Introduction to tensor categories
(Extract from my lecure in [QIIP-II])

1 Categories, functors, and natural transformations

Let us first recall the basic definitions and properties from category theory that we shall use. For a thorough introduction, see, e.g., [Mac98].

Definition 1.1. *A* category $\mathcal{C}$ *consists of*

(a) a class $\mathrm{Ob}\,\mathcal{C}$ *of* objects *denoted by* $A, B, C, \ldots$*,*
(b) a class $\mathrm{Mor}\,\mathcal{C}$ *of* morphism *(or* arrows*) denoted by* $f, g, h, \ldots$*,*
(c) mappings $\mathrm{tar}, \mathrm{src} : \mathrm{Mor}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{C}$ *assigning to each morphism* f *its* source *(or* domain*)* $\mathrm{src}(f)$ *and its* target *(or* codomain*)* $\mathrm{tar}(f)$*. We will say that* f *is a morphism in* $\mathcal{C}$ *from* A *to* B *or write "*$f : A \to B$ *is a morphism in* $\mathcal{C}$*" if* f *is a morphism in* $\mathcal{C}$ *with source* $\mathrm{src}(f) = A$ *and target* $\mathrm{tar}(f) = B$*,*
(d) a composition $(f, g) \mapsto g \circ f$ *for pairs of morphisms* f, g *that satisfy* $\mathrm{src}(g) = \mathrm{tar}(f)$*,*
(e) and a map $\mathrm{id} : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Mor}\,\mathcal{C}$ *assigning to an object* A *of* $\mathcal{C}$ *the* identity morphism $\mathrm{id}_A : A \to A$*,*

such that the

(1) associativity property*: for all morphisms* $f : A \to B$*,* $g : B \to C$*, and* $h : C \to D$ *of* $\mathcal{C}$*, we have*

$$(h \circ g) \circ f = h \circ (g \circ f),$$

and the
(2) identity property*:* $\mathrm{id}_{\mathrm{tar}(f)} \circ f = f$ *and* $f \circ \mathrm{id}_{\mathrm{src}(f)} = f$ *holds for all morphisms* f *of* $\mathcal{C}$*,*

are satisfied.

Let us emphasize that it is not so much the objects, but the morphisms that contain the essence of a category (even though categories are usually named after their objects). Indeed, it is possible to define categories without referring to the objects at all, see the definition of "arrows-only metacategories" in [Mac98, Page 9]. The objects are in one-to-one correspondence with the identity morphisms, in this way $\mathrm{Ob}\,\mathcal{C}$ can always be recovered from $\mathrm{Mor}\,\mathcal{C}$.

We give an example.

Example 1.2. Let $\mathrm{Ob}\,\mathfrak{Set}$ be the class of all sets (of a fixed universe) and $\mathrm{Mor}\,\mathfrak{Set}$ the class of total functions between them. Recall that a *total function* (or simply *function*) is a triple (A, f, B), where A and B are sets, and $f \subseteq A \times B$ is a subset of the cartesian product of A and B such that for a given $x \in A$ there exists a unique $y \in B$ with $(x, y) \in f$. Usually one denotes this unique element by $f(x)$, and writes $x \mapsto f(x)$ to indicate $\big(x, f(x)\big) \in f$. The triple (A, f, B) can also be given in the form $f : A \to B$. We define

$$\operatorname{src}\big((A, f, B)\big) = A, \qquad \text{and} \qquad \operatorname{tar}\big((A, f, B)\big) = B.$$

The composition of two morphisms (A, f, B) and (B, g, C) is defined as

$$(B, g, C) \circ (A, f, B) = (A, g \circ f, C),$$

where $g \circ f$ is the usual composition of the functions f and g, i.e.

$$g \circ f = \{(x, z) \in A \times C;\ \text{there exists a } y \in B \text{ s.t. } (x, y) \in f \text{ and } (y, z) \in g\}.$$

The identity morphism assigned to an object A is given by (A, id_A, A), where $\mathrm{id}_A \subseteq A \times A$ is the identity function, $\mathrm{id}_A = \{(x, x); x \in A\}$. It is now easy to check that these definitions satisfy the associativity property and the identity property, and therefore define a category. We shall denote this category by $\mathfrak{Set}$.

Definition 1.3. *Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ in $\mathcal{C}$ is called an* isomorphism *(or* invertible*), if there exists a morphism $g : B \to A$ in $\mathcal{C}$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. Such a morphism g is uniquely determined, if it exists, it is called the* inverse *of f and denoted by $g = f^{-1}$. Objects A and B are called* isomorphic*, if there exists an isomorphism $f : A \to B$.*

Morphisms f with $\operatorname{tar}(f) = \operatorname{src}(f) = A$ are called endomorphisms *of A. Isomorphic endomorphism are called* automorphisms*.*

For an arbitrary pair of objects $A, B \in \operatorname{Ob}\mathcal{C}$ we define $\operatorname{Mor}_{\mathcal{C}}(A, B)$ to be the collection of morphisms from A to B, i.e.

$$\operatorname{Mor}_{\mathcal{C}}(A, B) = \{f \in \operatorname{Mor}\mathcal{C}; \operatorname{src}(f) = A \text{ and } \operatorname{tar}(f) = B\}.$$

Often the collections $\operatorname{Mor}_{\mathcal{C}}(A, B)$ are also denoted by $\hom_{\mathcal{C}}(A, B)$ and called the *hom-sets* of $\mathcal{C}$. In particular, $\operatorname{Mor}_{\mathcal{C}}(A, A)$ contains exactly the endomorphisms of A, they form a semigroup with identity element with respect to the composition of $\mathcal{C}$ (if $\operatorname{Mor}_{\mathcal{C}}(A, A)$ is a set).

Compositions and inverses of isomorphisms are again isomorphisms. The automorphisms of an object form a group (if they form a set).

Example 1.4. Let $(G, \circ, e)$ be a semigroup with identity element e. Then $(G, \circ, e)$ can be viewed as a category. The only object of this category is G itself, and the morphisms are the elements of G. The identity morphism is e and the composition is given by the composition of G.

