A Brief on Hilbert Modules and von Neumann Modules

Michael Skeide

Dipartimento E.G.S.I.
Università degli Studi del Molise

Bengaluru, January 2, 2013

- Hilbert modules are ...
  - ... a generalization of Hilbert spaces (replacing $\mathbb{C}$ by $\mathcal{B}$).
  - ... a generalization of C^*–algebras (better: right ideals).
  - ... subsets of C^*–algebras.

- Hilbert bimodules (or correspondences) are ...
  - ... powerful functors that, under tensor product, transform given right or left modules into new ones.

- $\mathcal{B}^a(E)$, the algebra of adjointable operators on $E_{\mathcal{B}}$, is a C^* or von Neumann algebra ...
  - ... capturing (almost) all the simplicity of $\mathcal{B}(H)$.
  - ... being still sufficiently general to treat many, many problems.

- These things occur ...
  - ... from dynamical maps (CP-maps, endomorphisms) and dilation theory.
  - ... in representation theory.
  - ... in classification of C^* (and von Neumann) algebras.
  - ... and, and, and ...

An advice:

- Don't study them (too much) for their own sake;
- study them to solve your problems!
- Problems will guide you to what are interesting questions about Hilbert modules.
- Still, don't hesitate to study them if you **do** have a problem where Hilbert modules **do** occur!

  Examples:

  - Stinespring construction *versus* GNS-construction for CP-maps.)
  - Tensor product of Connes correspondences *versus* tensor product of von Neumann correspondences.

Outline

Hilbert spaces and Hilbert modules

Correspondences and tensor products

Morita equivalence and representations

Von Neumann modules and von Neumann correspondences

More exercises

A complex vector space H with a sesquilinear map $\langle \bullet, \bullet \rangle \colon H \times H \to \mathbb{C}$ is a:

- Semi-Hilbert space if $\langle x, x \rangle \geq 0$.
- Pre-Hilbert space if $\langle x, x \rangle = 0 \Rightarrow x = 0$.
- Hilbert space if it is complete wrt the norm $\|x\| := \sqrt{\langle x, x \rangle}$.

Exercise

Prove that $\langle x, y \rangle = \overline{\langle y, x \rangle}$. Is this so for real semi-Hilbert spaces?

$a \colon G \to H$ is adjointable if $\exists a^* \colon H \to G$ with $\langle ax, y \rangle = \langle x, a^* y \rangle$.

- $\mathcal{L}(G, H)$:= linear maps $a \colon G \to H$.
- $\mathcal{L}^a(G, H)$:= adjointable maps $a \colon G \to H$.
- $\mathcal{B}(G, H)$:= bounded linear maps $a \colon G \to H$.
- $\mathcal{B}^a(G, H)$:= bounded adjointable maps $a \colon G \to H$.

Exercise

Not two of the spaces $\mathcal{L}(H)$, $\mathcal{L}^a(H)$, $\mathcal{B}(H)$, and $\mathcal{B}^a(H)$ are equal, if H is not complete.

Two fundamental results

Cauchy-Schwarz inequality (semi!): $\langle x, y\rangle\langle y, x\rangle \leq \langle y, y\rangle\langle x, x\rangle$.

- Seminorm $\|\bullet\|$. ($\rightsquigarrow$ quotients. Also GNS!)
- $\|x\| = \sup_{\|y\|\leq 1} |\langle y, x\rangle|$. ($\rightsquigarrow$ $\mathcal{B}^a(H)$ is (pre-)C^*–algebra.)
- $\langle\bullet, \bullet\rangle$ continuous. ($\rightsquigarrow$ completion.)

Self-duality (Hilbert!): $f \in \mathcal{B}(H, \mathbb{C}) \Rightarrow \exists(!) y \in H\colon f = \langle y, \bullet\rangle$.

- $\mathcal{B}(G, H) = \mathcal{B}^a(G, H)$ $(= \mathcal{L}^a(G, H))$.
- G Hilbert subspace of $H \Rightarrow p = p^* = p^2 \in \mathcal{B}(H)\colon pH = G$.
- G subspace of Hilbert $H \Rightarrow \overline{G} = G^{\perp\perp}$ and $H = G^{\perp\perp} \oplus G^{\perp}$.
- Hilbert spaces have ONBs.

Exercise

- *For p with $pH = G$, is it necessary that H is Hilbert?*
- *G, H pre-Hilbert.* span $S = G$, span $T = H$. *Suppose $a\colon S \to H$ and $a^*\colon T \to G$ with $\langle as, t\rangle = \langle s, a^*t\rangle$. Then a and a^* extend as mutually adjoint elements $a \in \mathcal{L}^a(G, H)$ and $a^* \in \mathcal{L}^a(H, G)$.*

Let $\mathcal{B}$ denote C^*–algebra. A right(!!) $\mathcal{B}$–module E (often indicated as $E_{\mathcal{B}}$) with a sesquilinear map $\langle \bullet, \bullet \rangle \colon E \times E \to \mathcal{B}$ is a:

- Semi-Hilbert $\mathcal{B}$–module if $\langle x, x \rangle \geq 0$ and $\langle x, yb \rangle = \langle x, y \rangle b$.
- Pre-Hilbert $\mathcal{B}$–module if $\langle x, x \rangle = 0 \Rightarrow x = 0$.
- Hilbert $\mathcal{B}$–module if complete wrt the norm $\|x\| := \sqrt{\|\langle x, x \rangle\|}$.

