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Quantum E(2) group

Due to Woronowicz : • Lett. Math. Phys. 23 (1991), 251–263
• Comm Math Phys. 136 (1991) 399–432.

Baaj : 2 papers (Astérisque, CRAS)

Armand Jacobs : PhD thesis under van Daele, Leuven (320p.)

(1) Affiliation: A C^*-algebra, $A \subset B(H)$

T unbounded closed operator on H.

We say that T is affiliated to A ($T \eta A$) if:

let $z_T = T(1+T^*T)^{-\frac{1}{2}}$: $\|z_T\| \leq 1$ as $\|Tx\|^2 = \langle T^*Tx, x\rangle$

$\|z_T\| < 1 \iff T$ bounded

$T = z_T(1 - z_T^* z_T)^{-\frac{1}{2}}$

Def.: Let A be a C^*-algebra, T closed: $T \eta A$ iff $z_T \in M(A)$

$(1+T^*T)^{-\frac{1}{2}}A$ dense in A.

Properties: • Let $A^\eta = \{T : T\eta A\}$. Then $A \subset M(A) \subset A^\eta$ and $M(A) = A^\eta \Rightarrow A$ is unital.

• $T\eta A \Rightarrow T^* \eta A$

• $z_{T^*} = z_T^*$

Ex: If $A = C_0(X)$, X l.c., then $M(A) = C_b(X)$ and $A^\eta = C(X)$.

Th: Let T be a closed normal operator, A a C^*-algebra. TFAE

• $T \eta A$

• $\{f(T) : f \in C_0(\sigma(T))\} \subset M(A)$ and contains an approximate unit for A.

Th: Let $\pi : A \longrightarrow M(B)$ be a nondegenerate $*$-homomorphism and $S \eta A$.

There is a unique $T \eta B$ such that $\pi(z_S) = z_T$. We will write $T = \pi(S)$.

(2) Definition of $E_\mu(2)$, $\mu \in]0,1[$ or $\mu > 1$.

Let $H = \ell_2(\mathbb{Z}\times\mathbb{Z})$ with basis e_{kl} : define $\begin{cases} v e_{kl} = e_{k-1,l} \\ n e_{kl} = \mu^k e_{k,l+1} \end{cases}$ v unitary, n unbounded normal.

$\sigma(n) \subset \bar{\mathbb{C}}^\mu := \{0\} \cup \{\mu^k S^1 : k \in \mathbb{Z}\}$.

We have $vnv^* = \mu n$. Let $A = \{\sum'_{k\in\mathbb{Z}} v^k f_k(n) : f_k \in \mathcal{C}_0(\overline{\mathbb{C}}^\mu)\}^{\|\cdot\|}$

$\simeq \mathcal{C}_0(\overline{\mathbb{C}}^\mu) \rtimes_\mu \mathbb{Z}$, where $\mu_k f(z) = f(\mu^{-k} z)$.

One can see that $v, n \,\eta\, A$.

<u>Theorem</u> (universality property): For any representation π of A on a Hilbert space $\hat{H}$ let $\tilde{v} = \pi(v)$, $\tilde{n} = \pi(n)$. Then $(*)$ $\tilde{n}$ is normal, $\tilde{v}$ unitary, we have $\tilde{v}\tilde{n}\tilde{v}^* = \tilde{n}$ and $\sigma(\tilde{n}) \subset \overline{\mathbb{C}}$

Conversely, let $\tilde{v}, \tilde{n}$ satisfying $(*)$; there is a unique representation π of A s.t. $\pi(n) = \tilde{n}$, $\pi(v) = \tilde{v}$.
Also $\tilde{v}, \tilde{n} \,\eta\, \tilde{A} \Leftrightarrow \pi \in \mathrm{Mor}(A, \tilde{A})$, i.e. $\pi : A \longrightarrow M(\tilde{A})$.

Let now $\tilde{v} = v \otimes v$ and $\tilde{n} = v \otimes n + n \otimes v^*$. By definition, $\mathcal{D}(n) \otimes_{alg} \mathcal{D}(n)$ is a core for $\tilde{n}$

[Recall: C is a core for T if $C \subset \mathcal{D}(T)$ and $\overline{T|_C} = \overline{T}$.]

Then $\tilde{v}, \tilde{n}$ satisfy $(*)$: thus $\tilde{v}, \tilde{n} \,\eta\, A \otimes A$; by the Theorem,

there is a unique morphism $\phi : A \longrightarrow M(A \otimes A)$ s.t. $\tilde{n} = \phi(n)$, $\tilde{v} = \phi(v)$.

By definition, ϕ is the comultiplication on $A = E_\mu(2)$.

The Haar weight is, for $x \in A^+$, $\varphi(x) = \sum_{k\in\mathbb{Z}} \mu^{-2k} \langle x e_{0k}, e_{0k} \rangle$

One can write an explicit formula for W, the multiplicative unitary.