Definition 1.5. *For every category $\mathcal{C}$ we can define its* dual or opposite category *$\mathcal{C}^{\mathrm{op}}$. It has the same objects and morphisms, but target and source are interchanged, i.e.*

$$\operatorname{tar}_{\mathcal{C}^{\mathrm{op}}}(f) = \operatorname{src}_{\mathcal{C}}(f) \ \textit{and}\ \operatorname{src}_{\mathcal{C}^{\mathrm{op}}}(f) = \operatorname{tar}_{\mathcal{C}}(f)$$

and the composition is defined by $f \circ_{\mathrm{op}} g = g \circ f$. We obviously have $\mathcal{C}^{\mathrm{op}\,\mathrm{op}} = \mathcal{C}$.

Dualizing, i.e. passing to the opposite category, is a very useful concept in category theory. Whenever we define something in a category, like an epimorphism, a terminal object, a product, etc., we get a definition of a "co-something", if we take the corresponding definition in the opposite category. For example, an *epimorphism* or *epi* in $\mathcal{C}$ is a morphism in $\mathcal{C}$ which is right cancellable, i.e. $h \in \operatorname{Mor}\mathcal{C}$ is called an epimorphism, if for any morphisms $g_1, g_2 \in \operatorname{Mor}\mathcal{C}$ the equality $g_1 \circ h = g_2 \circ h$ implies $g_1 = g_2$. The dual notion of a epimorphism is a morphism, which is an epimorphism in the category $\mathcal{C}^{\mathrm{op}}$, i.e. a morphism that is left cancellable. It could therefore be called a "co-epimorphism", but the generally accepted name is *monomorphism* or *monic*. The same technique of dualizing applies not only to definitions, but also to theorems. A morphism $r : B \to A$ in $\mathcal{C}$ is called a *right inverse* of $h : A \to B$ in $\mathcal{C}$, if $h \circ r = \mathrm{id}_B$. If a morphism has a right inverse, then it is necessarily an epimorphism, since $g_1 \circ g = g_2 \circ h$ implies $g_1 = g_1 \circ g \circ r = g_2 \circ h \circ r = g_2$, if we compose both sides of the equality with a right inverse r of h. Dualizing this result we see immediately that a morphism $f : A \to B$ that has a *left inverse* (i.e. a morphism $l : B \to A$ such that $l \circ f = \mathrm{id}_A$) is necessarily a monomorphism. Left inverses are also called *retractions* and right inverses are also called *sections*. Note that one-sided inverses are usually not unique.

Definition 1.6. *A category $\mathcal{D}$ is called a* subcategory *of the category $\mathcal{C}$, if*

(1) the objects of $\mathcal{D}$ form a subclass of $\operatorname{Ob}\mathcal{C}$, and the morphisms of $\mathcal{D}$ form a subclass of $\operatorname{Mor}\mathcal{C}$,
(2) for any morphism f of $\mathcal{D}$, the source and target of f in $\mathcal{C}$ are objects of $\mathcal{D}$ and agree with the source and target taken in $\mathcal{D}$,
(3) for every object D of $\mathcal{D}$, the identity morphism id_D of $\mathcal{C}$ is a morphism of $\mathcal{D}$, and
(4) for any pair $f : A \to B$ and $g : B \to C$ in $\mathcal{D}$, the composition $g \circ f$ in $\mathcal{C}$ is a morphism of $\mathcal{D}$ and agrees with the composition of f and g in $\mathcal{D}$.

A subcategory $\mathcal{D}$ of $\mathcal{C}$ is called full*, if for any two objects $A, B \in \operatorname{Ob}\mathcal{D}$ all $\mathcal{C}$-morphisms from A to B belong also to $\mathcal{D}$, i.e. if*

$$\operatorname{Mor}_{\mathcal{D}}(A, B) = \operatorname{Mor}_{\mathcal{C}}(A, B).$$

Remark 1.7. If D is an object of $\mathcal{D}$, then the identity morphism of D in $\mathcal{D}$ is the same as that in $\mathcal{C}$, since the identity element of a semigroup is unique, if it exists.

Exercise 1.8. Let $(G, \circ, e)$ be a unital semigroup. Show that a subsemigroup G_0 of G defines a subcategory of $(G, \circ, e)$ (viewed as a category), if and only if $e \in G_0$.

Definition 1.9. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A* covariant functor *(or simply* functor*) $T : \mathcal{C} \to \mathcal{D}$ is a map for objects and morphisms, every object $A \in \operatorname{Ob}\mathcal{C}$ is mapped to an object $T(A) \in \operatorname{Ob}\mathcal{D}$, and every morphism $f : A \to B$ in $\mathcal{C}$*

is mapped to a morphism $T(f) : T(A) \to T(B)$ in $\mathcal{D}$, such that the identities and the composition are respected, i.e. such that

$$T(\mathrm{id}_A) = \mathrm{id}_{T(A)}, \qquad \textit{for all } A \in \mathrm{Ob}\,\mathcal{C}$$
$$T(g \circ f) = T(g) \circ T(f), \qquad \textit{whenever } g \circ f \textit{ is defined in } \mathcal{C}.$$

We will denote the collection of all functors between two categories $\mathcal{C}$ and $\mathcal{D}$ by $\mathrm{Funct}(\mathcal{C}, \mathcal{D})$.

A contravariant functor *$T : \mathcal{C} \to \mathcal{D}$ maps an object $A \in \mathrm{Ob}\,\mathcal{C}$ to an object $T(A) \in \mathrm{Ob}\,\mathcal{D}$, and a morphism $f : A \to B$ in $\mathcal{C}$ to a morphism $T(f) : T(B) \to T(A)$ in $\mathcal{D}$, such such that*

$$T(\mathrm{id}_A) = \mathrm{id}_{T(A)}, \qquad \textit{for all } A \in \mathrm{Ob}\,\mathcal{C}$$
$$T(g \circ f) = T(f) \circ T(g), \qquad \textit{whenever } g \circ f \textit{ is defined in } \mathcal{C}.$$

Example 1.10. Let $\mathcal{C}$ be a category. The *identity functor* $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ is defined by $\mathrm{id}_{\mathcal{C}}(A) = A$ and $\mathrm{id}_{\mathcal{C}}(f) = f$.

