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0 - COMPACT QUANTUM GROUPS

Definition : $\mathbb{G} = (A, \Delta)$ A unital C^*-algebra
$\Delta : A \to A \otimes A$ comultiplication st
(i) $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
(ii) $\Delta(A)(1 \otimes A)$ and $\Delta(A)(A \otimes 1)$ is lin. dense in $A \otimes A$

Theorem: $\exists h : A \to \mathbb{C}$ Haar state
$(id \otimes h)\Delta(a) = h(a)1 = (h \otimes id)\Delta(a) \quad \forall a \in A.$

Definition: A corepresentation matrix of $\mathbb{G}$ is $u \in M_n(A)$ st
$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

We consider the subalgebra $\mathcal{A} \subset A$ defined by
$$\mathcal{A} = \text{span}\left\{ u_{ij}^{\alpha} : 1 \leq i,j \leq d_\alpha, \alpha \in I \right\}$$
composed of matrix coefficients of unitary irreducible corepresentations.

Proposition :
- $\overline{\mathcal{A}}^{\|\cdot\|} = A$
- h is faithful on $\mathcal{A}$.

Def-Proposition : Let $\mathbb{G} = (A, \Delta)$ be a CQG. TFAE

(i) h is a trace on A
(ii) $S^2_{|\mathcal{A}} = id$

In this case we say that $\mathbb{G}$ is of Kac type.

Orthogonality relations : $h\left(c_{kp} S(c_{ij})\right) = h\left(c_{kp} c_{ji}^*\right) = \dfrac{\delta_{kj}\delta_{ip}}{\dim(c)}$

where $c = (c_{ij})$ is a irreducible corepresentation.

1. HAAGERUP PROPERTY FOR COMPACT QUANTUM GROUPS

Let $\mathbb{G} = (A, \Delta)$ be a CQG (of Kac type), h Haar state.

Denote $(\pi_h, L^2(\mathbb{G}), \Lambda)$ the GNS triple obtained from $(x,y) \mapsto h(x^*y)$

$\pi_h : A \to B(L^2(\mathbb{G}))$ $\qquad$ $\pi_h(a)\Lambda(b) = \Lambda(ab)$

$\Lambda : A \to L^2(\mathbb{G})$ $\qquad$ $\langle \Lambda(a), \Lambda(b) \rangle = h(a^*b)$

$h(a) = \langle \pi_h(a)\xi, \xi \rangle \quad (\xi = \Lambda(1))$

- Reduced C^*-algebra:

$$C_r(\mathbb{G}) = \pi_h(A) \subset B(L^2(\mathbb{G})) \simeq A/I$$

where $I = \ker \pi_h = \{x \in A : h(x^*x) = 0\}$

$$\Delta_r : C_r(\mathbb{G}) \to C_r(\mathbb{G}) \otimes C_r(\mathbb{G})$$
$$\pi_h(x) \mapsto (\pi_h \otimes \pi_h)\Delta(x)$$

$$h = h_r \circ \pi_h$$

- von Neumann algebra: $L^\infty(\mathbb{G}) = C_r(\mathbb{G})'' = \overline{C_r(\mathbb{G})}^{\sigma w} \subset B(L^2\mathbb{G})$
- Fact: Δ_r, h_r extend to normal $*$-homomorphism / faithful state on $L^\infty(\mathbb{G})$
- Remark: we have the inclusions

$$\mathcal{A} \underset{(1)}{\subset} C_r(\mathbb{G}) \underset{(2)}{\subset} L^\infty(\mathbb{G}) \underset{(3)}{\subseteq} L^2(\mathbb{G})$$

(1) h is faithful on $\mathcal{A}$ so π_h is injective on $\mathcal{A}$.

(3) $\Lambda : A \to L^2(\mathbb{G})$ factors through $C_r(\mathbb{G})$ since $\pi_h(a) = 0 \Rightarrow \Lambda(a) = \pi_h(a)\Lambda(1) = 0$

- $$\left.\begin{array}{l} \overline{\mathcal{A}}^{\|\cdot\|} = A \\ \overline{C_r(\mathbb{G})}^{\sigma w} = L^\infty(\mathbb{G}) \end{array}\right\} \Rightarrow \overline{\mathcal{A}}^{\sigma w} = L^\infty(\mathbb{G})$$

- $\{u^\alpha_{ij} : 1 \le i,j \le d_\alpha, \alpha \in I\}$ $\qquad$ basis of $\mathcal{A}$

  $\{d_\alpha^{1/2} u^\alpha_{ij} : 1 \le i,j \le d_\alpha, \alpha \in I\}$ $\qquad$ orthonormal basis of $L^2(\mathbb{G})$

