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Examples of QG / integrability theory

Integrating over O_N (or O_N^+): $G = O_N = \{ V = [V_{ij}] \in M_N(\mathbb{C}) : V \text{ unitary}, V^t = V^* = V^{-1}\}$

It has a Haar measure h: $h : C(G) \to \mathbb{C}$, $h(f) = \int f\,dg$.

Consider $Pol(G)$, the $*$-algebra generated by the V_{ij} seen as coordinate f^{ns} on G, in $C(G)$. It is dense there. We want to compute $h(f) = \int_G f(g)\,dg$ for $f \in Pol(G)$.

I.e., we want to know $h(V_{i(1)j(1)} \cdots V_{i(k)j(k)})$ for all $k \in \mathbb{N}$, i.e. to know the joint distribution of $\{V_{ij}\}_{i,j=1}^N$ in $L^\infty(G, dg)$. This is a HARD problem!

A special case: one variable V_{11}: $\mathrm{Law}(V_{11})$ = hyperspherical

One approach for the joint distribution: <u>use representation theory!</u>

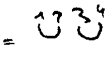
Fix K, $Q_K^{(N)} = [h(V_{i(1)j(1)} \cdots V_{i(K)j(K)})] = (id \otimes h)\, V^{\otimes K}$

$=$ projection onto the space $\mathrm{Fix}(V^{\otimes K})$ of fixed vectors:

$$\mathrm{Fix}(V^{\otimes K}) = \{ \xi \in (\mathbb{C}^N)^{\otimes K} : V^{\otimes K}(\xi \otimes 1) = (\xi \otimes 1)\}$$
$$= \mathrm{Hom}(\mathbb{C}, V^{\otimes K})$$

Considering $-$(orthogonal matrix), we get that if k is odd, $Q_K^{(N)} = 0$.

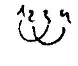
Let $P_2(K)$ be the collection of all pairings of $[K] = \{1, \dots, K\}$.

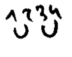
Ex: $\pi =$  $\sigma =$ 

For $\pi \in P_2(K)$, let $\xi_\pi = \sum'_{i:[K]\to[N]} \delta_\pi(i)\, e_{i(1)} \otimes e_{i(2)} \otimes \cdots \otimes e_{i(k)}$;

$$\delta_\pi(i) = \begin{cases} 1 & \text{if } i \text{ is constant on the blocks of } \pi \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum'_{\substack{i:[K]\to[N] \\ \ker i \geq \pi}} e_{i(1)} \otimes \cdots \otimes e_{i(k)}$$

ex.: $\pi =$  : $\xi_\pi = \sum_{a,b=1}^N e_a \otimes e_b \otimes e_a \otimes e_b$

$\sigma =$  : $\xi_\sigma = \sum_{a,b=1}^N e_a \otimes e_a \otimes e_b \otimes e_b$.

Th (Brauer): $\mathrm{Fix}\,(V^{\otimes k}) = \mathrm{span}\,\{\xi_\pi : \pi \in P_2(k)\}$

N.B.: In general, the ξ_π are linearly dependent, (unless $\frac{k}{2} \leq N$)

Recently, the relations of linear dependence have been described.

Let $\tilde{\xi}_\pi = N^{-\frac{k}{4}}\xi_\pi$: it is a unit vector and $\langle \tilde{\xi}_\pi | \tilde{\xi}_\sigma \rangle = N^{-\frac{k}{2}} \langle \xi_\pi | \xi_\sigma \rangle$

Diagrammatically:  : $\#\,\ell(\pi,\sigma) = 1$ (nb of closed paths $=$ loops)

$= N^{-\frac{k}{2}} N^{\#\ell(\pi,\sigma)} = \begin{cases} 1 & \pi = \sigma \\ O(N^{-1}) & \pi \neq \sigma \end{cases}$

Thus, if $N \to \infty$, k fixed, the $\{\tilde{\xi}_\pi\}$ are asymptotically orthogonal.