Example 1.11. The *inclusion* of a subcategory $\mathcal{D}$ of $\mathcal{C}$ into $\mathcal{C}$ also defines a functor, we can denote it by $\subseteq : \mathcal{D} \to \mathcal{C}$ or by $\mathcal{D} \subseteq \mathcal{C}$.

Example 1.12. The functor $\mathrm{op} : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ that is defined as the identity map on the objects and morphisms is a contravariant functor. This functor allows to obtain covariant functors from contravariant ones. Let $T : \mathcal{C} \to \mathcal{D}$ be a contravariant functor, then $T \circ \mathrm{op} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ and $\mathrm{op} \circ T : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ are covariant.

Example 1.13. Let G and H be unital semigroups, then the functors $T : G \to H$ are precisely the identity preserving semigroup homomorphisms from G to H.

Functors can be composed, if we are given two functors $S : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{C}$, then the composition $T \circ S : \mathcal{A} \to \mathcal{C}$,

$$(T \circ S)(A) = T(S(A)), \qquad \text{for } A \in \mathrm{Ob}\,\mathcal{A},$$
$$(T \circ S)(f) = T(S(f)), \qquad \text{for } f \in \mathrm{Mor}\,\mathcal{A},$$

is again a functor. The composite of two covariant or two contravariant functors is covariant, whereas the composite of a covariant and a contravariant functor is contravariant. The identity functor obviously is an identity w.r.t. to this composition. Therefore we can define categories of categories, i.e. categories whose objects are categories and whose morphisms are the functors between them.

Definition 1.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $S, T : \mathcal{C} \to \mathcal{D}$ be two functors between them. A* natural transformation *(or* morphism of functors*)*

$\eta : S \to T$ assigns to every object $A \in \mathrm{Ob}\,\mathcal{C}$ of $\mathcal{C}$ a morphism $\eta_A : S(A) \to T(A)$ such that the diagram

$$\begin{tikzcd}
S(A) \arrow[r, "\eta_A"] \arrow[d, "S(f)"'] & T(A) \arrow[d, "T(f)"] \\
S(B) \arrow[r, "\eta_B"] & T(B)
\end{tikzcd}$$

is commutative for every morphisms $f : A \to B$ in $\mathcal{C}$. The morphisms η_A, $A \in \mathrm{Ob}\,\mathcal{C}$ are called the components *of η. If every component η_A of $\eta : S \to T$ is an isomorphism, then $\eta : S \to T$ is called a* natural isomomorphism *(or a* natural equivalence*), in symbols this is expressed as $\eta : S \cong T$.*

We will denote the collection of all natural transformations between two functors $S, T : \mathcal{C} \to \mathcal{D}$ by $\mathrm{Nat}(S, T)$.

Exercise 1.15. Let G_1 and G_2 be two groups (regarded as categories as in Example 1.4). $S, T : G_1 \to G_2$ are functors, if they are group homomorphisms, see Example 1.13. Show that there exists a natural transformation $\eta : S \to T$ if and only if S and T are conjugate, i.e. if there exists an element $h \in G$ such that $T(g) = hS(g)h^{-1}$ for all $g \in G_1$.

Definition 1.16. *Natural transformations can also be composed. Let $S, T, U : \mathcal{B} \to \mathcal{C}$ and let $\eta : S \to T$ and $\vartheta : T \to U$ be two natural transformations. Then we can define a natural transformation $\vartheta \cdot \eta : S \to U$, its components are simply $(\vartheta \cdot \eta)_A = \vartheta_A \circ \eta_A$. To show that this defines indeed a natural transformation, take a morphism $f : A \to B$ of $\mathcal{B}$. Then the following diagram is commutative, because the two trapezia are.*

$$\begin{tikzcd}
S(A) \arrow[rr, "(\vartheta\cdot\eta)_A=\vartheta_A\circ\eta_A"] \arrow[dr, "\eta_A"] \arrow[ddd, "S(f)"'] & & U(A) \arrow[ddd, "U(f)"] \\
& T(A) \arrow[ur, "\vartheta_A"'] \arrow[d, "T(f)"] & \\
& T(B) \arrow[dr, "\vartheta_B"] & \\
S(B) \arrow[ur, "\eta_B"] \arrow[rr, "(\vartheta\cdot\eta)_B=\vartheta_B\circ\eta_B"'] & & U(B)
\end{tikzcd}$$

For a given functor $S : \mathcal{B} \to \mathcal{C}$ there exists also the identical natural transformation *$\mathrm{id}_S : S \to S$ that maps $A \in \mathrm{Ob}\,\mathcal{B}$ to $\mathrm{id}_{S(A)} \in \mathrm{Mor}\,\mathcal{C}$, it is easy to check that it behaves as a unit for the composition defined above.*

Therefore we can define the functor category *$\mathcal{C}^{\mathcal{B}}$ that has the functors from $\mathcal{B}$ to $\mathcal{C}$ as objects and the natural transformations between them as morphisms.*

Remark 1.17. Note that a natural transformation $\eta : S \to T$ has to be defined as the triple $(S, (\eta_A)_A, T)$ consisting of its the source S, its components $(\eta_A)_A$ and its target T. The components $(\eta_A)_A$ do not uniquely determine the functors S and T, they can also belong to a natural transformation between another pair of functors (S', T').

Definition 1.18. *Two categories $\mathcal{B}$ and $\mathcal{C}$ can be called* isomorphic*, if there exists an invertible functor $T : \mathcal{B} \to \mathcal{C}$. A useful weaker notion is that of* equivalence *or* categorical equivalence*. Two categories $\mathcal{B}$ and $\mathcal{C}$ are equivalent, if there exist functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{B}$ and natural isomorphisms $G \circ F \cong \mathrm{id}_{\mathcal{B}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{C}}$.*

2 Products and coproducts

We will look at products and coproducts of objects in a category. The idea of the product of two objects is an abstraction of the Cartesian product of two sets. For any two sets M_1 and M_2 their Cartesian product $M_1 \times M_2$ has the property that for any pair of maps (f_1, f_2), $f_1 : N \to M_1$, $f_2 : N \to M_2$, there exists a unique map $h : N \to M_1 \times M_2$ such that $f_i = p_i \circ h$ for $i = 1, 2$, where $p_i : M_1 \times M_2 \to M_i$ are the canonical projections $p_i(m_1, m_2) = m_i$. Actually, the Cartesian product $M_1 \times M_2$ is characterized by this property up to isomorphism (of the category $\mathfrak{Set}$, i.e. set-theoretical bijection).

