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<u>Definition</u>: A l.c.q.g. (M,Δ) is a vna M, a normal unital $*$-homomorphism Δ such that $\boxed{(\Delta\otimes 1)\circ\Delta = (1\otimes\Delta)\circ\Delta}$, two n.s.f. weights $\varphi, \psi : M^+ \longrightarrow [0,\infty]$ such that

$$\varphi(\omega\otimes 1)\Delta(x) = \omega(1)\varphi(x) \quad \text{for } x\in \mathfrak{M}_\varphi^+ \text{ and } \omega\in M_*^+$$
$$\psi(1\otimes\omega)\Delta(x) = \omega(1)\psi(x) \quad \text{for } x\in \mathfrak{M}_\psi^+ \text{ and } \omega\in M_*^+.$$

<u>Theorem</u>: There is a unique unitary $W\in M\otimes B(H_\varphi)$ such that $\boxed{W^*(\Lambda_\varphi\otimes\Lambda_\varphi)(x\otimes y) = \Lambda_\varphi\otimes\Lambda_\varphi(\Delta(y)(x\otimes 1))}$

leg notation:

<u>Prop</u>:
$$(\Delta\otimes 1)W = W_{13}W_{23} \in M\otimes M\otimes B(H_\varphi)$$
$$\Delta(x) = W^*(1\otimes x)W$$

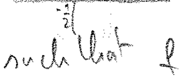
<u>Theorem</u> [Pentagon equation]: W satisfies $W_{12}W_{13}W_{23} = W_{23}W_{12}$

$\longrightarrow$ In fact, $(\Delta\otimes 1)(W) = W_{13}W_{23}$
$$\parallel$$
$$W_{12}^*W_{23}W_{12}$$

<u>Theorem</u>: There is a σ-weakly (σ-strong-$*$) closed map $S : \operatorname{dom} S \subseteq M \longrightarrow M$ such that for all $\omega\in B(H_\varphi)_*$ $(1\otimes\omega)(W)\in \operatorname{Dom} S$ and
$$S(1\otimes\omega)(W) = (1\otimes\omega)(W^*)$$
[van Daele introduces S in another way]

Moreover, $\{(1\otimes\omega)(W) : \omega\in B(H)_*\}$ form a σ-weak core for S

Another <u>Th</u> we won't prove: There is a unique strongly continuous 1-parameter group of automorphisms of M, i.e. $\mathbb{R} \longrightarrow \operatorname{Aut}(M)$, $t \longmapsto \tau_t$ and a unique $*$-anti-homomorphism $R : M \longrightarrow M$ such that $S = R\,\tau_{-\frac{i}{2}}$, where

$\operatorname{Dom}(\tau_{-\frac{i}{2}}) = \{x\in M : \exists f_x :$  $\longrightarrow M$ continuous, analytic in the interior such that $f_x(t) = \tau_t(x)$ for $t\in\mathbb{R}\}$

[One formula that always permits extension to the complex plane:
$$\frac{n}{\sqrt{\pi}}\int e^{-n^2t^2}\tau_t(x)\,dt \;]$$

This is viewed as a sort of polar decomposition.

N.B.: $S = R \circ \tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}} \circ R$

Idea of proof: Last time we considered the operator K; formally, $\Lambda_\varphi(x) \longmapsto \Lambda_\varphi(S(x)^*)$. Let $K = IL^{\frac{1}{2}}$ be the polar decomposition. Then define

$$R(x) = I x^* I$$
$$\tau_t(x) = L^{it} x L^{-it};$$

$$\begin{aligned} R \circ \tau_{-i/2}(y) &= K y^* K \Lambda_\varphi(x) = K y^* \Lambda_\varphi(S(x)^*) \\ &= \Lambda_\varphi(S(y^* S(x)^*)^*) = \Lambda_\varphi((S(x)^* S(y^*))^*) \\ &= S(y^*)^* \Lambda_\varphi(S(x)) \end{aligned}$$?