Corollary: [Weingarten 1978, Statistical physics:] Let $(g_{ij})_{ij=1}^{\infty}$ be standard i.i.d. real $\mathcal{N}(0,1)$ variables. Then $\{\underbrace{\sqrt{N}\, V_{ij}}_{\text{Variance} = \frac{1}{N} \text{ for all}}\}_{1\leq i,j\leq N} \xrightarrow[\text{distr.}]{N\to\infty} \{g_{ij}\}$

Why: $k = 2\ell$: $h(\sqrt{N}\, V_{i(1)j(1)} \cdots \sqrt{N}\, V_{i(2\ell)j(2\ell)}) = N^\ell \langle e_i | Q_{2\ell}^{(N)} e_j \rangle$

(here $e_i = e_{i(1)} \otimes \cdots e_{i(2\ell)}$)

and $Q_{2\ell}^{(N)}$ projects onto $\mathrm{Fix}(V^{\otimes 2\ell})$

$$\simeq \sum_{\pi \in P_2(2\ell)} |\tilde{\xi}_\pi\rangle\langle\tilde{\xi}_\pi| + O(N^{-1}).$$

therefore $M_k = \cancel{N^\ell} \sum_{\pi\in P_2(2\ell)} \underbrace{\langle e_i | \tilde{\xi}_\pi \rangle}_{\cancel{N^{\frac{\ell}{2}}}\langle e_i|\xi_\pi\rangle = \delta_\pi(i)} \underbrace{\langle \tilde{\xi}_\pi | e_j \rangle}_{\cancel{N^{\frac{\ell}{2}}}\langle \xi_\pi | e_j\rangle = \delta_\pi(j)} + O(N^{-1})$

independent of N!

If $N \to \infty$, $M_k^{(N)} \to \sum_{\pi \in P_2(2\ell)}' \delta_\pi(i)\,\delta_\pi(j) = \#\{\pi \in P_2(2\ell) : (i,j) \text{ constant on blocks of } \pi\}$

$\to$ this is exactly the Wick formula for the Gaussian!!

$= \mathbb{E}(g_{i(1)j(1)} \cdots g_{i(2\ell)j(2\ell)})$.

But for finite N, it is much more complicate!

Now let us consider the <u>free orthogonal group O_N^+</u>:

quantize $C(O_N)$: $C(O_N)$: C^*(coord f^{ns} on O_N : $[V_{ij}]$ unitary, $V_{ij} = V_{ij}^*$)

Then consider $C(O_N^+) = C^*_{\substack{\text{universal}\\ \text{unital}}}\left(\{U_{ij}\}_{1\leq i,j\leq N} : \begin{matrix} U = [U_{ij}] \text{ unitary} \\ U_{ij} = U_{ij}^* \end{matrix}\right)$

The QG structure is given by $\Delta : C(O_N^+) \to C(O_N^+)^{\otimes_{\min} 2}$ unital, $*$homomorphisms,

that is coassociative $(\Delta \otimes id)\circ\Delta = (id \otimes \Delta)\circ\Delta$. It is $\Delta U_{ij} = \sum_{k=1}^{N} U_{ik} \otimes U_{kj}$

$O_N^+ = (C(O_N^+), \Delta)$ is called the "free orthogonal group".

$\mathrm{Pol}(O_N^+) = *\text{-alg}\langle U_{ij} : 1 \leq i,j \leq N \rangle \underset{\text{dense}}{\subseteq} C(O_N^+)$

Haar state : there is a unique state $h : C(O_N^+) \to \mathbb{C}$ s.t.

$$(h \otimes id) \circ \Delta = (id \otimes h) \circ \Delta = h(\cdot) 1_{C(O_N^+)} .$$

We can do NC probability on O_N^+ !

Q: Compute the ~~distribution~~ of $\{U_{ij}\}$ in $(C(O_N^+), h)$

The representation theory approach works again!