Definition 2.1. *A triple $(A \mathit{\Pi} B, \pi_A, \pi_B)$ is called a* product *(or* binary product*) of the objects A and B in the category $\mathcal{C}$, if for any object $C \in \mathrm{Ob}\,\mathcal{C}$ and any morphisms $f : C \to A$ and $g : C \to B$ there exists a unique morphism h such that the following diagram commutes,*

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
A & A \mathit{\Pi} B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

We will also denote the mediating morphism $h : C \to A \mathit{\Pi} B$ by $[f, g]$.

Often one omits the morphisms π_A and π_B and simply calls $A \mathit{\Pi} B$ the product of A and B. The product of two objects is sometimes also denoted by $A \times B$.

Proposition 2.2. *(a) The product of two objects is unique up to isomorphism, if it exists.*

(b) Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be two morphisms in a category $\mathcal{C}$ and assume that the products $A_1 \mathit{\Pi} A_2$ and $B_1 \mathit{\Pi} B_2$ exist in $\mathcal{C}$. Then there exists a unique morphism $f_1 \mathit{\Pi} f_2 : A_1 \mathit{\Pi} A_2 \to B_1 \mathit{\Pi} B_2$ such that the following diagram commutes,

$$
\begin{tikzcd}
 & A_1 \arrow[r, "f_1"] & B_1 & \\
A_1 \mathit{\Pi} A_2 \arrow[ur, "\pi_{A_1}"] \arrow[dr, "\pi_{A_2}"'] \arrow[rrr, dotted, "f_1 \mathit{\Pi} f_2"] & & & B_1 \mathit{\Pi} B_2 \arrow[ul, "\pi_{B_1}"'] \arrow[dl, "\pi_{B_2}"] \\
 & A_2 \arrow[r, "f_2"'] & B_2 &
\end{tikzcd}
$$

(c) *Let $A_1, A_2, B_1, B_2, C_1, C_2$ be objects of a category $\mathcal{C}$ and suppose that the products $A_1 \mathit{\Pi} A_2$, $B_1 \mathit{\Pi} B_2$ and $C_1 \mathit{\Pi} C_2$ exist in $\mathcal{C}$. Then we have*

$$\mathrm{id}_{A_1} \mathit{\Pi}\, \mathrm{id}_{A_2} = \mathrm{id}_{A_1 \mathit{\Pi} A_2} \text{ and } (g_1 \mathit{\Pi} g_2) \circ (f_1 \mathit{\Pi} f_2) = (g_1 \circ f_1) \mathit{\Pi} (g_2 \circ f_2)$$

for all morphisms $f_i : A_i \to B_i$, $g_i : B_i \to C_i$, $i = 1, 2$.

Proof. (a) Suppose we have two candidates (P, π_A, π_B) and (P', π'_A, π'_B) for the product of A and B, we have to show that P and P' are isomorphic. Applying the defining property of the product to (P, π_A, π_B) with $C = P'$ and to (P', π'_A, π'_B) with $C = P$, we get the following two commuting diagrams,

$$
\begin{tikzcd}
 & P' \arrow[dl, "\pi'_A"'] \arrow[d, dotted, "h"] \arrow[dr, "\pi'_B"] & \\
A & P \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[dl, "\pi_A"'] \arrow[d, dotted, "h'"] \arrow[dr, "\pi_B"] & \\
A & P' \arrow[l, "\pi'_A"] \arrow[r, "\pi'_B"'] & B
\end{tikzcd}
$$

We get $\pi_A \circ h \circ h' = \pi'_A \circ h' = \pi_A$ and $\pi_B \circ h \circ h' = \pi'_B \circ h' = \pi_B$, i.e. the diagram

$$
\begin{tikzcd}
 & P \arrow[dl, "\pi_A"'] \arrow[d, dotted, "h \circ h'"] \arrow[dr, "\pi_B"] & \\
A & P \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
$$

is commutative. It is clear that this diagram also commutes, if we replace $h \circ h'$ by id_P, so the uniqueness implies $h \circ h' = \mathrm{id}_P$. Similarly one proves $h' \circ h = \mathrm{id}_{P'}$, so that $h : P' \to P$ is the desired isomorphism.

(b) The unique morphism $f_1 \mathit{\Pi} f_2$ exists by the defining property of the product of B_1 and B_2, as we can see from the diagram

$$
\begin{tikzcd}
 & A_1 \mathit{\Pi} A_2 \arrow[dl, "f_1 \circ \pi_{A_1}"'] \arrow[d, dotted, "f_1 \mathit{\Pi} f_2"] \arrow[dr, "f_2 \circ \pi_{A_2}"] & \\
B_1 & B_1 \mathit{\Pi} B_2 \arrow[l, "\pi_{B_1}"] \arrow[r, "\pi_{B_2}"'] & B_2
\end{tikzcd}
$$

(c) Both properties follow from the uniqueness of the mediating morphism in the defining property of the product. To prove $\mathrm{id}_{A_1}\,\Pi\,\mathrm{id}_{A_2} = \mathrm{id}_{A_1\,\Pi\,A_2}$ one has to show that both expressions make the diagram

$$\begin{tikzcd} & A_1\,\Pi\,A_2 \arrow[dl, "\mathrm{id}_{A_1}"'] \arrow[d, dotted] \arrow[dr, "\mathrm{id}_{A_2}"] & \\ A_1 & A_1\,\Pi\,A_2 \arrow[l, "\pi_{A_1}"] \arrow[r, "\pi_{A_2}"'] & A_2 \end{tikzcd}$$

commutative, for the the second equality one checks that $(g_1\,\Pi\,g_2) \circ (f_1\,\Pi\,f_2)$ and $(g_1 \circ f_1)\,\Pi\,(g_2 \circ f_2)$ both make the diagram

$$\begin{tikzcd} & A_1\,\Pi\,A_2 \arrow[dl, "g_1 \circ f_1"'] \arrow[d, dotted] \arrow[dr, "g_2 \circ f_2"] & \\ C_1 & C_1\,\Pi\,C_2 \arrow[l, "\pi_{C_1}"] \arrow[r, "\pi_{C_2}"'] & C_2 \end{tikzcd}$$

commutative.