- $k_r(u^\alpha_{ij}) = u^{\alpha *}_{ji}$ $\qquad$ $(u^{\alpha *}_{ji})$ is also unitary

  $(k_r(u^{\alpha *}_{ji}) = u^\alpha_{ij})$ $\qquad$ Notation: $k_r(u^\alpha_{ij}) = u^{\bar\alpha}_{ji}$

Definition: ① (M,h) is a finite ONA. We say that (M,h) has the HAP if there exists a net $(\phi_x)_x$ of NUCP maps $h\circ\phi_x = h$ st
(i) $\forall x$, $\phi_x : L^2(M) \to L^2(M)$ is compact
(ii) $\forall a \in M$, $\|\phi_x a - a\|_{L^2} \xrightarrow[x]{} 0$

② We say that $\mathbb{G}$ has HAP if $L^\infty(\mathbb{G})$ has HAP.

2. CONSTRUCTION OF NUCP h-PRESERVING MAPS.

Theorem (Brannan) $\mathbb{G} = (A, \Delta)$ Kac type.

$$\mathcal{B} = C^*\langle \sum_{i=1}^{d_\alpha} u^\alpha_{ii} = \chi_\alpha, \alpha \in \mathcal{I} \rangle \quad \text{(central subalgebra)}$$

$\varphi \in \mathcal{B}^*$ state, (i) $T\varphi = \sum_{\alpha\in\mathcal{I}} \frac{\varphi(\chi_{\bar\alpha})}{d_\alpha} P_\alpha$

where $P_\alpha : L^2(\mathbb{G}) \to L^2_\alpha(\mathbb{G}) = \mathrm{span}\{u^\alpha_{ij} : 1\leq i,j\leq d_\alpha\}$

$T\varphi$ is a unital contraction on $L^2(\mathbb{G})$

(ii) $T\varphi|_{L^\infty(\mathbb{G})}$ is a NUCP h preserving maps $T\varphi \in \mathcal{B}(L^\infty(\mathbb{G}))$

Proof: (i) $\|T\varphi\|_{\mathcal{B}(L^2(\mathbb{G}))} = \sup_{\alpha\in\mathcal{I}} \left|\frac{\varphi(\chi_{\bar\alpha})}{d_\alpha}\right| \leq \frac{\|\varphi\|}{d_\alpha}\|\chi_{\bar\alpha}\|_A \leq \frac{1}{d_\alpha}\sum_{i=1}^{d_\alpha}\|u^\alpha_{ii}\|_A \leq 1$

hence $T\varphi$ is a contraction

(ii) <u>Lemma</u>: $\varphi \in A^*$ state, $C\varphi : A \to A$
$x \mapsto (\varphi\otimes id)\Delta(x)$

There exists a UCP-h preserving map $S\varphi : C_r(\mathbb{G}) \to C_r(\mathbb{G})$ st $S\varphi(\pi_R(x)) = \pi_R(C\varphi x)$ and this map extends to $S\varphi : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ NUCP-h-preserving.

$\varphi \in \mathcal{B}^*$, take $\tilde\varphi \in A^*$ Hahn-Banach extension.
Consider $S\tilde\varphi \in CB(L^\infty\mathbb{G})$ given by the lemma.

$$\Phi\varphi = \Delta_r^* \circ ((k_r \circ S\tilde\varphi \circ k_r)\otimes id) \circ \Delta_r \quad \text{is NUCP } h\text{-preserving.}$$

We want to show that $\Phi\varphi = T\varphi$ on $L^\infty(\mathbb{G})$. It suffices to prove that $\Phi\varphi(u^\alpha_{ij}) = T\varphi(u^\alpha_{ij})$

$$\Phi\varphi(u^\alpha_{ij}) \in L^2(\mathbb{G})$$

$$\| $$

$$\sum_{\beta,p,q} \langle \Phi\varphi(u^\alpha_{ij}), d_\beta^{1/2} u^\beta_{pq}\rangle d_\beta^{1/2} u^\beta_{pq} = \sum_{\beta,p,q} \langle (k_r \circ S\tilde\varphi \circ k_r \otimes id)\underbrace{\Delta_r(u^\alpha_{ij})}_{= \sum_k u^\alpha_{ik}\otimes u^\alpha_{kj}}, \underbrace{\Delta_r(u^\beta_{pq})}_{= \sum_\ell u^\beta_{p\ell}\otimes u^\beta_{\ell q}}\rangle d_\beta u^\beta_{pq}$$

