let $\xi\in\mathcal{H}_\psi$. We say that $\underline{\xi\in \mathrm{Dom}(K)}$ if $\exists\,\xi_1\in\mathcal{H}_\psi\ \forall\varepsilon>0\ \forall\eta_1,\dots,\eta_n\in\mathcal{H}_\psi$ $\exists\,p_1,\dots,p_m,q_1,\dots,q_m\in\mathcal{M}_\psi$ $\|\xi\otimes\eta_k - V\sum_{j=1}^m\Lambda_\psi(p_j)\otimes q_j^*\eta_k\|<\varepsilon$

$\|\xi_1\otimes\eta_k - V\sum_{j=1}^m\Lambda(q_j)\otimes p_j^*\eta_k\|<\varepsilon$

[Idea: construct a map $\underline{K:\xi\mapsto\xi_1}$] [At some point, we shall define an antipode $S:\mathrm{dom}(S)\to M$]
[But we will do it first on the GNS level]

Lemma: If $\xi=0$, then $\xi_1=0$

The proof will use the fact that V is an isometry and some Tomita-Takesaki theory

Proposition: (i) If $\xi\in\mathrm{Dom}(K)$, then $K\xi\in\mathrm{Dom}(K)$ and $K(K(\xi))=\xi$.
(ii) K is closed, $K:\mathrm{dom}\,K\to\mathcal{H}_\psi$.

[this follows from a "2ε"-argument].

Now that the antipode is defined on the Hilbert-space level, we shall define it on the qg level.

Definition. For $x\in M$, we say that $x\in D_0$ if $\exists\,x_1\in M\ \forall\varepsilon>0\ \forall\xi_1,\dots,\xi_n,\eta_1,\dots,\eta_n\in\mathcal{H}_\psi$ $\exists\,p_1,\dots,p_m,q_1,\dots,q_m\in M\ \forall 1\le k\le n$ $\|x\xi_k\otimes\eta_k - \sum_j\Delta(p_j)\xi_k\otimes q_j^*\eta_k\|<\varepsilon$

$\|x_1\xi_k\otimes\eta_k - \sum_j\Delta(q_j)\xi_k\otimes p_j^*\eta_k\|<\varepsilon$.

[The idea is that $x\mapsto x_1$ is $S(\ )^*$]
[But there is a problem in defining this map. Van Daele lets $x_1=S_0(x)^*\dots$]

Prop: (i) If $x\in\mathcal{D}_0$, then the x_1 that witnesses this will satisfy $x_1\in\mathcal{D}_0$ and x witnesses this
(ii) "$x\mapsto x_1$ is σ-weak – σ-weak closed." I.e., the set of pairs (x,x_1) is closed.

Remark: The set of x's is dense. But it's not obvious!

Prop: (i) If $x, y \in D_0$, then $xy \in D_0$ and "$(xy)_1 = y_1 x_1$", i.e. $S(xy)^* = S(y)^* S(x)^*$

(ii) If $x \in D_0$, $\xi \in \mathcal{D}om(K)$, then $x\xi \in \mathcal{D}om(K)$ and $Kx\xi = S(x)^* K\xi$

Cor: If $\mathrm{Dom}(K)$ is dense, then $x = 0 \Rightarrow S(x)^* = 0$!

Let us prove (ii): Let $\varepsilon > 0$ and $\eta_1, \dots, \eta_n \in H_\varphi$. There are $p_1, \dots, p_m, q_1, \dots, q_m \in M$ s.t.

$$\| x\xi \otimes \eta_k - \sum_{j=1}^{m} \Delta(p_j)(\xi \otimes q_j^* \eta_k) \| < \varepsilon \quad (1)$$

$$\| S(x)^* \xi \otimes \eta_k - \sum_{j=1}^{m} \Delta(q_j)(\xi \otimes p_j^* \eta_k) \| < \varepsilon \quad (2)$$

As $\xi \in \mathrm{Dom}(K)$, there are $r_i, s_i \in \mathcal{N}_{\varphi_*}$ s.t. $\forall k, j$ $\| \xi \otimes q_j^* \eta_k - V \sum_{i=1}^{?} \Lambda_\varphi(r_i) \otimes s_i^* q_j^* \eta_k \| < \varepsilon/m \; (1')$

$$\| K\xi \otimes p_j^* \eta_k - V \sum_{i=1}^{?} \Lambda_\varphi(s_i) \otimes r_i^* p_j^* \eta_k \| < \varepsilon/m \quad (2')$$

Compare (1) and (1'): $\| x\xi \otimes \eta_k - \sum' \Delta(p_j) V \Lambda_\varphi(r_i) \otimes s_i^* q_j^* \eta_k \| < \varepsilon \times \cancel{m}$

Comparing (2) and (2'): $\| S(x)^* K\xi \otimes \eta_k - \sum_{j,i}' \Delta(q_j) V \Lambda_\varphi(s_i) \otimes r_i^* p_j^* \eta_k \| < \varepsilon$

But $\Delta(q_j) V = V(q_j \otimes 1)$, and the theorem follows from this.

Remark: If we knew dom K to be dense, this would be much easier.

Q: Why these formulas? Is it approximation by simple tensors in the case of $\otimes^{w}$?