Exercise

Prove that $\langle x, y \rangle = \langle y, x \rangle^*$.

$a \colon E_{\mathcal{B}} \to F_{\mathcal{B}}$ is adjointable if $\exists a^* \colon F \to E$ with $\langle ax, y \rangle = \langle x, a^* y \rangle$.

- $\mathcal{L}^r(E, F)$:= right linear maps $a \colon E \to F$.
- $\mathcal{L}^a(E, F)$:= adjointable maps $a \colon E \to F$.
- $\mathcal{B}^r(E, F)$:= right linear bounded maps $a \colon E \to F$.
- $\mathcal{B}^a(E, F)$:= bounded adjointable maps $a \colon E \to F$.

Exercise

- *$a \in \mathcal{L}^a(E, F)$ is right linear and closeable, and a^* is unique.*
- *Repeat exercise on $\mathcal{B}$–spanning subsets* span $S\mathcal{B} = E$.

Examples:

- A Hilbert space is a Hilbert $\mathbb{C}$–module.
- $\mathcal{B}$ with $\langle b, b' \rangle := b^* b'$.
- More generally, each closed right ideal of $\mathcal{B}$.
- Direct sum $\bigoplus_i E_i$ completion of $\underline{\bigoplus}_i E_i$ with $\langle x, y \rangle := \sum_i \langle x_i, y_i \rangle$.
  (vNm: There are no others than direct sums of right ideals.)
  Special case: $E^{\mathfrak{n}} = \mathbb{C}^{\mathfrak{n}} \otimes E$. (Here, $\mathfrak{n} = \dim \mathbb{C}^{\mathfrak{n}}$.)
- Closed left ideal L in $\mathcal{A}$. Then L is a Hilbert module over the heriditary subalgebra $\mathcal{B} := \overline{\operatorname{span}}\, L^* L$ of $\mathcal{A}$.
- $E^* := \{x^* := \langle x, \bullet \rangle \colon x \in E\} \subset \mathcal{B}^a(E, \mathcal{B})$. (Find their adjoints!)
  E^* is Hilbert $\mathcal{K}(E)$–module,
  where $\mathcal{K}(E) := \overline{\operatorname{span}}\{xy^* \colon x, y \in E\} \subset \mathcal{B}^a(E)$.

Note: E is self-dual if $E^* = \mathcal{B}^r(E, \mathcal{B})$.
Ideals are rarely self-dual. (Say when! $\rightsquigarrow$ examples.)

Exercise

- *$\mathcal{B}$ unital: E not self-dual, then $\mathcal{B}^r(\mathcal{B} \oplus E) \neq \mathcal{B}^a(\mathcal{B} \oplus E)$*
- *Is the opposite true? (E self-dual $\Rightarrow \mathcal{B}^r(E) = \mathcal{B}^a(E)$?)*

We see: HM behave to quite an extent like pre-HS. Fortunately:

Cauchy-Schwarz inequality (semi!): $\langle x, y\rangle\langle y, x\rangle \leq \|\langle y, y\rangle\| \langle x, x\rangle$.

Proof: As for (semi-)Hilbert spaces.

- For $\langle y, y\rangle \neq 0$, take $z = \|\langle y, y\rangle\|^2 x - y\langle y, x\rangle$ and use $\langle z, z\rangle \geq 0$. (You may need the C^*–inequality $a^*bb^*a \leq \|bb^*\| a^*a$.)
- What about $\langle y, y\rangle = 0 = \langle x, x\rangle$? $\square$

Consequences:

- Seminorm $\|\bullet\|$. ($\rightsquigarrow$ quotients. Also GNS!)
- $\|x\| = \sup_{\|y\|\leq 1} \|\langle y, x\rangle\|$. ($\rightsquigarrow$ $\mathcal{B}^a(E)$ is (pre-)C^*–algebra.)
- $\langle \bullet, \bullet\rangle$ continuous. ($\rightsquigarrow$ completion.)

Attention: No problem pre-HM over pre-C^*.
However, when completing E complete also $\mathcal{B}$.

Exercise (1st part accessible, 2nd part not exactly easy)

- $\bigoplus_i E_i = \Big\{(x_i) \colon \sum_i \langle x_i, x_i\rangle$ *exists*$\Big\}$. *(Hint: Why here?)*
- $\mathcal{B}$ *unital. Can you identify* $\mathcal{B}^r\big(\mathcal{B}^\infty, \mathcal{B}\big)$*?*

From now: Everything completed. (Unless, ...)

Exercise

$$\mathcal{B}^a\binom{\mathcal{B}}{E} = \begin{pmatrix} \mathcal{B}^a(\mathcal{B}) & \mathcal{B}^a(E,\mathcal{B}) \\ \mathcal{B}^a(\mathcal{B},E) & \mathcal{B}^a(E) \end{pmatrix} \supset \underbrace{\begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}}_{\text{(extended)}} \supset \underbrace{\begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{pmatrix} = \mathcal{K}\binom{\mathcal{B}}{E}}_{\text{linking algebra}}$$

What if $\mathbf{1} \in \mathcal{B}$?

Corollary: $xx^* \geq 0$ in $\mathcal{K}\binom{\mathcal{B}}{E}$, hence, in $\mathcal{K}(E) \subset \mathcal{K}\binom{\mathcal{B}}{E}$.