For all k, let $Q_k^{(N)} = [h(U_{i(1)j(1)} \cdots U_{i(k)j(k)})]$

$= \text{Proj. onto } \mathrm{Fix}(U^{\otimes k}) = \{\xi \in (\mathbb{C}^N)^{\otimes k} : U^{\otimes k}(\xi \otimes 1) = \xi \otimes 1\}$

Th (Banica, 1996) $\mathrm{Fix}(U^{\otimes k}) = \mathrm{span}\{\xi_\pi : \pi \in NC_2(k)\}$ non crossing pair partitions of $[k]$

Fact (Banica) $(\xi_\pi)_{\pi \in NC_2(k)}$ is a basis for $N \geq 2$ and k.

Th (Banica, Collins) : Let $\{S_{ij}\}_{i,j=1}^\infty$ be a free semicircular system in (M, τ) some finite von Neumann algebra

i.e. : (S_{ij}) are $*$-free with respect to τ and identically distributed, and $S_{ij} = S_{ij}^*$ and the law (S_{ij}) is the semicircle law $d\mu(t) = 1_{[-2,2]} \frac{\sqrt{4-t^2}}{2\pi} dt$

As $h(U_{ij}^2) = \frac{1}{N}$, consider $\{\sqrt{N} U_{ij}\} \xrightarrow[\text{dist}]{N \to \infty} \{S_{ij}\}_{ij}$.

And the same proof works !

$$\lim_N h(\sqrt{N} U_{i(1)j(1)} \cdots \sqrt{N} U_{i(k)j(k)}) = \sum_{\pi \in NC_2(k)} \underbrace{\delta_\pi(i)\delta_\pi(j)}_{\substack{= 1 \text{ if } (i,j) \text{ constant on } \pi \\ = 0 \text{ otherwise}}}$$

and this is the Wick formula for $\tau(S_{i(1)j(1)} \cdots S_{i(k)j(k)})$

N.B. In the free case, you get a bounded limiting distribution

Ask : for $P \in \mathbb{C}\langle X_{ij} \rangle$ (P noncommutative polynomial),

$$x_N = P(\{\sqrt{N} U_{ij}\}) \in \mathrm{Pol}(O_N^+) \xrightarrow{?} P(\{S_{ij}\})$$

It converges in distribution. But does it so strongly ?

I.e., $\|P(\sqrt{N} U_{ij})\|_{L^\infty(O_N^+)} \to \|P(S_{ij})\|_M$ "a.s. convergence in norm"

Another question: for fixed N, what is $\|P(\sqrt{N}\,U_{ij})\|_{L^\infty(O_N^+)}$?

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{GNS}(C(O_N^+), h)$

There is a partial answer:

Th (Banica, Collins, Zinn-Justin 2009) [One variable case]

Consider $U_{11} \in \mathrm{Pol}(O_N^+)$. It has law: $\mathrm{Law}(\sqrt{N+2}\,U_{11}) = \mathrm{Law}(W)$:

for all $N \geq 2$, the $W = W_{11} + W_{12} + W_{21} + W_{22}$, sum of the generators of $SU_q(2)$:

$\begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$ = "fundamental representation" of $C(SU_q(2))$, with $q + q^{-1} = -N$

The spectral measure of W is: for $z \in \mathbb{T}$, $d\mu(z+z^{-1}) = \sum_{r \in \mathbb{Z}} \frac{(-1)^r q^{r-1}(1+q)^2}{(1+q^{r+1})(1+q^{r-1})} z^{2r}$

$$\sigma_{L^\infty(O_N^+)}(U_{11}) = \left[-\frac{2}{\sqrt{N+2}}, \frac{2}{\sqrt{N+2}}\right]$$: it has no atoms! on $[-2,2]$

There is a general Q of whether spectral measures of nc polynomials contain atoms! $\longrightarrow$ Atiyah conjecture!

Consequence: $\|\sqrt{N}\,U_{11}\|_{L^\infty(O_N^+)} = \frac{2\sqrt{N}}{\sqrt{N+2}} \longrightarrow 2 = \|$ semicircle $\|$.

