20/11/2012

Yulia

(s'inscrit dans la suite de l'exposé de Martin)

⓪ $\boxed{\text{Recall}}$: the <u>multiplicative unitary</u> for a LCG G is $W, V \in B(L_2(G)^{\otimes 2}) = B(L_2(G\times G))$ s.t. $Vx(s,t) = x(st,t)$ and $Wx(s,t) = x(s,st)$. Then $\Delta(x) = W(1\otimes x)W^*$

① $\boxed{\text{Twisting by cocycles}}$: a way to obtain non-co-commutative LCQG from co-commutative ones, e.g. from $VN(G)$.

This is due to Drinfeld for the algebraic part:

Let $\Omega \in M\bar{\otimes}M$ be a unitary, where M is a Hopf-von-Neumann-algebra.

And change the comultiplication of M: $\Delta_\Omega(x) = \Omega\,\Delta(x)\,\Omega^{-1} = \Omega\,\Delta(x)\,\Omega^*$

Q: When is Δ_Ω a comultiplication, i.e. when is it coassociative?

Let $\partial_2\Omega = (\iota\otimes\Delta)(\Omega^*)(1\otimes\Omega^*)(\Omega\otimes 1)(\Delta\otimes\iota)(\Omega)$

<u>Def</u>: Ω is a <u>2-cocycle</u> if $\partial_2\Omega = 1$, i.e. if $(\Omega\otimes 1)(\Delta\otimes\iota)(\Omega) = (1\otimes\Omega)(\iota\otimes\Delta)(\Omega)$

Ω is a <u>2-pseudococycle</u> if $\partial_2\Omega(\Delta\otimes\iota)\Delta(x) = (\Delta\otimes\iota)\Delta(x)\,\partial_2\Omega$ for $x\in M$.

<u>Prop</u>: Ω is a pseudococycle iff Δ_Ω is coassociative

<u>Proof</u>: We need to check $(\Delta_\Omega\otimes\iota)\circ\Delta_\Omega = (\iota\otimes\Delta_\Omega)\circ\Delta_\Omega$

We have $(\Delta_\Omega\otimes\iota)(a\otimes b) = \Omega\Delta(a)\Omega^{-1}\otimes b = (\Omega\otimes 1)(\Delta(a)\otimes b)(\Omega^*\otimes 1)$

N.B.: $(\Delta_\Omega\otimes\iota)(\xi) = (\Omega\otimes 1)(\Delta\otimes\iota)(\xi)(\Omega^*\otimes 1)$

Then $(\Delta_\Omega\otimes\iota)\Delta_\Omega(x) = (\Omega\otimes 1)(\Delta\otimes\iota)(\Omega\Delta(x)\Omega^*)(\Omega^*\otimes 1)$

$= (\Omega\otimes 1)(\Delta\otimes\iota)(\Omega)\underbrace{(\Delta\otimes\iota)\Delta(x)}(\Delta\otimes\iota)(\Omega^*)(\Omega^*\otimes 1)$

because $\Delta\otimes\iota$ is an algebra homomorphism.

$= (1\otimes\Omega)(\iota\otimes\Delta)(\Omega)(\iota\otimes\Delta)(\Omega^*)(1\otimes\Omega^*)$ —"—

$= (1\otimes\Omega)(\iota\otimes\Delta)(\Omega)(\iota\otimes\Delta)\Delta(x)(\iota\otimes\Delta)(\Omega^*)(1\otimes\Omega^*)(\Omega\otimes 1)\cdot$
$\cdot(\Delta\otimes\iota)(\Omega)(\Delta\otimes\iota)(\Omega^*)(\Omega^*\otimes 1)$

Now let us use that Ω is a unitary:

$(\iota\otimes\Delta_\Omega)\Delta_\Omega(x) = (1\otimes\Omega)(\iota\otimes\Delta)(\Omega\Delta(x)\Omega^*)(1\otimes\Omega^*)$.

This shows one direction. The other will be admitted.

Q: 1) How to construct it

2) What properties are preserved?