$\boxed{\text{The dual qg}}$ $(\hat{M}, \hat{\Delta}, \hat{\varphi}, \hat{\psi})$

Definition: $\hat{M} = \overline{\{(\omega \otimes \iota)(W) : \omega \in M_*\}}^{\sigma\text{-weak}}$

Claim: $\hat{M}$ is a vna: – It is an algebra, as, if $\omega, \vartheta \in M_*$, then

$$\begin{aligned} (\omega \otimes \iota)(W)(\vartheta \otimes \iota)(W) &= (\omega \otimes \vartheta \otimes 1)(W_{13} W_{23}) \\ &= ((\omega \otimes \vartheta) \circ \Delta \otimes \iota) W \in \hat{M}. \end{aligned}$$

(N.B.: multiplication is σ-weakly continuous; you can also consider bounded nets)

(one should start by proving stability under involution.)

– It is a $*$-algebra: Let $\omega \in M_*$ and suppose that there is a $\omega^* \in M_*$ such that $\omega^*(x) = \bar{\omega} \circ S(x)$ for $x \in \mathrm{Dom}\, S$.

Then $(\omega^* \otimes 1)(W) = (\bar{\omega} \otimes 1)(W) = (\omega \otimes 1)(W)^*$.

The proof is finished if such ω are dense in M_*. Suppose that $\omega \in M_*$ is arbitrary, then $\frac{n}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-n^2t^2}\, \omega \circ \tau_t \in M_*$ and it $\xrightarrow[t \to \infty]{\text{weakly}} \omega$

Then let $M_*^{\#} = \{\omega \in M_* : \exists \omega^* \in M_* \;\; \omega^*(x) = \bar{\omega}\, S(x) \text{ for } x \in \mathrm{Dom}\, S\}$
$= \{ \text{———} \;\vartheta\; \text{———} \; (\vartheta \otimes \iota)(W) = (\omega \otimes 1)(W^*) \}$

Then you can work on the simple elements of this form.

Definition of the dual comultiplication: $\hat{\Delta} : \hat{M} \to \hat{M} \otimes \hat{M}$, $x \longmapsto \chi(W(x \otimes 1)W^*)$, where $\chi : \hat{M} \otimes \hat{M} \longrightarrow \hat{M} \otimes \hat{M}$ is the flip, is a unital, normal $*$-homomorphism such that $(\hat{\Delta} \otimes \iota)\hat{\Delta} = (\iota \otimes \hat{\Delta})\hat{\Delta}$.

Proof: Let $x = (\omega \otimes \iota)(W)$ and consider $W(x \otimes 1)W^* = (\omega \otimes \iota \otimes \iota)(W_{23} W_{12} W_{23}^*)$
$= (\omega \otimes \iota \otimes \iota)(W_{12} W_{13})$

(First consider $\omega = \langle \cdot\, \xi, \eta \rangle$ and check it:) $\in \hat{M} \otimes \hat{M}$.

(The commutant of the tensor product is the tensor product of the commutants)

Let e_j be an onb of H_φ:

$(\omega \otimes \iota \otimes \iota)(W_{12} W_{13}) = \sum_i (\langle \ldots e_j, \eta \rangle \otimes \iota)(W) \otimes (\langle \ldots \xi, e_j \rangle \otimes \iota)(W)$

so that it is in $\hat{M} \otimes \hat{M}$

An interlude of weight theory: Ref: Takesaki II.

Idea: If we have something like a GNS rep, that is a "Hilbert algebra", then it defines a weight.

Def: An involutive algebra $\mathfrak{A}$ with involution $\xi \longmapsto \xi^\#$ is called a left Hilbert algebra if $\mathfrak{A}$ admits an inner product $\langle\, ,\, \rangle : \longrightarrow \mathbb{C}$ s.t.

① If $\xi \in \mathfrak{A}$, the map $\eta \longmapsto \xi\eta$ is bounded, and associative

② $\langle \xi \cdot \eta, \zeta \rangle = \langle \eta, \xi^\# \zeta \rangle$

③ $\xi \longmapsto \xi^\#$ is "preclosed" when one extends it to the completion of $\mathfrak{A}$

④ $\mathfrak{A}^2 = \{ \xi \cdot \eta : \xi, \eta \in \mathfrak{A} \} \subseteq \mathfrak{A}$ is dense.