A different approach: We want to see whether $\|P(\sqrt{N}\,U_{ij})\|_{L^\infty(O_N^+)} \longrightarrow \|P(S_{ij})\|_M$ by using a NC-Khinchin $\leq$.

Vergnioux's Haagerup $\leq$: $L^2(O_N^+) = \bigoplus_{k \in \mathbb{N}} L^2_k(O_N^+)$ span of $(U_{ij}^k)_{ij}$ of the k^{th} irreducible representation U^k of O_N^+.

$\qquad\qquad\qquad\qquad\qquad \cup \qquad\qquad L^\infty(O_N^+)$

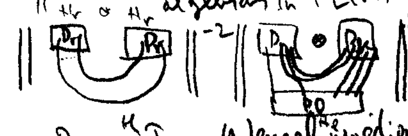
If $x \in L^2_k(O_N^+)$, $y \in L^2_m(O_N^+)$, then $\|xy\|_{L^\infty} \leq D_N \|x\|_2 \|y\|_2$

The natural length on L^∞ is, for $x \in \mathrm{Pol}(O_N^+)$, $\ell(x) = \min\{n : x \in \bigoplus_{k=0}^{n} L^2_k\}$.

If $x \in \mathrm{Pol}(O_N^+)$ and $\ell(x) = r$, $\|x\| \leq D_N (r+1)^{3/2} \|x\|_{L^2}$

You want to know how D_N behaves: $D_N \simeq \sup_{(m,k,\ell)} C_{(m,k,\ell)}$ of norms of diagrams in Temperley Lieb algebras in $TL(N)$

Here $\ell = m+k-2r = (m-r)+(k-r)$



$U^k = (P_k \otimes 1)\, U^{\otimes k} (P_k \otimes 1)$.

U_ℓ acts on $H_\ell = P_\ell (\mathbb{C}^N)^{\otimes \ell}$

P_n n^{th} Jones Wenzel projection "highest weight projection"

$\longrightarrow$ look at knot theory:

D_N is uniformly bounded! $D_N \asymp 1$ indep. of N!

Now adapt some ideas of Pisier's: $x_N = P(\sqrt{N}\,U_{ij})$, $\ell(x_N)\, r$:

For all m, $\|x_N\|_{L^{4m}} \leq \|x_N\|_{L^\infty} = \|(x_N^* x_N)^m\|_\infty^{\frac{1}{2m}} \leq \left(D_N (2mr+1) \|(x_N^* x_N)^m\|_{L^2}\right)^{\frac{1}{2m}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ell(\quad) \leq 2rm$

$\leq D_N^{\frac{1}{2m}} (2mr+1)^{\frac{3}{4m}} \|x_N\|_{L^{4m}}$

In particular, for $\varepsilon>0$, take m large enough so that

$$\|P(S_{ij})\|_{L^{4m}(M)} \geq \|P(S_{ij})\| - \varepsilon$$

$R_N^{\frac{1}{2m}}(2rm+1)^{\frac{3}{4m}} \leq 1+\varepsilon$. Therefore, for $N \longrightarrow \infty$, $\|P(S_{ij})\|_\infty - \varepsilon \leq \overline{\lim_{N\to\infty}} \|x_N\|_\infty \leq (1+\varepsilon)\|P(S_{ij})\|_\infty$.

<u>Questions</u>: How to further compare the U_{ij}'s to the S_{ij}'s.
$\rightarrow$ questions of interest in free probability!

This is all done on O_N^+. You can use this in U_N^+.

And strong cv is preserved by free products!

Large N asymptotics in the non unimodular case

Consider $A_o(F) = C^*((U_{ij})_{ij} \mid [U_{ij}] = U$ unitary: $U = F\overline{U}F^{-1}\}$
$\overline{U} = [U_{ij}^*]$
for $F \in GL_m(\mathbb{C})$, and $L^\infty(A_o(F))$: it is type III.

What about the distribution of generators $\{u_{ij}\}$ as $N \longrightarrow \infty$.

Cf Freslon's truncation $\leq$ ies.