2 Some constructions:

1) Let $u \in M$. Then $\partial_1 u = (u^* \otimes u^*)\Delta(u)$ is a 2-cocycle. Let us compute

$$(u^* \otimes u^* \otimes 1)(\cancel{\Delta(u)} \otimes 1)(\cancel{\Delta(u^*)} \otimes u^*)(\Delta \otimes \iota)\Delta(u):$$

It is $= (1 \otimes u^* \otimes u^*)(1 \otimes \Delta(u))(u^* \otimes \Delta(u^*))(\iota \otimes \Delta)(\Delta(u))$

and also $= (u^* \otimes u^* \otimes u^*)(\iota \otimes \Delta)(\Delta(u))$ by canceling out.

2) Let $M = L^\infty(G)$ and $\omega \in L^\infty(G \times G)$. It is unitary iff $|\omega| = 1$.

N.B.: $(\Omega \otimes 1)(\Delta \otimes \iota)(\Omega) = (1 \otimes \Omega)(\iota \otimes \Delta)(\Omega)$.

$$(\omega \otimes 1)(\Delta \otimes \iota)(\omega)(r,s,t) = \boxed{\omega(r,s)\,\omega(rs,t)}$$
$$= (1 \otimes \omega)(\iota \otimes \Delta)(\omega)(r,s,t) = \boxed{\omega(s,t)\,\omega(r,st)}$$

This expresses that ω is a 2-cocycle

Since M is commutative, every pseudococycle is a 2-cocycle.

- A bicharacter (i.e., $\forall s$ $\omega_s : t \longmapsto \omega(s,t)$ and $\omega^s : t \longmapsto \omega(t,s)$ are characters) is a 2-cocycle: in fact, $\omega(rs,t) = \omega(r,t)\,\omega(s,t)$, $\omega(r,st) = \omega(r,s)\,\omega(r,t)$ hold.

On $\mathbb{R}^n$, if $B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a bilinear form, then $\omega(s,t) = e^{iB(s,t)}$ is a bicharacter.

3 Theorems and references

Quantum symmetry in noncommutative geometry, to be published by EMS (Hajac, ed.): it is good but outdated

[EV] Enock, Vainerman, Comm. Math. Physics (1996)

[dC] K. de Commer, JFA (2009): On cocycle twisting of CQG.

JOT (2011): Galois objects and cocycle twisting for LCQG

Th (dC) Every twisting of a LCQG by a 2-cocycle is a LCQG. In particular, for every 2-cocycle Ω there is a unitary $u \in M$ such that

$$\Theta\,(R \otimes R)\,\Omega^* = (u^* \otimes u^*)\,\Omega\,\Delta(u)$$

(flip) (unitary antipode)

Theorem (EV): If M is discrete and Kac, then M_Ω is discrete and Kac
compact and Kac, then M_Ω is compact and Kac.

Th (dC), there is a compact M such that M_Ω is not compact.

- M_Ω is a twisting of $\bigotimes_{n=1}^{\infty} SU_{q_n}(2)$
- Here compact means that the Haar weight is a state.

(4) Twisting of SU_q's

Recall: $SU_q(2)$ is the CQG generated by a, b such that $\begin{cases} a^*a + b^*b = 1 \\ aa^* + q^2 bb^* = 1 \\ ab = qba \\ a^*b = q^{-1} b a^* \\ bb^* = b^*b \end{cases}$

and $\Delta(a) = a \otimes a - q b^* \otimes b$
$\Delta(b) = b \otimes a + a^* \otimes b$, where $q \in]-\infty, 0[\cup]0, +\infty[$.

If $q = 1$, one gets the function space $SU(2)$.

We now choose $q = (q_n) \in \mathbb{N}^{]-1,0[\cup]0,1[}$ with and $\sum q_n^2 < \infty$

and let $\mathcal{A}$ be the inductive limit of $\bigotimes_{k=1}^{n} \mathrm{Pol}(SU_{q_n}(2))$ (Pol = polynomials in a, a^*, b, b^*)

and $A = SU_q(2) = C^*(\mathcal{A})$: A is the universal infinite tensor product of $SU_{q_n}(2)$.
it also is the reduced C^*-tensor product wrt Haar state.