□

The notion of product extends also to more then two objects.

Definition 2.3. *Let $(A_i)_{i\in I}$ be a family of objects of a category $\mathcal{C}$, indexed by some set I. The pair $\left(\prod_{i\in I} A_i, \left(\pi_j : \prod_{i\in I} A_i \to A_j\right)_{j\in I}\right)$ consisting of an object $\prod_{i\in I} A_i$ of $\mathcal{C}$ and a family of morphisms $\left(\pi_j : \prod_{i\in I} A_i \to A_j\right)_{j\in I}$ of $\mathcal{C}$ is a* product *of the family $(A_i)_{i\in I}$ if for any object C and any family of morphisms $(f_i : C \to A_i)_{i\in I}$ there exists a unique morphism $h : C \to \prod_{i\in I} A_i$ such that*

$$\pi_j \circ h = f_j, \qquad \text{for all } j \in I$$

holds. The morphism $\pi_j : \prod_{i\in I} A_i \to A_j$ for $j \in I$ is called the jth product projection. *We will also write $[f_i]_{i\in I}$ for the morphism $h : C \to \prod_{i\in I} A_i$.*

An object T of a category $\mathcal{C}$ is called *terminal*, if for any object C of $\mathcal{C}$ there exists a unique morphism from C to T. A terminal object is unique up to isomorphism, if it exists. A product of the empty family is a terminal object.

Exercise 2.4. (a) We say that a category $\mathcal{C}$ has finite products if for any family of objects indexed by a finite set there exists a product. Show that this is the case if and only if it has binary products for all pairs of objects and a terminal object.

(b) Let $\mathcal{C}$ be a category with finite products, and let

$$\begin{tikzcd}
 & & C_1 \arrow[r, "h_1"] & D_1 \\
A \arrow[r, "f"] & B \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & & \\
 & & C_2 \arrow[r, "h_2"'] & D_2
\end{tikzcd}$$

be morphisms in $\mathcal{C}$. Show

$$(h_1 \mathit{\Pi}\, h_2) \circ [g_1, g_2] = [h_1 \circ g_1, h_2 \circ g_2] \text{ and } [g_1, g_2] \circ f = [g_1 \circ f, g_2 \circ f].$$

Remark 2.5. Let $\mathcal{C}$ be a category that has finite products. Then the product is associative and commutative. More precisely, there exist natural isomorphisms $\alpha_{A,B,C} : A \mathit{\Pi}\, (B \mathit{\Pi}\, C) \to (A \mathit{\Pi}\, B)\, C$ and $\gamma_{A,B} : B \mathit{\Pi}\, A \to A \mathit{\Pi}\, B$ for all objects $A, B, C \in \operatorname{Ob} \mathcal{C}$.

The notion *coproduct* is the dual of the product, i.e.

$$\left(\coprod_{i \in I} A_i, \left(\imath_j : A_j \to \coprod_{i \in I} A_i \right)_{j \in I} \right)$$

is called a coproduct of the family $(A_i)_{i \in I}$ of objects in $\mathcal{C}$, if it is a product of the same family in the category $\mathcal{C}^{\mathrm{op}}$. Formulated in terms of objects and morphisms of $\mathcal{C}$ only, this amounts to the following.

Definition 2.6. *Let $(A_i)_{i \in I}$ be a family of objects of a category $\mathcal{C}$, indexed by some set I. The pair $\left(\coprod_{i \in I} A_i, \left(\imath_j : A_k \to \prod_{i \in I} A_i \right)_{j \in I} \right)$ consisting of an object $\coprod_{i \in I} A_i$ of $\mathcal{C}$ and a family of morphisms $\left(\imath_j : A_j \to \coprod_{i \in I} A_i \right)_{j \in I}$ of $\mathcal{C}$ is a coproduct of the family $(A_i)_{i \in I}$ if for any object C and any family of morphisms $(f_i : A_i \to C)_{i \in I}$ there exists a unique morphism $h : \coprod_{i \in I} A_i \to C$ such that*

$$h \circ \imath_j = f_j, \qquad \textit{for all } j \in I$$

holds. The morphism $\imath_j : A_j \to \prod_{i \in I} A_i$ for $j \in I$ is called the jth coproduct injection. *We will write $[f_i]_{i \in I}$ for the morphism $h : \prod_{i \in I} A_i \to C$.*

A coproduct of the empty family in $\mathcal{C}$ is an *initial object*, i.e. an object I such that for any object A of $\mathcal{C}$ there exists exactly one morphism from I to A.

It is straightforward to translate Proposition 2.2 to its counterpart for the coproduct.

Example 2.7. In the trivial unital semigroup $(G = \{e\}, \cdot, e)$, viewed as a category (note that is is isomorphic to the discrete category over a set with one element) its only object G is a terminal and initial object, and also a product and coproduct for any family of objects. The product projections and coproduct injections are given by the unique morphism e of this category.

In any other unital semigroup there exist no initial or terminal objects and no binary or higher products or coproducts.

Example 2.8. In the category $\mathfrak{Set}$ a binary product of two sets A and B is given by their Cartesian product $A \times B$ (together with the obvious projections) and any set with one element is terminal. A coproduct of A and B is defined by their disjoint union $A \dot{\cup} B$ (together with the obvious injections) and the empty set is an initial object. Recall that we can define the disjoint union as $A \dot{\cup} B = (A \times \{A\}) \cup (B \times \{B\})$.

Exercise 2.9. Let $\mathfrak{Vek}$ be the category that has as objects all vector spaces (over some field $\mathbb{K}$) and as morphisms the $\mathbb{K}$-linear maps between them. The trivial vector space $\{0\}$ is an initial and terminal object in this category. Show that the direct sum of (finitely many) vector spaces is a product and a coproduct in this category.