We get $\varphi_\varphi(u_{ij}^\alpha) = \sum'_{\beta,p,q} \langle (k_r \circ S_\varphi \circ k_r) \otimes id \left(\sum_k u_{ik}^\alpha \otimes u_{kj}^\alpha \right), \sum_e u_{pe}^\beta \otimes u_{eq}^\beta \rangle d_\beta u_{pq}^\beta$

$$= \sum'_{\substack{\beta,p,q\\k,e}} \langle \underbrace{k_r \circ S_\varphi(u_{ki}^{\bar\alpha})}_{\substack{k_r \circ (\varphi \otimes id)\Delta_r(u_{ki}^{\bar\alpha}) \\ \| \\ k_r \circ (\varphi \otimes id)\left(\sum_n u_{kn}^{\bar\alpha} \otimes u_{ni}^{\bar\alpha}\right) \\ \| \\ \sum_n k_r\left(\varphi(u_{kn}^{\bar\alpha}) \otimes u_{ni}^{\bar\alpha}\right) \\ \| \\ \sum_n \varphi(u_{kn}^{\bar\alpha})\, u_{in}^{\alpha}}} \otimes u_{kj}^\alpha, u_{pe}^\beta \otimes u_{eq}^\beta \rangle d_\beta u_{pq}^\beta$$

$$= \sum_{\substack{\beta,p,q\\k,e,n}} \varphi(u_{kn}^{\bar\alpha}) \underbrace{\langle u_{in}^\alpha \otimes u_{kj}^\alpha, u_{pe}^\beta \otimes u_{eq}^\beta \rangle}_{\delta_{\alpha\beta}\delta_{ip}\delta_{np}\delta_{ke}\delta_{jq}\frac{1}{d_\alpha^2}} d_\beta u_{pq}^\beta$$

$$= \sum_k \frac{\varphi(u_{kk}^{\bar\alpha})}{d_\alpha} u_{ij}^\alpha = T_\varphi(u_{ij}^\alpha)$$

CQFD ■

3. FEW FACTS ABOUT S_N^+

$S_N^+ = (A_s(N), \Delta)$ is the quantum permutation group of dimension N given by

- $A_s(N)$ universal C^*-alg. generated by N^2 generators v_{ij}

and the relations $v = (v_{ij})$ is unitary, $v_{ij}^* = v_{ij} = v_{ij}^2$ (v is a magic unitary)

- $\Delta(v_{ij}) = \sum_{k=1}^N v_{ik} \otimes v_{kj}$ (v is the fundamental corepresentation)

Motivation:

$C(S_N)$ is the universal C^*-alg. generated by the N^2 elements $p_{ij} = 1\!\!1(\sigma \in S_N / \sigma(j) = i)$

$N = 1,2,3 \rightarrow$ same thing (for S_N^+ commutative)

$N = 4 \rightarrow \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$ magic unitary: the entries do not necessarily commute

p, q projections

$\Downarrow$

we get a different algebra.

Theorem (Banica): There exists a maximal family $(v^{(t)})_{t \in \mathbb{N}}$ of irreducible corepresentation of $A_s(N)$ st

(i) $v^{(0)} = 1 \qquad v = 1 \oplus v^{(1)}$

(ii) $\overline{v^{(t)}} \simeq v^{(t)}$

(iii) $v^{(s)} \otimes v^{(t)} = \bigoplus_{k=0}^{2\min(s,t)} v^{(s+t-k)}$

(iv) $d_0 = 1,\ d_1 = N-1,\ d_1 d_k = d_{k+1} + d_k + d_{k-1}$

Proposition: Let $\chi \in A_S(N)$ be the character of ν. Then $\chi^* = \chi$ and

$$\varphi : C^*_-\langle \chi_k, k\in\mathbb{N}\rangle \longrightarrow C(\underset{\sigma}{\mathrm{Spec}\,\chi})$$
$$\chi_k \longmapsto \pi_k|_\sigma$$

π_k are the Tchebichev polynomials.

$$\pi_0 = 1,\quad \pi_1 = x-1,\quad \pi_1\pi_k = \pi_{k+1} + \pi_k + \pi_{k-1}$$

Proof: • $\nu = 1 \oplus \nu^{(1)}$ so $\chi = 1 + \chi_1 \Rightarrow \chi^* = 1 + \chi_1^* = 1 + \chi_1 = \chi$

• $\chi_1\chi_k = \chi_{k+1} + \chi_k + \chi_{k-1}$ so $C^*_-\langle \chi_k, k\in\mathbb{N}\rangle = C^*_-\langle 1, \chi\rangle$ and we can consider

$$\varphi : C^*_-\langle \chi_k\rangle \xrightarrow{\sim} C(\mathrm{Spec}(\chi))$$
$$1 \longmapsto 1$$
$$\chi \longmapsto x_{|\sigma}$$

so $\forall x\in\sigma, \forall k\geq 1$ $\quad \varphi(\chi_1)(x)\varphi(\chi_k)(x) = \varphi(\chi_k)(x) + \varphi(\chi_{k+1})(x) + \varphi(\chi_{k-1})(x)$