It is a CQG but not a CMQG.

N.B. $SU_q(2)$ is nuclear: cf Woronowicz

Now consider $M = L^\infty(SU_q(2))$.

In every $SU_q(2)$, construct a cocycle ω. Consider a representation π on $\ell_2(\mathbb{N} \times \mathbb{Z})$.
its basis is $(\xi_{nk})_{n \geq 0, k \in \mathbb{Z}}$ and $\pi(a)\xi_{nk} = \sqrt{1 - q^{2n}}\, \xi_{n-1,k}$
$\pi(b)\xi_{n,k} = q^n \xi_{n,k+1}$

$\mathbb{Z}$
$e_0 \otimes \ell_2(\mathbb{Z})$
$\pi(a)$
$e_n \otimes \ell_2(\mathbb{Z})$
$\pi(b)$
$\mathbb{N}$

The Haar state is $\varphi(x) = (1-q^2) \sum_{n \in \mathbb{N}} q^{n} \langle \pi(x) \xi_{n0}, \xi_{n0} \rangle$

π is faithful; in $B(\ell_2(\mathbb{N} \times \mathbb{Z}))$ define $f_{k\ell}\xi_{nm} = \delta_{n\ell}\xi_{km}$:

$f_{k\ell} : e_\ell \otimes \ell^2(\mathbb{Z}) \longmapsto e_k \otimes \ell^2(\mathbb{Z})$
$e_{\ell'} \otimes \ell^2(\mathbb{Z}) \longmapsto 0$ if $\ell' \neq \ell$.

Every $f_{k\ell}$ is in the unital C^*-algebra generated by $\pi(a)$: We can consider
$\pi(a) e_n = \sqrt{1 - q^{2n}}\, e_{n-1}$ and $f_{k\ell} e_n = \delta_{\ell n} e_k$: $\pi(a^*) e_n = \lambda_{n+1} e_{n+1}$
$= \lambda_n e_{n-1}$ $\pi(a) e_0 = 0$
$\pi(a^{*m} a^m) e_n = \begin{cases} 0 & \text{if } n < m \\ \lambda \ldots e_n & \text{if } n \geq m \end{cases}$

Let $C = 1 - \pi(aa^*)$: $Ce_n = e_n - \lambda^2_{n+1} e_n = (1-\lambda^2_{n+1})e_n$ and note that $\lambda_n \longrightarrow 1$

Also, $c^k e_0 = (1-\lambda_1^2)^k e_0$ and $\dfrac{c^k}{(1-\lambda_1^2)^k} e_n = \dfrac{(1-\lambda_n^2)^k}{(1-\lambda_1^2)^k} e_n \longrightarrow 0$ if $n>0$.

Thus $p_0 \in C^*(\pi(a))$ and $f_{k\ell} = \text{cst} \times \pi(a^*)^k p_0 \pi(a)^\ell$

N.B.: c is compact and selfadjoint, so one could get the conclusion without computation.

Now we can define $e_{k\ell} = \pi^{-1}(f_{k\ell})$: $\mathrm{Id} = \sum_{k=0}^{\infty} f_{kk}$ [check on ξ_{mn}].

Consider $\omega = \left(\begin{array}{ccc|c} 0&1&0& \\ 0&0&1& \\ 1&0&0& \\ \hline &&& \mathrm{Id} \end{array}\right) = e_{01} + e_{12} + e_{20} + \sum_{k=3}^{\infty} e_{kk}$ and ω_n for every n.

[Up to characters, we get all rep π this way]

The twisting is strange because it is trivial at finite level.

$$\begin{tikzcd} \cap \arrow[r, "\Delta"] \arrow[d, "u\omega u"'] & \cap \otimes M \arrow[d, "u^*\otimes u^*"] \\ \cap \arrow[r, "\Delta_\Omega"] & \cap \otimes M \end{tikzcd}$$

Let $\Omega = \bigotimes_{n=1}^{\infty} \left((\omega_n \otimes \omega_n)\Delta_n(\omega_n^*)\right)$ it exists as a σ-strong limit.