Reduced linking algebra $\mathcal{K}\binom{\mathcal{B}_E}{E}$, with range ideal $\mathcal{B}_E := \overline{\operatorname{span}}\langle E, E\rangle$.
E is full if $\mathcal{B}_E = \mathcal{B}$.

Note: $\mathcal{B}_E$ ideal in $\mathcal{A}$ $\rightsquigarrow$ E is Hilbert $\mathcal{A}$–module.
Consequently: $\mathcal{K}(E)$ ideal in $\mathcal{A}$ $\rightsquigarrow$ E^* is Hilbert $\mathcal{A}$–module.

Exercise

- *$M_n(\mathcal{B}) \subset \mathcal{B}(\mathcal{B}^n)$ is C^*–algebra. The row space $E_n := ((E^*)^n)^*$ is Hilbert $M_n(\mathcal{B})$–module with $\langle X_n, Y_n\rangle = (\langle x_i, y_j\rangle)_{i,j}$ ($\rightsquigarrow$ $(\langle x_i, x_j\rangle)_{i,j} \geq 0$ in $M_n(\mathcal{B})$!) and $\mathcal{B}^a(E_n) = \mathcal{B}^a(E)$.*
- *Do the same for $(E_m)^n = (E^n)_m =: M_{n,m}(E)$.*

Exercise

- *A double centralizer of a C^*–algebra $\mathcal{B}$ is pair (L, R) of maps on $\mathcal{B}$ such that $\quad bL(b') = R(b)b'$. $\qquad$ Show $\mathcal{B}^a(\mathcal{B}) = M(\mathcal{B})$ where $M(\mathcal{B}) := \big\{(L, R)\big\}$ is the multiplier algebra.*
- *(Can you characterize $\mathcal{L}^a(\mathcal{B})$ and $\mathcal{B}^a(\mathcal{B})$ when $\mathcal{B}$ is only pre?)*
- *Show $\mathcal{B}^a(E) = \mathcal{B}^a(\mathcal{K}(E))$.*

Corollary (Kasparov 1980)

$$\mathcal{B}^a(E) = M(\mathcal{K}(E))$$

- Strict topology: $\qquad \|\bullet k\|, \|k\bullet\|$ with $k \in \mathcal{K}(E)$.
- $*$–Strong topology: $\|\bullet x\|, \|x^*\bullet\|$ with $x \in E$.

Exercise

Strict and $*$–strong topology coincide on bounded subsets.

- Substitute for normality. Depends on E.
- Recognizing $\mathcal{A}$ as $\mathcal{B}^a(E) \rightsquigarrow$ equipping $\mathcal{A}$ with a “good” topology.

A correspondence from $\mathcal{A}$ to $\mathcal{B}$ (or a Hilbert $\mathcal{A}$–$\mathcal{B}$–bimodule) E, denoted frequently ${}_{\mathcal{A}}E_{\mathcal{B}}$, is:

- A Hilbert $\mathcal{B}$–module E.
- An $\mathcal{A}$–$\mathcal{B}$–bimodule such that the left action defines a nondegenerate(!) ($*$–)homomorphism $\mathcal{A} \to \mathcal{B}^a(E)$.

Example (Paschke's GNS-construction for CP-maps, 1973(!))

- *$\mathcal{A}$, $\mathcal{B}$ unital. (In particular, $\mathcal{A}$!)*
- *$T: \mathcal{A} \to \mathcal{B}$ a CP-map, that is, $\sum_{i,j} b_i^* T(a_i^* a_j) b_j \geq 0$.*
- *$\rightsquigarrow$ semiinner product on $\mathcal{A} \otimes \mathcal{B}$*

$$\langle a \otimes b, a' \otimes b' \rangle := b^* T(a^* a') b'.$$

- *Quotient $\mathcal{N} := \{x : \langle x, x \rangle = 0\}$ and complete $\rightsquigarrow$ ${}_{\mathcal{A}}E_{\mathcal{B}}$. (CSI!)*
- *$\xi := \mathbf{1} \otimes \mathbf{1} + \mathcal{N}$ fulfills $\langle \xi, a\xi \rangle = T(a)$ and $\overline{\operatorname{span}}\, \mathcal{A}\xi\mathcal{B} = E$.*

Example

Unital endomorphism ϑ of $\mathcal{B}$ $\Leftrightarrow$ ${}_{\vartheta}\mathcal{B}$ (that is, $b.x = \vartheta(b)x$).

- Let ${}_{\mathcal{A}}E_{\mathcal{B}}$ and ${}_{\mathcal{B}}F_{\mathcal{C}}$.
- On $E \otimes F$ define semiinner product by

$$\left\langle x \otimes y, x' \otimes y' \right\rangle := \left\langle y, \langle x, x' \rangle y' \right\rangle.$$

Is this positive?

- $0 \leq \langle x, x \rangle = b^* b \quad \rightsquigarrow$
  $\langle x \otimes y, x \otimes y \rangle = \langle y, \langle x, x \rangle y \rangle = \langle y, b^* b y \rangle = \langle by, by \rangle \geq 0.$
- For $\sum_i x_i \otimes y_i$ put $X_n = \begin{pmatrix} x_1, \ldots, x_n \end{pmatrix} \in E_n$ and $Y^n = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in F^n$.
- Then $\langle \sum_i x_i \otimes y_i, \sum_j x_j \otimes y_j \rangle = \langle X_n \otimes Y^n, X_n \otimes Y^n \rangle \geq 0.$

Definition (Rieffel 1974a(?))