A typical example: M vna, φ n.s.f. weight, $\mathfrak{n}_\varphi = \{ x \in M : \varphi(x^* x) < \infty \}$.
Then $\Lambda_\varphi(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ is a left Hilbert algebra.

Let $\mathfrak{A}$ be a Hilbert algebra; for $\xi \in \mathfrak{A}$, denote $\pi_\ell(\xi)$ for the map $\mathfrak{A} \to \mathfrak{A}$, $\eta \mapsto \xi\eta$

Then $\{ \pi_\ell(\xi) : \xi \in \mathfrak{A} \}''$ is a vnA. If you started from a vna, you get an isomorphic vna.

There is a "canonical" way to define a nsf weight on M, determined by the following property: If $\xi \in \mathfrak{A}$, $\varphi(\pi_\ell(\xi)^* \pi_\ell(\xi)) = \langle \xi, \xi \rangle$

There is a notion of full Hilbert algebra, where you can put $\varphi(\ ) = \infty$ for all remaining elements.

Let $\mathfrak{H}$ be the Hilbert space completion of $\mathfrak{A}$.

Let $\Lambda_0 : \pi_\ell(\mathfrak{A}) \to \mathfrak{H}$, $\pi_\ell(\xi) \mapsto \xi$: this map is σ-strongly*/norm closable and let Λ be its closure.

Then $(\mathfrak{H}, \Lambda, \pi_\ell)$ is the GNS-construction of φ.

<u>Remark</u>: $\mathfrak{n}_\varphi = \mathrm{Dom}\,\Lambda$

<u>Idea</u>: Use this technique to construct a weight $\hat{\varphi}$ on $\hat{M}$:

<u>Def</u>: Let $I \subseteq M_*$ be the set of all $\omega \in M_*$ for which the map

$$\Lambda_\varphi(x) \mapsto \omega(x^*) \quad (\mathfrak{n}_\varphi \to \mathbb{C})$$ is bounded.

Let $\xi(\omega) \in H_\varphi$ be the vector such that $\langle \xi(\omega), \Lambda_\varphi(x) \rangle = \omega(x^*)$

<u>Prop</u>: If $a, b \in \{x \in M : x$ is analytic wrt σ^φ and $\sigma_z^\varphi(x) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ for $z \in \mathbb{C}\}$

Then $\varphi(a \cdot\cdot b) \in I$ and $\xi(\omega) = \Lambda(b\,\sigma_{-i}(a))$

proof: $\varphi(a x^* b) = \varphi(x^* b\, \sigma_{-i}(a)) = \langle \Lambda(b\,\sigma^\varphi_{-i}(a)), \Lambda(x) \rangle$

<u>Cor</u>: ① $\{\xi(\omega) : \omega \in I\} \subseteq H_\varphi$ is dense.

② $I \subseteq M_*$ is dense

<u>Th</u>: Let $J = I \cap M_*^\# \cap (I \cap M_*^\#)^*$: $(\omega^* \otimes \iota)(W) = (\omega \otimes 1)(W^*)$

Set $\mathfrak{A} = \{(\omega \otimes \iota)(W) : \omega \in J\}$. Then $\mathfrak{A}$ is a Hilbert algebra.

Proof: Notation: $\lambda(\omega) \equiv (\omega \otimes \iota)(W)$ for $\omega \in J$

$$\lambda(\omega)^\# = \lambda(\omega)^* = \lambda(\omega^*) \in \mathfrak{A}$$

Inner product: $\langle \lambda(\omega), \lambda(\vartheta) \rangle = \langle \xi(\omega), \xi(\vartheta) \rangle$

Algebra: $\lambda(\omega)\lambda(\vartheta) = \lambda((\omega \otimes \vartheta) \circ \Delta) \overset{?}{\in} \mathfrak{A}$