At finite steps, we get the 2-cocycle identity, so also at the limit.

[Q. is the ∞ $\otimes$ of the multiplicative unitaries also one; can you extend Δ normally?]

The whole theory of Galois objects is needed to prove that is is compact. We shall not do it but prove that the twisted group is not.

N.B.: for $\underset{x \geq 0}{x \in} SU_q(2)$, $\varphi(x) \leq q^{-2}\varphi(\omega^* x \omega)$

One can check that $\varphi_\Omega(\cdot) = \lim\limits_{n\to\infty}\left(\prod_{k=1}^{n} q_k^{-2}\right)\left(\left(\bigotimes_{k=1}^{n} \varphi_k(\omega_k^* \cdot \omega_k)\right) \otimes \left(\bigotimes_{k=n+1}^{\infty} \varphi_k(\cdot)\right)\right)$

is the Haar weight for $SU_q(2)_\Omega$. It is normal as the limit of an increasing sequence; one has to prove that φ_Ω is invariant and semifinite. It is not finite.

Recall that
$$\varphi(1) = (1-q^2)\sum_{m=0}^{\infty} q^{2m}\langle \pi(1)\xi_{m0}, \xi_{m0}\rangle = (1-q^2)\sum_{m=0}^{\infty} q^{2m} = \frac{1-q^2}{1-q^2} = 1.$$

Also $\varphi_\Omega(1) = \lim\limits_{n\to\infty} \prod_{k=1}^{n} q_k^{-2} = +\infty$.

It is semifinite as it is finite on $1 \otimes \bigotimes_{k=n+1}^{\infty} p_k$, where $p_k \in SU_{q_k}(2)$ is the element e_{00}; note that $\omega^* p \omega = e_{11}$.

One also checks invariance.

5 Classical and twisted Heisenberg group $H_n(\mathbb{R})$

$H_n(\mathbb{R}) = \mathbb{R}^{n+1} \rtimes \mathbb{R}^n$, where for $a, b, v, w \in \mathbb{R}^n$ and $s, t, u \in \mathbb{R}$:

$$(b_1, t_1, a_1)(b_2, t_2, a_2) = (b_1 + b_2,\ t_1 + t_2 + \langle a_1, b_2 \rangle,\ a_1 + a_2)$$

Thus $\mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1} \times \{0\} \hookrightarrow H_n(\mathbb{R})$ and $M = VN(H_n(\mathbb{R})) \longleftarrow VN(\mathbb{R}^{n+1}) \simeq L^\infty(\widehat{\mathbb{R}^{n+1}})$, $\mathbb{R}^{n+1} \xrightarrow{\lambda} \widehat{\mathbb{R}^{n+1}}$

Realise M as generated by a representation π of $H_n(\mathbb{R})$, "quasi-isomorphic" to the left regular representation.

Consider $\beta(\lambda(b,t)) = \pi(b,t,0)$, here $\lambda : \mathbb{R}^{n+1} \to L^\infty(\widehat{\mathbb{R}^{n+1}})$ is the regular representation of $\mathbb{R}^{n+1}$: $\lambda(b,t) f(u,v) = e^{i(\langle b, v\rangle + tu)} f(u,v) = \lambda_{b,t}(u,v) \cdot f(u,v)$

Before defining π, consider first the left regular representation λ of $H_n(\mathbb{R})$.

It acts on $L_2(H_n(\mathbb{R})) \simeq L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}^n) \ni f$:

$$\lambda(b,t,a) f(v,s) = f(v-b,\ s-t-\langle a, v-b\rangle,\ w-a)$$

Define a unitary operator $\mathcal{U}$ on $L_2(H_n(\mathbb{R}))$ as $\mathcal{U} f(\hat{v}, \hat{s}, \omega) = \sqrt{\frac{|\hat{s}|}{2}} \int f(v, s, \hat{v} - w) e^{i\langle v, \hat{v}\rangle \hat{s} + is\hat{s}}\, ds\, dv$ (partial Fourier transform)