The following example shall be used throughout this Section and the following.

Example 2.10. The coproduct in the category of unital algebras $\mathfrak{Alg}$ is the *free product* of $*$-algebras *with* identification of the units. Let us recall its defining universal property. Let $\{\mathcal{A}_k\}_{k\in I}$ be a family of unital $*$-algebras and $\coprod_{k\in I} \mathcal{A}_k$ their free product, with canonical inclusions $\{i_k : \mathcal{A}_k \to \coprod_{k\in I} \mathcal{A}_k\}_{k\in I}$. If $\mathcal{B}$ is any unital $*$-algebra, equipped with unital $*$-algebra homomorphisms $\{i'_k : \mathcal{A}_k \to \mathcal{B}\}_{k\in I}$, then there exists a unique unital $*$-algebra homomorphism $h : \coprod_{k\in I} \mathcal{A}_k \to \mathcal{B}$ such that

$$h \circ i_k = i'_k, \qquad \text{for all} \quad k \in I.$$

It follows from the universal property that for any pair of unital $*$-algebra homomorphisms $j_1 : \mathcal{A}_1 \to \mathcal{B}_1$, $j_2 : \mathcal{A}_2 \to \mathcal{B}_2$ there exists a unique unital $*$-algebra homomorphism $j_1 \coprod j_2 : \mathcal{A}_1 \coprod \mathcal{A}_2 \to \mathcal{B}_1 \coprod \mathcal{B}_2$ such that the diagram

$$\begin{tikzcd}
 & \mathcal{A}_1 \arrow[r, "j_1"] \arrow[dl, "i_{\mathcal{A}_1}"'] & \mathcal{B}_1 \arrow[dr, "i_{\mathcal{B}_1}"] & \\
\mathcal{A}_1 \coprod \mathcal{A}_2 \arrow[rrr, dotted, "j_1 \coprod j_2"] & & & \mathcal{B}_1 \coprod \mathcal{B}_2 \\
 & \mathcal{A}_2 \arrow[r, "j_2"'] \arrow[ul, "i_{\mathcal{A}_2}"] & \mathcal{B}_2 \arrow[ur, "i_{\mathcal{B}_2}"'] &
\end{tikzcd}$$

commutes.

The free product $\coprod_{k\in I}\mathcal{A}_k$ can be constructed as a sum of tensor products of the $\mathcal{A}_k$, where neighboring elements in the product belong to different algebras. For simplicity, we illustrate this only for the case of the free product of two algebras. Let

$$\mathbb{A}=\bigcup_{n\in\mathbb{N}}\{\epsilon\in\{1,2\}^n|\epsilon_1\neq\epsilon_2\neq\cdots\neq\epsilon_n\}$$

and decompose $\mathcal{A}_i=\mathbb{C}\mathbf{1}\oplus\mathcal{A}_i^0$, $i=1,2$, into a direct sum of vector spaces. As a coproduct $\mathcal{A}_1\coprod\mathcal{A}_2$ is unique up to isomorphism, so the construction does not depend on the choice of the decompositions.

Then $\mathcal{A}_1\coprod\mathcal{A}_2$ can be constructed as

$$\mathcal{A}_1\coprod\mathcal{A}_2=\bigoplus_{\epsilon\in\mathbb{A}}\mathcal{A}^\epsilon,$$

where $\mathcal{A}^\emptyset=\mathbb{C}$, $\mathcal{A}^\epsilon=\mathcal{A}^0_{\epsilon_1}\otimes\cdots\otimes\mathcal{A}^0_{\epsilon_n}$ for $\epsilon=(\epsilon_1,\ldots,\epsilon_n)$. The multiplication in $\mathcal{A}_1\coprod\mathcal{A}_2$ is inductively defined by

$$(a_1\otimes\cdots\otimes a_n)\cdot(b_1\otimes\cdots\otimes b_m)=\begin{cases}a_1\otimes\cdots\otimes(a_n\cdot b_1)\otimes\cdots\otimes b_m & \text{if } \epsilon_n=\delta_1,\\ a_1\otimes\cdots\otimes a_n\otimes b_1\otimes\cdots\otimes b_m & \text{if } \epsilon_n\neq\delta_1,\end{cases}$$

for $a_1\otimes\cdots\otimes a_n\in\mathcal{A}^\epsilon$, $b_1\otimes\cdots\otimes b_m\in\mathcal{A}^\delta$. Note that in the case $\epsilon_n=\delta_1$ the product $a_n\cdot b_1$ is not necessarily in $\mathcal{A}^0_{\epsilon_n}$, but is in general a sum of a multiple of the unit of $\mathcal{A}_{\epsilon_n}$ and an element of $\mathcal{A}^0_{\epsilon_n}$. We have to identify $a_1\otimes\cdots a_{n-1}\otimes 1\otimes b_2\otimes\cdots b_m$ with $a_1\otimes\cdots\otimes a_{n-1}\cdot b_2\otimes\cdots b_m$.

Since $\coprod$ is the coproduct of a category, it is commutative and associative in the sense that there exist natural isomorphisms

$$\begin{aligned}\gamma_{\mathcal{A}_1,\mathcal{A}_2}&:\mathcal{A}_1\coprod\mathcal{A}_2\overset{\cong}{\to}\mathcal{A}_2\coprod\mathcal{A}_1,\\ \alpha_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3}&:\mathcal{A}_1\coprod\Big(\mathcal{A}_2\coprod\mathcal{A}_3\Big)\overset{\cong}{\to}\Big(\mathcal{A}_1\coprod\mathcal{A}_2\Big)\coprod\mathcal{A}_3\end{aligned}\tag{2.1}$$

for all unital $*$-algebras $\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3$. Let $i_\ell:\mathcal{A}_\ell\to\mathcal{A}_1\coprod\mathcal{A}_2$ and $i'_\ell:\mathcal{A}_\ell\to\mathcal{A}_2\coprod\mathcal{A}_1$, $\ell=1,2$ be the canonical inclusions. The commutativity constraint $\gamma_{\mathcal{A}_1,\mathcal{A}_2}:\mathcal{A}_1\coprod\mathcal{A}_2\to\mathcal{A}_2\coprod\mathcal{A}_1$ maps an element of $\mathcal{A}_1\coprod\mathcal{A}_2$ of the form $i_1(a_1)i_2(b_1)\cdots i_2(b_n)$ with $a_1,\ldots,a_n\in\mathcal{A}_1$, $b_1,\ldots,b_n\in\mathcal{A}_2$ to

$$\gamma_{\mathcal{A}_1,\mathcal{A}_2}\big(i_1(a_1)i_2(b_1)\cdots i_2(b_n)\big)=i'_1(a_1)i'_2(b_1)\cdots i'_2(b_n)\in\mathcal{A}_2\coprod\mathcal{A}_1.$$