The tensor product of E and F is the unique ${}_{\mathcal{A}}E \odot F_{\mathcal{C}}$ generated by $x \odot y$ subject to

$$\left\langle x \odot y, x' \odot y' \right\rangle = \left\langle y, \langle x, x' \rangle y' \right\rangle, \qquad a(x \odot y) = (ax) \odot y.$$

Exercise

Show that $M_{n,\ell}(E) \odot M_{\ell,m}(F) = M_{n,m}(E \odot F)$.

Example (Bhat-MS 2000)

- $(E,\xi) =$ GNS of $T\colon \mathcal{A} \to \mathcal{B}$ and $(F,\zeta) =$ GNS of $S\colon \mathcal{B} \to \mathcal{C}$.
- Then $\langle \xi \odot \zeta, a\xi \odot \zeta \rangle = \langle \zeta, \langle \xi, a\xi \rangle \zeta \rangle = S \circ T(a)$.
- So, GNS-$(S \circ T) = (\overline{\operatorname{span}}\, \mathcal{A}\xi \odot \zeta\mathcal{C}\,,\, \xi \odot \zeta)$.

Gives rise to product systems from CP-semigroups and units.

Example (Rieffel 1974b, Murphy 1997, MS 2000)

- $E_{\mathcal{B}}$ and ${}_{\mathcal{B}}G_{\mathbb{C}}$ (that is, G a representation space of $\mathcal{B}$).
- $H := E \odot G \quad \rightsquigarrow \quad x \odot \mathrm{id}_G \in \mathscr{B}(G,H)$ and $a \odot \mathrm{id}_G \in \mathscr{B}(H)$.
- $\mathcal{B} \subset \mathscr{B}(G) \quad \rightsquigarrow \quad E \subset \mathscr{B}(G,H)$ and $\mathscr{B}^a(E) \subset \mathscr{B}(H)$.
- ${}_{\mathcal{A}}E_{\mathcal{B}} \rightsquigarrow \rho\colon \mathcal{A} \to \mathscr{B}^a(E) \to \mathscr{B}(H)$ Stinespring representation.
- Indeed, (E,ξ) GNS of $T\colon \mathcal{A} \to \mathcal{B} \subset \mathscr{B}(G) \rightsquigarrow \xi \in \mathscr{B}(G,H)$ with

$$\xi^* \rho(a) \xi \;=\; T(a).$$

Note: With ${}_{\mathcal{A}}E_{\mathcal{B}}$, ${}_{\mathcal{B}}F_{\mathcal{C}}$ and $\mathcal{C} \subset \mathscr{B}(L)$, obviously,
$(x \odot y) \odot \mathrm{id}_L \;= (x \odot \mathrm{id}_{F \odot L})(y \odot \mathrm{id}_L) \in \mathscr{B}(L, E \odot F \odot L)$.

Recall: $E_{\mathcal{B}} \quad \rightsquigarrow \quad E^* \quad \text{is} \quad {}_{\mathcal{B}}E^*_{\mathcal{K}(E)} \qquad (\text{or } {}_{\mathcal{B}_E}E^*_{\mathcal{B}^a(E)}).$

Exercise

Show $\quad E^* \odot E \cong \mathcal{B}_E \quad$ *and* $\quad E \odot E^* \cong \mathcal{K}(E), \quad$ *canonically.*

Definition (Rieffel 1974b, MS 2009 (preprint))

$\mathcal{A}$ and $\mathcal{B}$ are (~~strongly~~) Morita equivalent if $\exists \ {}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}_{\mathcal{B}}N_{\mathcal{A}}$ such that

$$ {}_{\mathcal{A}}M \odot N_{\mathcal{A}} \ \cong \ {}_{\mathcal{A}}\mathcal{A}_{\mathcal{A}}, \qquad\qquad {}_{\mathcal{B}}N \odot M_{\mathcal{B}} \ \cong \ {}_{\mathcal{B}}\mathcal{B}_{\mathcal{B}}. $$

M is a Morita equivalence from $\mathcal{A}$ to $\mathcal{B}$, and N its inverse.

- M is full and faithful (=left action faithful).
- Morita equivalence is an equivalence relation.

Theorem (MS 2009 (preprint))

A full ${}_{\mathcal{A}}M_{\mathcal{B}}$ is a Morita equivalence if and only if the left action defines an isomorphism onto $\mathcal{K}(E)$. $\qquad$ ($\rightsquigarrow$ "standard" definition.)

- $\vartheta\colon \mathcal{B}^a(E_{\mathcal{B}}) \to \mathcal{B}^a(F_{\mathcal{C}})$ a unital strict homomorphism.
- Strict=strictly continuous on bounded subsets.
- **For us:** $\vartheta(EE^*)F$ is total in F.
  (Use a bounded approximate unit for $\mathcal{K}(E)$ in $\operatorname{span} EE^*$.)