Now set $\tilde{\pi}(b,t,a) = \mathcal{U} \lambda(b,t,a) \mathcal{U}^*$, equivalent to the original one; $\mathbb{R}^n \times \mathbb{R}$

$$\tilde{\pi}(b,t,a) f(v,s,w) = e^{i(\langle b, v\rangle + t)s} f(v+a, s, w) \quad : \tilde{\pi} \simeq \pi \otimes id$$

Then define π on $L^2(\mathbb{R}^{n+1})$ as $\pi(b,t,a) f(v,s) = e^{i(\langle b,v\rangle + t)s} f(v+a, s)$

If $a = 0$, $\pi(b,t,0) f(v,s) = e^{i(\langle b,v\rangle + t)s} f(v,s) = \lambda_{b,t}(s, sv) f(v,s)$

So
$$\beta : L^\infty(\mathbb{R}^{n+1}) \longrightarrow B(L^2(\mathbb{R}^{n+1}))$$
$$f \longmapsto (\beta f : (v,s) \mapsto f(vs, s)) \in L^\infty(\mathbb{R}^{n+1}) \subset VN(H_n(\mathbb{R}))$$

Now we can lift a cocycle on the commutative $\mathbb{R}^{n+1}$ to the vN algebra.

We will construct a cocycle ω on $L^\infty(\mathbb{R}^{n+1})$ and then put $\Omega = (\beta \otimes \beta)\omega$:

ω is defined as $\omega(v,s,u,t) = e^{iB(v,s,u,t)}$ with B nonsymmetric bilinear (real) form, for example $\omega(\cdot) = e^{i \sum'_{j<k} q_{jk}(v_j u_k - v_k u_j)}$: this is a cocycle on $\mathbb{R}^{n+1}$.

Since $\Delta\beta = (\beta \otimes \beta)\Delta$, we have
$$(\Omega \otimes 1)(\Delta \otimes \iota)\underset{((\beta\otimes\beta)\omega)}{(\Omega)} = ((\beta\otimes\beta)\omega \otimes \underset{\beta(1)}{1})(\beta\otimes\beta\otimes\beta)(\Delta\otimes\iota)(\omega)$$
$$= (\beta\otimes\beta\otimes\beta)((\omega\otimes 1)(\Delta\otimes\iota)(\omega)) = (\beta\otimes\beta\otimes\beta)((1\otimes\omega)(\iota\otimes\Delta)(\omega))$$
$$= (1\otimes\Omega)\cdot(\iota\otimes\Delta)(\Omega)$$

(homomorphism!)

Thus Ω is a cocycle. Explicitly, $\Omega(v,s,u,t) = e^{i\sum_{j<k} q_{jk}\, st(v_j u_k - v_k u_j)}$ is an element of $L^\infty(\mathbb{R}^{n+1}) \otimes L^\infty(\mathbb{R}^{n+1}) \subset VN(H_n(\mathbb{R})) \otimes VN(H_n(\mathbb{R}))$.

One might also consider the enveloping Lie algebra and observe the effect of twisting. and the corresponding infinitesimal generators of the representation π:

$$P_k\, \varphi(u,v) = \frac{\partial}{\partial a_k}\Big(\pi(b,t,a)\,\varphi(u,v)\Big)_{\substack{a=b=0\\ t=0}} = \frac{\partial \varphi}{\partial v_k}(u,v)$$

$$R\, \varphi(u,v) = \frac{\partial}{\partial t}\Big(\qquad \Big) = iu\,\varphi(u,v)$$

$$Q_k\, \varphi(u,v) = \frac{\partial}{\partial b_k}\Big(\qquad \Big) = iu\, v_k\, \varphi(u,v).$$

then $\Delta(P_k) = P_k \otimes 1 + 1 \otimes P_k$; after twisting

$$\Delta_\Omega(P_k) = \Delta(P_k) + \sum_{j\neq k} q_{jk}\left(Q_j \otimes iR - iR \otimes Q_j\right)\cdot \varepsilon_{jk}, \text{ where } \varepsilon_{jk} = \begin{cases} 1, & j<k \\ -1, & j>k \end{cases}$$