Is $(\omega \otimes \vartheta) \circ \Delta$ in I? Take $x \in \mathfrak{n}_\varphi$: $(\omega \otimes \vartheta)\Delta(x^*) = \vartheta((\omega \otimes \iota)\Delta(x^*))$

(Recall $\bar{\omega}(x) = \overline{\omega(x^*)}$)

$$= \vartheta(\underbrace{((\bar{\omega} \otimes \iota)\Delta(x))}_{\in \mathfrak{n}_\varphi}{}^*)$$

$$= \langle \xi(\vartheta), \Lambda_\varphi((\bar{\omega} \otimes \iota)\Delta(x)) \rangle$$

$$= \langle \xi(\vartheta), (\bar{\omega} \otimes \iota)(W^*)\Lambda_\varphi(x) \rangle$$

$$= \langle (\omega \otimes \iota)(W)\,\xi(\vartheta), \Lambda_\varphi(x) \rangle$$

You can check that $((\omega \otimes \vartheta)\Delta)^* = (\vartheta^* \otimes \omega^*) \circ \Delta$

(slice first then second leg!)

[$\langle \xi(\omega), \Lambda_\varphi(x)\rangle = \omega(x^*)$
$\langle \Lambda_\varphi(x), \eta(\omega)\rangle = \omega(x)$?]

Let us agree on the fact that $\mathfrak{A}$ is an involutive algebra.

(1) $\lambda(\vartheta) \longmapsto \lambda(\omega)\lambda(\vartheta)$ is bounded [problematic notation: rather write $\xi(\vartheta) \longmapsto \xi((\omega\otimes\vartheta)\circ\Delta)$ is bounded.]. But $\xi((\omega\otimes\vartheta)\circ\Delta) = (\omega\otimes\iota)(W)\xi(\vartheta)$.

(2) $\langle \lambda(\omega)\xi(\vartheta), \xi(\rho)\rangle = \langle \xi(\vartheta), \lambda(\omega)^*\xi(\rho)\rangle$.

(3) Let (ω_i) be a net in J such that the $\xi(\omega_i)$ converge to 0 and st $(\xi(\omega_i^*))$ converges. Consider $\langle \xi(\omega_i)^*, \Lambda_\varphi(x)\rangle = \omega_i^*(x^*)$

$$\overset{\shortparallel}{(\iota\otimes\omega)(W)}$$

$$= (\omega_i^*\otimes\omega)(W) = \omega(\omega_i^*\otimes\iota)(W) = \omega(\omega_i\otimes\iota)(W)^* = (\omega_i\otimes\overline{\omega})(W)^*$$

$$= \overline{\omega_i}((\iota\otimes\overline{\omega})(W)^*) = \overline{\omega_i((\iota\otimes\overline{\omega})(W))} = \langle \Lambda_\varphi((\iota\otimes\omega)(W)), \xi(\omega_i)\rangle \downarrow 0$$

(4) We also have to check that $\mathfrak{A}\mathfrak{A}$ is dense in $\mathfrak{A}$. Consider $\mathfrak{A}$ sitting in the GNS space: $\lambda(\omega)\xi(\vartheta) \in H_\varphi$. Using the previous corollary and the fact that $\hat{M}$ acts nondegenerately on H_φ, (4) follows.

(5) Left invariance: cf van Daele's note.

There is also a right invariant weight: $(\hat{M}, \hat{\Delta}, \hat{\varphi}, \hat{\psi})$ will be the original idea behind Pontryagin duality: the W operator: $\hat{W} = \Sigma W^* \Sigma$ (flip)

Illustration: G group: $\lambda : G \to \mathcal{L}(G)$ is in $L^\infty(G; \mathcal{L}(G)) \simeq L^\infty(G)\otimes\mathcal{L}(G)$ and there, it simply is W_λ | $\hat{\hat{W}} = W$.

$(M, \Delta, \varphi, \psi) = (L^\infty(G), \ldots)$
If $f \in L^1(G)$, $\left(\int \cdot f(x)\,dx \otimes 1\right)(W_\lambda) = \int f(x)\lambda_x\,dx$; $\hat{M} = \mathcal{L}(G)$.