$(S \otimes 1)(\Delta(x)(1 \otimes y)) = \Delta(y)(1 \otimes x)$ was known on the Hopf algebra level; then it was transferred to the vNa-level

Prop: There is a bounded operator W on $H \otimes H_\varphi$ characterised by

$$(\omega \otimes \iota)(W^*) \Lambda_\varphi(x) = \Lambda_\varphi((\omega \otimes \iota)\Delta(x)) \text{ for } \omega \in B(H)_* \text{ and } x \in \mathcal{N}_\varphi.$$

Moreover, (i) $WW^* = 1$

(ii) $(1 \otimes x) W = W \Delta(x)$

(iii) $W \in M \otimes B(H_\varphi)$ and $(\Delta \otimes \iota)(W) = W_{13} W_{23}$, where

$W_{23} = 1 \otimes W$ and $W_{13} = (\Sigma \otimes 1) W_{23} (\Sigma \otimes 1)$

Sketch of iii: If $y, z \in \mathcal{N}_\varphi$, $\omega_{\Lambda(y), \Lambda(z)}(x) = \langle x\Lambda(y), \Lambda(z) \rangle$

$$(\iota \otimes \omega_{\Lambda_\varphi(y), \Lambda(z)})(W^*) = \sum' y_{(1)} \varphi(z^* y_{(2)}) \text{ "=" } (\iota \otimes \varphi(z^* \cdot\cdot))\Delta(y) \in M$$

where $\Delta(y) = \sum' a_i \otimes b_i = \sum' y_{(1)} \otimes y_{(2)}$

Remark: $M \otimes B(H_\varphi) = (M_* \hat{\otimes} N(H_\varphi))^* = CB(M_*, N(H_\varphi))$; therefore we deal with an algebra homomorphism.

Lemma: If $\omega \in B(H_\varphi)_*$ and $x = (\iota \otimes \omega)(W)$, $x_1 = (\iota \otimes \bar{\omega})(W)$, where $\bar{\omega}(x) = \overline{\omega(x^*)}$, then $x \in \mathcal{D}_0$ and you can choose $S(x^*) = x$. ← excellent formula for recalling the antipode

Proof: Let (e_j) be an onb for H_φ. First assume that $\omega = \omega_{\xi,\eta}$.

Put $p_j = (\iota \otimes \omega_{e_j,\eta})(W)$, $q_j = (\iota \otimes \omega_{e_j,\xi})(W)$. Then

$$\sum_j \Delta(p_j)(1 \otimes q_j^*) = (\iota \otimes \iota \otimes \omega_{\xi,\eta})(W_{13}W_{23}W_{23}^*)$$
$$= (\iota \otimes \iota \otimes \omega)(W_{13})$$
$$= x \otimes 1$$

and this formula still makes sense for strongly converging sums.

Also $\sum_j' \Delta(q_j)(1 \otimes p_j^*) = (\iota \otimes \iota \otimes \bar{\omega})(W_{13}W_{23}W_{23}^*) = (x_1 \otimes 1)$.

Lemma: Let $c \in \mathcal{N}_\psi$ and $\omega \in B(H_\varphi)_*$. Then $x = (1 \otimes \omega(c \cdots))(W)$
$= (1 \otimes \omega)((1 \otimes c)W) \in \mathcal{N}_\psi$.

Lemma: Let $c, d \in \mathcal{N}_\psi$ and $\omega \in B(H_\varphi)_*$. Set $\xi = \Lambda_\psi(((\iota \otimes \omega(c \cdots d^*))W)$.

Then $\xi \in \mathrm{Dom}(K)$ and $K\xi = \Lambda_\psi(((\iota \otimes \bar{\omega}(d \cdots c^*))(W))$

Proof: Assume $\omega = \omega_{\xi',\eta'}$ and (e_j) is an onb of H_φ.

$p_j = (\iota \otimes \omega_{e_j, c^*\eta'})(W) \in \mathcal{N}_\psi$
$q_j = (\iota \otimes \omega_{e_j, d^*\xi'})(W) \in \mathcal{N}_\psi$
and then follow the previous proof.

Theorem: V is unitary and $\mathrm{Dom}(K)$ is dense.

Proof: "Kustermans's trick" according to van Daele: $\mathcal{K} = \mathrm{span}\,\{\Lambda_\psi(\iota \otimes \omega(c \cdots))(W) : c \in \mathcal{N}_\psi, \omega \in B(H)_*\}$

Then ① $V(\mathcal{K} \otimes H_\varphi) \supseteq \mathcal{K} \otimes H_\varphi$ [n.b. you can intertwine H_φ and H_ψ]

② $V(H_\psi \otimes H_\varphi) \subseteq \mathcal{K} \otimes H_\varphi$.

We have $\mathcal{K} \otimes H_\varphi \overset{(i)}{\subseteq} V\mathcal{K} \otimes H_\varphi \subseteq V H_\psi \otimes H_\varphi \overset{(ii)}{\subseteq} \mathcal{K} \otimes H_\varphi$.

As of (i), put $\xi = \Lambda_\psi((\iota \otimes \omega(c \cdots))(W))$, $c \in \mathcal{N}_\psi$. Then, for all $\eta \in H_\varphi$,

$\xi \otimes \eta \leftarrow V \sum_{j=1}^{n} \Lambda_\psi(p_j) \otimes q_j^* \eta$ with the p_j, q_j of the proof of the previous lemma.

(ii) Note that $(\iota \otimes \omega_{\Lambda_\varphi(y), \Lambda_\varphi(x)})(W^{*}) = (\iota \otimes \varphi)(\Delta(x^*)(1 \otimes y))$?

This shows that $\mathcal{K} = \overline{\text{span}}\,\{\Lambda_\psi(\iota\otimes\omega)(\Delta(x)) : x\in \mathcal{M}_\psi \text{ and } \omega\in M_*\}$

<u>Fazit</u>: We have def'd lcqg^s, we have the antipode, S.

W is also a unitary: rewrite in terms of left and right...

Q: is there an opposite structure on a qg.

Recall the following formula: $\Delta(x) = W(1\otimes x)W^*$
$M = \{(1\otimes\omega)(W) : \omega\in B(H_\psi)_*\}$.

Next time: • elts of weight theory
• example.