With $F_\vartheta := E^* \odot {}_\vartheta F$, we get the chain of isomorphisms

$$F \;=\; \overline{\operatorname{span}}\,\vartheta(EE^*)F \;=\; \mathcal{K}(E) \odot {}_\vartheta F \;=\; E \odot E^* \odot {}_\vartheta F \;=\; E \odot F_\vartheta.$$

Theorem (Muhly-MS-Solel 2006 (preprint 2004))

$$u\colon x' \odot (x^* \odot y) \;\longmapsto\; \vartheta(x'x^*)y$$

defines a unitary $E \odot F_\vartheta \to F$ such that $\vartheta(a) = u(a \odot \mathrm{id}_{F_\vartheta})u^*$.

Exercise

If E is full, then F_ϑ is unique. (Hint: $u \in \mathcal{B}^{a,bil}(E \odot F_\vartheta, {}_\vartheta F)$. If (F'_ϑ, u') is another pair, consider $\mathrm{id}_{E^*} \odot (u^* u')$.)

Exercise

Do the same for normal $\vartheta\colon \mathcal{B}(G) \to \mathcal{B}(H)$. *(Hilbert spaces.)*

E_0-Semigroups and product systems

- E full.
- $\vartheta_t \colon \mathcal{B}^a(E) \to \mathcal{B}^a(E)$ a strict E_0–semigroup.
- $E_t := E^* \odot_t E$ with $v_t \colon x \odot (y^* \odot_t z) \mapsto \vartheta_t(xy^*)z$ so that $\vartheta_t = v_t(\bullet \odot \mathrm{id}_t)v_t^*$.

Exercise (MS 2002 and 2009 (preprint 2004))

The E_t form a product system $E^\odot$, that is:

- $E_0 = E^* \odot_0 E = E^* \odot E = \mathcal{B}$.
- $E_s \odot E_t = (E^* \odot_s E) \odot (E^* \odot_t E) = E^* \odot_s ({}_{\vartheta_t}E) = E^* \odot_{s+t} E = E_{s+t}$ *via bilinear*

$$u_{s,t} \colon (x^* \odot_s x') \odot (y^* \odot_t y') \longmapsto x^* \odot_{s+t} \vartheta_t(x'y^*)y'.$$

- *The product* $x_s y_t := u_{s,t}(x_s \odot y_t)$ *is associative.*
- $E_0 \ni x_0 = b \in \mathcal{B} \quad \rightsquigarrow \quad x_0 y_t = b y_t$ *and* $y_t x_0 = y_t b$.

Recall: $\mathcal{K}\binom{\mathcal{B}}{E} = \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{K}(E) \end{pmatrix} \subset \begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix} \subset \mathcal{B}^a\binom{\mathcal{B}}{E}$

MS 2000 (preprint 1997): A $*$–algebra $\mathcal{A}$ is a matrix $*$–algebra if

$$\mathcal{A} = \bigoplus_{i,j=1}^{2} \mathcal{A}_{i,j} = \begin{pmatrix} \mathcal{A}_{1,1} & \mathcal{A}_{1,2} \\ \mathcal{A}_{2,1} & \mathcal{A}_{2,2} \end{pmatrix}$$

such that $\mathcal{A}_{i,k}\mathcal{A}_{\ell,j} \subset \delta_{k,\ell}\mathcal{A}_{i,j}$ and $\mathcal{A}_{i,j}^* \subset \mathcal{A}_{j,i}$.

Similarly, for (pre-)C^*–, W^*–, and von Neumann algebras.

Moreover, if $\mathcal{A}$ is a matrix (pre-)C^*–algebra (etc.), then:

- $\mathcal{A}_{i,i}$ is (pre-)C^*–algebra.
- $\mathcal{A}_{i,j}$ is (pre-)Hilbert $\mathcal{A}_{j,j}$–module with $\langle a, a' \rangle = a^* a'$.
- Action of $\mathcal{A}_{i,i}$ on $\mathcal{A}_{i,j}$ turns it into (pre-)correspondence from $\mathcal{A}_{i,i}$ to $\mathcal{A}_{j,j}$. (Note: $\mathcal{A}_{i,i} \supset \mathcal{F}(\mathcal{A}_{i,j})$.)

Proposition

Let $\mathcal{A} = (\mathcal{A}_{i,j})$ *be a matrix pre-*C^**–algebra. Then the subspace* $\mathcal{B}_{2,1}$ *of* $\mathcal{A}_{2,1}$ *is the* 21*–corner of a matrix pre-*C^**–algebra* $\mathcal{B} \subset \mathcal{A}$ *iff one of the following (obviously equivalent) conditions holds:*

- $\mathcal{B}_{2,1}\mathcal{B}_{2,1}^*\mathcal{B}_{2,1} \subset \mathcal{B}_{2,1}$.
- $\mathcal{B}_{2,1}(\operatorname{span}\mathcal{B}_{2,1}^*\mathcal{B}_{2,1}) \subset \mathcal{B}_{2,1}$.

More precisely, in either case, $\mathcal{B}_{1,1} := \operatorname{span}\mathcal{B}_{2,1}^*\mathcal{B}_{2,1}$ *and* $\mathcal{B}_{2,2} := \operatorname{span}\mathcal{B}_{2,1}\mathcal{B}_{2,1}^*$ *are* $*$*–subalgebras of* $\mathcal{A}_{1,1}$ *and* $\mathcal{A}_{2,2}$ *and with* $\mathcal{B}_{1,2} := \mathcal{B}_{2,1}^*$*, the* $*$*–subalgebra of* $\mathcal{A}$ *generated by* $\mathcal{B}_{2,1}$ *is the matrix* $*$*–algebra* $(\mathcal{B}_{i,j})$.