Exercise 2.11. We also consider non-unital algebras. Show that the *free product* of $*$-algebras *without* identification of units is a coproduct in the category $\mathfrak{nuAlg}$ of non-unital (or rather not necessarily unital) algebras. Give an explicit construction for the free product of two non-unital algebras.

Exercise 2.12. Show that the following defines a a functor from the category of non-unital algebras $\mathfrak{nuAlg}$ to the category of unital algebras $\mathfrak{Alg}$. For an algebra $\mathcal{A} \in \operatorname{Ob} \mathfrak{nuAlg}$, $\tilde{\mathcal{A}}$ is equal to $\tilde{\mathcal{A}} = \mathbb{C}\mathbf{1} \oplus \mathcal{A}$ as a vector space and the multiplication is defined by

$$(\lambda\mathbf{1} + a)(\lambda'\mathbf{1} + a') = \lambda\lambda'\mathbf{1} + \lambda' a + \lambda a' + aa'$$

for $\lambda, \lambda' \in \mathbb{C}$, $a, a' \in \mathcal{A}$. We will call $\tilde{\mathcal{A}}$ the *unitization* of $\mathcal{A}$. Note that $\mathcal{A} \cong 0\mathbf{1} + \mathcal{A} \subseteq \tilde{\mathcal{A}}$ is not only a subalgebra, but even an ideal in $\tilde{\mathcal{A}}$.

How is the functor defined on the morphisms?

Show that the following relation holds between the free product *with* identification of units $\coprod_{\mathfrak{Alg}}$ and the free product *without* identification of units $\coprod_{\mathfrak{nuAlg}}$,

$$\widetilde{\mathcal{A}_1 \coprod_{\mathfrak{nuAlg}} \mathcal{A}_2} \cong \tilde{\mathcal{A}}_1 \coprod_{\mathfrak{Alg}} \tilde{\mathcal{A}}_2$$

for all $\mathcal{A}_1, \mathcal{A}_2 \in \operatorname{Ob} \mathfrak{nuAlg}$.

Note furthermore that the range of this functor consists of all algebras that admit a decomposition of the form $\mathcal{A} = \mathbb{C}\mathbf{1} \oplus \mathcal{A}_0$, where $\mathcal{A}_0$ is a subalgebra. This is equivalent to having a one-dimensional representation. The functor is not surjective, e.g., the algebra $\mathcal{M}_2$ of 2×2-matrices can not be obtained as a unitization of some other algebra.

3 Tensor categories and tensor functors

Let us now come to the definition of a tensor category.

Definition 3.1. *A category* $(\mathcal{C}, \Box)$ *equipped with a bifunctor* $\Box : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, called* tensor product*, that is associative up to a natural isomorphism*

$$\alpha_{A,B,C} : A\Box(B\Box C) \stackrel{\cong}{\to} (A\Box B)\Box C, \qquad \text{for all } A, B, C \in \operatorname{Ob}\mathcal{C},$$

and an element E that is, up to natural isomorphisms

$$\lambda_A : E\Box A \stackrel{\cong}{\to} A, \quad \text{and} \quad \rho_A : A\Box E \stackrel{\cong}{\to} A, \quad \text{for all } A \in \operatorname{Ob}\mathcal{C},$$

a unit for $\Box$*, is called a* tensor category *or* monoidal category*, if the* pentagon axiom

$$\begin{tikzcd}
 & (A\Box B)\Box(C\Box D) \arrow[dr, "\alpha_{A\Box B,C,D}"] & \\
A\Box\big(B\Box(C\Box D)\big) \arrow[ur, "\alpha_{A,B,C\Box D}"] \arrow[d, "\mathrm{id}_A\Box\alpha_{B,C,D}"'] & & \big((A\Box B)\Box C\big)\Box D \\
A\Box\big((B\Box C)\Box D\big) \arrow[rr, "\alpha_{A,B\Box C.D}"'] & & \big(A\Box(B\Box C)\big)\Box D \arrow[u, "\alpha_{A,B,C}\Box\mathrm{id}_D"']
\end{tikzcd}$$

and the triangle axiom

$$\begin{tikzcd} A\Box(E\Box C) \arrow[rr, "\alpha_{A,E,C}"] \arrow[dr, "\mathrm{id}_A\Box\lambda_C"'] & & (A\Box E)\Box C \arrow[dl, "\rho_A\Box\mathrm{id}_C"] \\ & A\Box C & \end{tikzcd}$$

are satisfied for all objects A, B, C, D of $\mathcal{C}$.

Example 3.2. If a category has products or coproducts for all finite sets of objects, then the universal property guarantees the existence of the isomorphisms α, λ, and ρ that turn it into a tensor category.

A functor between tensor categories, that behaves "nicely" with respect to the tensor products, is called a *tensor functor* or *monoidal functor*, see, e.g., Section XI.2 in MacLane[Mac98].