$\varphi(\chi_0)(x) = 1$

$\varphi(\chi_1)(x) = x-1$

$\Rightarrow \varphi(\chi_k)(x) = \pi_k|_\sigma$ ∎

Remark: $\pi_k(x) = A_{2k}(\sqrt{x})$ where the A's are the polynomials $\begin{cases} A_0 = 1 \\ A_1 = x \\ A_1A_n = A_{n+1} + A_n \end{cases}$

Proposition: ① $\forall t, s \geq 1$, $A_tA_s = A_{t+s} + A_{t-1}A_{s-1}$

② $\exists\, 2 < x_0 < 3$, $\exists\, C_{x_0} > 0$ depending only on x_0 st

$$\forall x \in [x_0, N], \forall t \quad \frac{A_t(x)}{A_t(N)} \leq C_{x_0}\left(\frac{x}{N}\right)^t$$

Proof: ② $q(x) = \dfrac{x+\sqrt{x^2-4}}{2}$ $\qquad q(x) + q(x)^{-1} = x$

(i) $\forall t$, $A_t(x) = \dfrac{q(x)^{t+1} - q(x)^{-t-1}}{q(x) - q(x)^{-1}}$ $\qquad$ (exercise)

(ii) $$\frac{A_t(x)}{A_t(N)} = \frac{q(x)^{t+1} - q(x)^{-t-1}}{q(x) - q(x)^{-1}}\; \frac{q(N) - q(N)^{-1}}{q(N)^{t+1} - q(N)^{-t-1}} = \left(\frac{q(x)}{q(N)}\right)^t \frac{1-q(x)^{-2t-2}}{1-q(x)^{-2}}\; \underbrace{\frac{1-q(N)^{-2}}{1-q(N)^{-2t-2}}}_{\leq 1}$$
$$\leq \left(\frac{q(x)}{q(N)}\right)^t (1-q(x)^{-2})^{-1}$$

if $x_0 \in]2,3[$, $x_0 \leq x \leq N$ we get

$$\frac{A_t(x)}{A_t(N)} \leq \left(\frac{q(x)}{q(N)}\right)^t \underbrace{(1-q(x_0)^{-2})^{-1}}_{C_{x_0}} = \left(\frac{x}{N}\right)^t C_{x_0}\left(\frac{1+\sqrt{1-4/x^2}}{1+\sqrt{1-4/N^2}}\right) \leq \left(\frac{x}{N}\right)^t C_{x_0}.$$

∎

CONCLUSION :

Theorem: S_N^+ has the HAP for $N \geq 2$.

Proof: • $N = 2,3,4$ $\mathbb{G} = S_N^+$ is coamenable $\Rightarrow L^\infty(\mathbb{G})$ is injective $\Rightarrow \mathbb{G}$ has HAP

• $N \geq 5$ Consider the states on the central algebra of S_N^+ given by ev_x

$$\varphi : C^*\langle \chi_k \rangle \longrightarrow C(\mathrm{Spec}(\chi))$$
$$\chi_k \longmapsto \pi_{k|\sigma}$$

So we get NUCP trace preserving maps $T_{ev_x} \subset CB(L^\infty(\mathbb{G}))$

we need to show: (i) $T_{ev_x} : L^2 \to L^2$ compact

(ii) $\|T_{ev_x} a - a\|_{L^2} \xrightarrow[x \to N]{} 0 \quad \forall a \in \mathcal{B}(\mathbb{G})$

Theorem : $\mathrm{Spec}(\chi) = [0, N]$

(i) $$T_{ev_x} = \sum_{k \in \mathbb{N}} \frac{ev_x(\pi_k)}{d_k} P_k$$

$$\frac{ev_x(\pi_k)}{d_k} = \frac{\pi_k(x)}{\pi_k(N)} = \frac{A_{2k}(\sqrt{x})}{A_{2k}(\sqrt{N})} \leq C_{x_0} \left(\frac{\sqrt{x}}{\sqrt{N}}\right)^{2k} \xrightarrow[k \to \infty]{} 0 \qquad \text{we get compactness}$$

(ii) check this for $a = u_{ij}^k$

$$\| T_{ev_x} u_{ij}^k - u_{ij}^k \|_{L^2} = \| u_{ij}^k \|_{L^2} \left| 1 - \frac{\pi_k(x)}{d_k} \right| \xrightarrow[x \to N]{} 0.$$

∎