Note: The ternary product $(x, y, z) \mapsto x\langle y, z\rangle = xy^*z$ plays a role.

Abbaspour Tabadkan-MS 2007: For $u\colon E_{\mathcal{B}_E} \to F_{\mathcal{C}}$, tfae:

- u is a generalized isometry, that is, $\exists(!)$ homomorphism $\varphi\colon \mathcal{B} \to \mathcal{C}$ such that $\langle ux, uy\rangle = \varphi(\langle x, y\rangle)$.
- u is a ternary homomorphism, that is, u is linear and $u(xy^*z) = (ux)(uy)^*(uz)$.

Exercise

If $\mathcal{A} = (\mathcal{A}_{i,j}) \subset \mathbb{B}(H)$ nondegenerately, then $H = H_1 \oplus H_2$ where $H_i = \overline{\operatorname{span}}\, \mathcal{A}_{i,i} H$, and $\mathcal{A}_{i,j} \subset \mathbb{B}(H_j, H_i) \subset \mathbb{B}(H) = \left(\begin{smallmatrix} \mathbb{B}(H_1) & \mathbb{B}(H_2, H_1) \\ \mathbb{B}(H_1, H_2) & \mathbb{B}(H_2) \end{smallmatrix}\right)$.

Definition (Murphy 1997, MS 2006)

Let $\mathcal{B} \subset \mathbb{B}(G)$ a concrete C^*–algebra. The subspace $E \subset \mathbb{B}(G, H)$ is a concrete Hilbert (resp. von Neumann) $\mathcal{B}$–module if:

1. $E\mathcal{B} \subset E$.
2. $E^* E \subset \mathcal{B}$.
3. $\overline{\operatorname{span}}\, EG = H$.
4. $\overline{E} = E$ (resp. $\overline{E}^s = E$).

Definition (Bikram-Mukherjee-Srinivasan-Sunder 2012)

$E = \overline{\operatorname{span}}^s\, E \subset \mathbb{B}(G, H)$ *is a von Neumann corner if $EE^*E \subset E$.*

Recall: $E_{\mathcal{B}}$ and $\mathcal{B} \subset \mathcal{B}(G) \quad \rightsquigarrow \quad E \subset \mathcal{B}(G,H)$ and $\mathcal{B}^a(E) \subset \mathcal{B}(H)$ with $H = E \odot G$. $\qquad$ (E conc. module, $\rightsquigarrow H \cong E \odot G$, $xg \mapsto x \odot g$.)

Definition (MS 2000 (preprint 1997))

A (pre-)Hilbert module E over a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ is a von Neumann $\mathcal{B}$–module if the extended linking algebra is a matrix von Neumann algebra

$$\begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix} \subset \mathcal{B}\begin{pmatrix} G \\ H \end{pmatrix} = \begin{pmatrix} \mathcal{B}(G) & \mathcal{B}(H,G) \\ \mathcal{B}(G,H) & \mathcal{B}(H) \end{pmatrix}$$

or, equivalently, if $E = \overline{E}^s$ in $\mathcal{B}(G,H)$.

Like $\mathcal{B}^a(E) = \mathcal{B}^a(E) \odot \mathrm{id}_G \subset \mathcal{B}(H)$, there is the commutant lifting $\rho' \colon \mathcal{B}' = \mathcal{B}^{bil}(G) \to \mathrm{id}_E \odot \mathcal{B}'$.

Exercise (Double commutant theorem vor vN-modules)

(Pre-)Hilbert module E over a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$:

$$\begin{pmatrix} \mathcal{B} & E^* \\ E & \mathcal{B}^a(E) \end{pmatrix}'' = \left\{ \begin{pmatrix} b' & \\ & \rho'(b') \end{pmatrix} \colon b' \in \mathcal{B}' \right\}' = \begin{pmatrix} \mathcal{B} & C_{\mathcal{B}'}(\mathcal{B}(H,G)) \\ C_{\mathcal{B}'}(\mathcal{B}(G,H)) & \rho'(\mathcal{B}')' \end{pmatrix},$$

where for ${}_{\mathcal{B}}V_{\mathcal{B}}$ we set $C_{\mathcal{B}}(V) = \{x \in V \colon bx = xb \forall b \in \mathcal{B}\}$.

Exercise

Why is ρ' normal? *(Below: Where do we use that ρ' is normal?)*

Lemma (Muhly-Solel 2002)

$\rho' \colon \mathcal{B}' \xrightarrow[\text{unital}]{\text{normal}} \mathcal{B}(H)$ representation $\rightsquigarrow$ $\overline{\text{span}}\, C_{\mathcal{B}'}(\mathcal{B}(G,H))G = H$.

Proof: Define the vN-algebra $\mathcal{A}' := \left\{ \begin{pmatrix} b' & \\ & \rho'(b') \end{pmatrix} : b' \in \mathcal{B}' \right\}$.

We know its commutant is $\mathcal{A} := (\mathcal{A}')' = \begin{pmatrix} \mathcal{B} & C_{\mathcal{B}'}(\mathcal{B}(H,G)) \\ C_{\mathcal{B}'}(\mathcal{B}(G,H)) & \rho'(\mathcal{B}')' \end{pmatrix}$.