Definition 3.3. *Let $(\mathcal{C},\Box)$ and $(\mathcal{C}',\Box')$ be two tensor categories. A* cotensor functor *or* comonoidal functor $F : (\mathcal{C},\Box) \to (\mathcal{C}',\Box')$ *is an ordinary functor $F : \mathcal{C} \to \mathcal{C}'$ equipped with a morphism $F_0 : F(E_{\mathcal{C}}) \to E_{\mathcal{C}'}$ and a natural transformation $F_2 : F(\,\cdot\,\Box\,\cdot\,) \to F(\,\cdot\,)\Box' F(\,\cdot\,)$, i.e. morphisms $F_2(A,B) : F(A\Box B) \to F(A)\Box' F(B)$ for all $A, B \in \mathrm{Ob}\,\mathcal{C}$ that are natural in A and B, such that the diagrams*

$$\begin{tikzcd} F\big(A\Box(B\Box C)\big) \arrow[r, "F(\alpha_{A,B,C})"] \arrow[d, "F_2(A,B\Box C)"'] & F\big((A\Box B)\Box C\big) \arrow[d, "F_2(A\Box B,C)"] \\ F(A)\Box' F(B\Box C) \arrow[d, "\mathrm{id}_{F(A)}\Box' F_2(B,C)"'] & F(A\Box B)\Box' F(C) \arrow[d, "F_2(A,B)\Box'\mathrm{id}_{F(C)}"] \\ F(A)\Box'\big(F(B)\Box' F(C)\big) \arrow[r, "\alpha'_{F(A),F(B),F(C)}"'] & \big(F(A)\Box' F(B)\big)\Box' F(C) \end{tikzcd} \tag{3.1}$$

$$\begin{tikzcd} F(B\Box E_{\mathcal{C}}) \arrow[r, "F_2(B,E_{\mathcal{C}})"] \arrow[d, "F(\rho_B)"'] & F(B)\Box' F(E_{\mathcal{C}}) \arrow[d, "\mathrm{id}_B\Box' F_0"] \\ F(B) & F(B)\Box' E_{\mathcal{C}'} \arrow[l, "\rho'_{F(B)}"] \end{tikzcd} \tag{3.2}$$

$$\begin{tikzcd} F(E_{\mathcal{C}}\Box B) \arrow[r, "F_2(E_{\mathcal{C}},B)"] \arrow[d, "F(\lambda_B)"'] & F(E_{\mathcal{C}})\Box' F(B) \arrow[d, "F_0\Box'\mathrm{id}_B"] \\ F(B) & E_{\mathcal{C}'}\Box' F(B) \arrow[l, "\lambda'_{F(B)}"] \end{tikzcd} \tag{3.3}$$

commute for all $A, B, C \in \mathrm{Ob}\,\mathcal{C}$.

The functors in this definition are called **co**tensor functors, because w.r.t. the usual definition of tensor functors we have reversed the direction of F_0 and F_2. In the case of a strong tensor functor, i.e. when all the morphisms are isomorphisms, our definition of a cotensor functor[1]is equivalent to the usual definition of a tensor functor as, e.g., in MacLane[Mac98].

The conditions are exactly what we need to get morphisms

$$F_n(A_1,\ldots,A_n) : F(A_1\Box\cdots\Box A_n) \to F(A_1)\Box'\cdots\Box' F(A_n)$$

for all finite sets $\{A_1,\ldots,A_n\}$ of objects of $\mathcal{C}$ such that, up to these morphisms, the functor $F:(\mathcal{C},\Box)\to(\mathcal{C}',\Box')$ is a homomorphism.

4 Classical stochastic independence and the product of probability spaces

The product of probability spaces is also an example of a tensor product. This observation can be used to develop an abstract approach to the notion of stochastic independence, see [Fra02, QIIP-II].

Two random variables $X_1 : (\Omega,\mathcal{F},P)\to(E_1,\mathcal{E}_1)$ and $X_2 : (\Omega,\mathcal{F},P)\to(E_2,\mathcal{E}_2)$, defined on the same probability space $(\Omega,\mathcal{F},P)$ and with values in two possibly distinct measurable spaces $(E_1,\mathcal{E}_1)$ and $(E_2,\mathcal{E}_2)$, are called *stochastically independent* (or simply *independent*) w.r.t. P, if the σ-algebras $X_1^{-1}(\mathcal{E}_1)$ and $X_2^{-1}(\mathcal{E}_2)$ are independent w.r.t. P, i.e. if

$$P\big((X_1^{-1}(M_1)\cap X_2^{-1}(M_2)\big) = P\big((X_1^{-1}(M_1)\big)P\big(X_2^{-1}(M_2)\big)$$

holds for all $M_1\in\mathcal{E}_1$, $M_2\in\mathcal{E}_2$. If there is no danger of confusion, then the reference to the measure P is often omitted.

This definition can easily be extended to arbitrary families of random variables. A family $\big(X_j : (\Omega,\mathcal{F},P)\to(E_j,\mathcal{E}_j)\big)_{j\in J}$, indexed by some set J, is called independent, if

$$P\left(\bigcap_{k=1}^n X_{j_k}^{-1}(M_{j_k})\right) = \prod_{k=1}^n P\big(X_{j_k}^{-1}(M_{j_k})\big)$$

holds for all $n\in\mathbb{N}$ and all choices of indices $k_1,\ldots,k_n\in J$ with $j_k\neq j_\ell$ for $j\neq\ell$, and all choices of measurable sets $M_{j_k}\in\mathcal{E}_{j_k}$.

There are many equivalent formulations for independence, consider, e.g., the following proposition.

Proposition 4.1. *Let X_1 and X_2 be two real-valued random variables. The following are equivalent.*

[1] N.B.: In the groupe de travail we will need tensor functors as defined in MacLane[Mac98].

(i) X_1 *and* X_2 *are independent.*
(ii) *For all bounded measurable functions* f_1, f_2 *on* $\mathbb{R}$ *we have*

$$\mathbb{E}\big(f_1(X_1)f_2(X_2)\big) = \mathbb{E}\big(f_1(X_1)\big)\mathbb{E}\big(f_2(X_2)\big).$$

(iii) *The probability space* $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), P_{(X_1,X_2)})$ *is the product of the probability spaces* $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_{X_1})$ *and* $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_{X_2})$*, i.e.*

$$P_{(X_1,X_2)} = P_{X_1} \otimes P_{X_2}.$$

We see that stochastic independence can be reinterpreted as a rule to compute the joint distribution of two random variables from their marginal distribution. More precisely, their joint distribution can be computed as a product of their marginal distributions. This product is associative and can also be iterated to compute the joint distribution of more than two independent random variables.

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