(3) Why do we need the spectral condition.

Suppose that v, n, operators on H, satisfy the relations $\odot$ $\begin{cases} v \text{ unitary} \\ n \text{ normal} \\ v^* n v = \mu n. \end{cases}$

<u>Th</u> let (v_1, n_1) and (v_2, n_2) satisfy $\odot$. Let $N = v_1 \otimes n_2 + n_1 \otimes v_2^*$.
By definition, set $\mathcal{D}(N) = \mathcal{D}(v_1 \otimes n_2) \cap \mathcal{D}(n_1 \otimes v_2^*)$. Then

(1) If $|\sigma(n_1)| \neq |\sigma(n_2)|$, where $|\sigma(T)| = \{\mu z : \mu \in \sigma(n_1), z \in S^1\}$,
then N is closed, but not normal and has no normal extension.

(2) If $|\sigma(n_1)| = |\sigma(n_2)|$, then N is closable and the closure is normal
(in this case, $\sigma(\overline{N}) = |\sigma(n_1)| = |\sigma(n_2)|$.

Idea of proof: realise v_j, n_j concretely on $L_2(\mathbb{Z} \times S^1)$, find $\mathcal{D}(N)$, $\mathcal{D}(N^*)$ and show that $\dim \mathcal{D}(N^*)/\mathcal{D}(N) = 1$. If there were a normal extension $\tilde{N}$, we would have $\dim(\mathcal{D}(N^*)/\mathcal{D}(N)) = \dim(\mathcal{D}(N^*)/\mathcal{D}(\tilde{N}^*)) + \dim \mathcal{D}(\tilde{N})/\mathcal{D}(N)$
which is even $\lightning$.

④ The dual group $\hat{E}_\mu(2)$.

Set $\Sigma'_\mu = \{(s, \mu^{\tau}) : s \in \mathbb{Z},\ \tau - \frac{s}{2} \in \mathbb{Z}\}$

$K = \ell^2(\Sigma_\mu)$: let $\xi_{s,\lambda}$ be its canonical basis, $(s,\lambda) \in \Sigma_\mu$.

Set $\begin{cases} N \xi_{s,\lambda} = s\, \xi_{s,\lambda} \\ b\, \xi_{s,\lambda} = \lambda\, \xi_{s-2,\lambda} \end{cases}$

Then (**) $\begin{cases} N \text{ is selfadjoint, } b \text{ is normal, } N \text{ and } |b| \text{ strongly commute.} \\ \text{On } (\ker b)^{\perp},\ (\text{Phase } b)^* N (\text{Phase } b) = N + 2I. \\ \text{The joint spectrum } \sigma(N, |b|) \subset \overline{\Sigma_\mu}, \text{ the closure of } \Sigma_\mu. \end{cases}$

Moreover, $\ker b = \{0\}$, Phase b is unitary.

Def : Set $B = \hat{E}_\mu(2) = \overline{\left\{ \sum'_{k \in \mathbb{Z}} (\text{Phase } b)^k g_k(N, |b|) : g_k \in C_0(\overline{\Sigma_\mu}) \right\}}^{\|\cdot\|}$ $\quad g_k(s,0) = 0$ if $k \neq 0$.

then B is a nonunital C^*-algebra and $b, N \,\eta\, B$.

Theorem : For any representation ρ of B, $(\rho(N), \rho(b))$ satisfy (**); conversely, for all $\tilde{N}, \tilde{b}$ satisfying (**), there is a unique representation ρ of B such that $\tilde{N} = \rho(N)$, $\tilde{b} = \rho(b)$.

⑤ The multiplicative unitary :

Let $F_\mu(t) = \begin{cases} \prod \dfrac{1 + \mu^{2k} \bar{t}}{1 + \mu^{2k} t} & : t \neq -(\mu)^{-2k} \\ +1 & : t = -(\mu)^{-2k} \end{cases}$ (here one might have to suppose that $\mu > 1$)

F_μ is continuous on $\overline{\mathbb{C}_\mu}$, $W = F_\mu(\mu^{\frac{N}{2}} b \otimes vn)\,(I \otimes v)^{N \otimes I}$

is a unitary, an element of $M(B \otimes A)$, and $(\mathrm{Id} \otimes \Phi) W = W_{12} W_{13}$

In particular, $(\mathrm{Id} \otimes v)^{N \otimes I} = \chi(N \otimes \mathrm{Id}, \mathrm{Id} \otimes v)$, where $\chi \in C_b(\mathbb{Z} \times S^1)$, $\chi(k, z) = z^k$.

and W is « manageable » : $(\omega \otimes I)(W)$, $\omega \in B(H_1)_*$ is a Hopf algebra. $(I \otimes \omega)(W)$ also.

Take $\begin{pmatrix} v & n \\ 0 & v^{-1} \end{pmatrix}$ and let them act (rotation + translation.

$\to$ E like Euclidean.

2 dimensional symmetries