Let $P' \in \mathcal{A}'$ the projection onto $H_0 := \overline{\text{span}}\, \mathcal{A}G$.

Since $\mathcal{A}' \cong \mathcal{B}'$, there is unique $p' \in \mathcal{B}'$ such that $P' = \begin{pmatrix} p' & \\ & \rho'(p') \end{pmatrix}$.

$H_0 \supset G \rightsquigarrow p' = \mathbf{1} \rightsquigarrow H_0 = G \oplus H$.

Hence, $\overline{\text{span}}\, C_{\mathcal{B}'}(\mathcal{B}(G,H))G = \rho'(p')H_0 = H$. $\square$

Corollary (MS 2003, 2006)

There is a bijective functor between the categories
$\mathfrak{cvN}_{\mathcal{B}} \ni (H,E)$ and ${}_{\mathcal{B}'}\mathfrak{cvN} \ni (H,\rho')$.

Theorem (Rieffel 1974b, MS 2000 (preprint 1997), 2005b)

Von Neumann modules are self-dual (=W^*–module!).

All (known) proofs have two parts:

- $\Phi \in \mathcal{B}^r(E, \mathcal{B}) \quad \rightsquigarrow \quad \Phi \odot \mathrm{id}_G \in \mathcal{B}(E \odot G, G)$. (Boundedness!)
  - Rieffel 1974b: Banach modules.
  - MS 2000, 2005b: Cyclic decomposition of G and polar decomposition in E.
- $\Phi \odot \mathrm{id}_G \in E^* \subset \mathcal{B}(H, G)$.
  - MS 2000: QONBs.
  - MS 2005b (Rieffel 1974b): $\Phi \in C_{\mathcal{B}'}(\mathcal{B}(H, G)) = E$.
  - Bikram-Mukherjee-Srinivasan-Sunder 2012:
    (Only E strongly full, that is, $\overline{\operatorname{span}}^s \langle E, E \rangle = \mathcal{B}$.) $\Phi E \subset \mathcal{B}$ so $\Phi \in \overline{\operatorname{span}}^s \Phi E E^* \subset \overline{\operatorname{span}}^s \mathcal{B} E^* = E^*$. □

Exercise (MS 2000 (preprint 1997))

Formulate and prove all the pleasant properties of Hilbert spaces that we listed in the beginning.

A (concrete) von Neumann correspondence from a vN-algebra $\mathcal{A} \subset \mathcal{B}(K)$ to a vN-algebra $\mathcal{B} \subset \mathcal{B}(G)$ is a (concrete) von Neumann $\mathcal{B}$–module and an $\mathcal{A}$–$\mathcal{B}$–correspondence such that the Stinespring representation $\rho\colon \mathcal{A} \to \mathcal{B}^a(E) \subset \mathcal{B}(H)$ is normal.

Note: $[\rho'(\mathcal{B}'), \rho(\mathcal{A})] = \{0\}$.

Corollary (MS 2003, 2006 (Muhly-Solel 2005 for $\mathcal{A} \neq \mathcal{B}$))

There is a bijective functor between the categories
$_{\mathcal{A}}\mathfrak{cvN}_{\mathcal{B}} \ni (H, E)$ *and* $_{\mathcal{B}',\mathcal{A}}\mathfrak{cvN} \ni (H, \rho', \rho)$.
The chain $(H, E) \leftrightarrow (H, \rho', \rho) \equiv (H, \rho, \rho') \leftrightarrow (H, E')$
is a bijective functor, the commutant, between the categories
$_{\mathcal{A}}\mathfrak{cvN}_{\mathcal{B}} \ni (H, E)$ *and* $_{\mathcal{B}'}\mathfrak{cvN}_{\mathcal{A}'} \ni (H, E')$.

Note: $$C_{\mathcal{B}}(E) = E \cap E' = C_{\mathcal{B}'}(E').$$

Note: If $\mathcal{B}$ is in standard representation, then $\mathcal{B}' \cong \mathcal{B}^{op}$. So, H becomes a normal $\mathcal{A}$–$\mathcal{B}$–module. (Connes 1980.)

Let $\mathscr{A} \subset \mathcal{B}(K)$, $\mathscr{B} \subset \mathcal{B}(G)$, $\mathcal{C} \subset \mathcal{B}(L)$ and ${}_{\mathscr{A}}E_{\mathscr{B}}$, ${}_{\mathscr{B}}F_{\mathcal{C}}$. (All vN.)

Tensor product: (Note: $(E \odot F) \odot L = E \odot (F \odot L)$.)

- Compute $E \odot F$ and take $E \bar{\odot}^s F := \overline{E \odot F}^s \subset \mathcal{B}(L, E \odot F \odot L)$.
- Or: Compute $E \bar{\odot}^s F := \overline{\operatorname{span}}^s (E \odot \mathrm{id}_{F \odot L})(F \odot \mathrm{id}_L)$.
  (Cf. tensor product of Connes correspondences.)

Easy: Left action is normal!

There is an incredible lot of representations acting on $E \odot F \odot L$!

$$\underset{a}{E} \odot F \odot \underset{c'}{L} = \underset{a}{E} \odot (F \odot \underset{c'}{L}) = \underset{a}{E} \odot (\underset{c'}{F'} \odot G)$$
$$\cong \underset{c'}{F'} \odot (\underset{a}{E} \odot G) = \underset{c'}{F'} \odot (E' \odot \underset{a}{K}) = \underset{c'}{F'} \odot E' \odot \underset{a}{K}$$

Theorem (MS 2003 for $\mathscr{A} = \mathscr{B} = \mathcal{C}$, Muhly-Solel 2005)

$$(E \bar{\odot}^s F)' \cong F' \bar{\odot}^s E'.$$

No time: Recall $\vartheta \colon \mathcal{B}^a(E) \to \mathcal{B}^a(F) \rightsquigarrow F = E \bar{\odot}^s F_\vartheta$.
One may show: $(F_\vartheta)' \cong C_{\mathcal{B}^a(E)}(\mathcal{B}(E \odot G, {}_\vartheta F \odot L))$.
$\rightsquigarrow$ PS (E'_t) from E_0–semigroups à la Arveson. (MS 2003, 2005a)

More exercises

- $(\Omega, \mathfrak{F}, \mu)$ a measure space and E a Hilbert $\mathcal{B}$–module.
- $L^2(\Omega, E) := L^2(\Omega) \otimes E := \overline{L^2(\Omega) \underline{\otimes} E}$ (external tensor product.)

Exercise

- *Show that* $\ell^2(\ell^\infty) \supsetneq \left\{(x_n) \colon x_n \in \ell^\infty, \sum \|x_n\|^2 < \infty\right\}$.
  (That is, $L^2 \supset L^2_{Bochner}$ *but not always* $L^2 = L^2_{Bochner}$*.)*
- *Show that not every element in* $L^2\big(\,[0,1]\,,\ L^\infty[0,1]\,\big)$ *can be represented as a function* $[0,1] \to L^\infty[0,1]$.

Exercise

- $\mathcal{B}^a(E) \odot \mathrm{id}_F \subset \mathcal{B}^a(E \odot F)$. *(*$\cong \mathcal{B}^a(E)$ *if* ${}_{\mathcal{B}}F_{\mathcal{C}}$ *is faithful.)*
- $\mathrm{id}_E \odot \mathcal{B}^{a,bil}(F) = (\mathcal{B}^a(E) \odot \mathrm{id}_F)'$. *(*$\cong \mathcal{B}^{a,bil}(F)$ *if* E *is full.)*
- E *full. Then* $x \odot y = x \odot y'$ *for all* $x \in E \quad \Rightarrow \quad y = y'$.

More exercises

Exercise (MS-Sumesh 2012 (preprint))

- *For all* $x \in E$ *and* $0 < \alpha < 1$ $\exists!$ $x_\alpha \in \overline{xC^*(|x|)}$ *such that* $x_\alpha\, |x|^\alpha = x$. *(Corollary:* $E = E\mathcal{B} = \mathcal{K}(E)E$*. What about* $\mathcal{A}E$*?)*
- $\|x \odot y\| = \inf\Big\{\|x'\|\,\|y'\| : x' \odot y' = x \odot y\Big\}$. *(Hint:* $\alpha \to 1$*.)*
- $\|\sum_i x_i \odot y_i\| = \inf\Big\{\|X_n\|\,\|Y^n\| : X_n \odot Y^n = \sum_i x_i \odot y_i\Big\}$.

$\rightsquigarrow$ ***Blecher 1997:*** *Tensor product=(amalgamated) Haagerup tensor product (completely isomorphic).* *(*$\rightsquigarrow$ *universal property.)*

(Actually, last exercise:

Exercise (MS 2005a)

View G *and* H *as correspondences over* $\mathbb{C} = \mathbb{C}' = \mathcal{B}(\mathbb{C})$ *and convince yourself that* $(H \otimes G)' = G' \otimes H'$.

)

More exercises

Exercise (Reducing more statements to linking algebras)

Formulate the following results for vN-modules and and reduce their proofs to the known statements on vN-algebras:

- *Polar decomposition.*
- *The Kaplansky density theorem.*

Let $E_{\mathcal{B}}$ and $F_{\mathcal{C}}$ be vN-modules. Let E be strongly full.

- *A map $u\colon E \to F$ extends as normal homomorphism acting block-wise between the linking vN-algebras iff u is a σ–weak ternary homomorphism.*
- *A block-wise homomorphism between the linking vN-algebras is normal iff its to one (hence. all) of the corners is σ–weak.*

More exercises

Exercise (vN-modules and representations of vN–algebras (MS 2009 (preprint 2004))

$E \subset \mathcal{B}(G,H)$ vN-module frequently given by (H,ρ').

- *There exists a QONB, that is, partial isometries (e_i) in E such that $\sum_i e_i e_i^* = \mathrm{id}_E$.*
- *Corollary: $E = \overline{\bigoplus}^s_i p_i\mathcal{B} \subset \overline{\mathcal{B}^{\#S}}^s = \mathbb{C}^{\#S} \bar{\otimes}^s \mathcal{B}$.*
  Infer form this the amplification-induction theorem for ρ'.

Now let E be strongly full ($\Leftrightarrow \rho'$ faithful).

- *$\exists\mathfrak{n}$ such that $\overline{E^{\mathfrak{n}}}^s \ni \xi$ with $\langle\xi,\xi\rangle = \mathbf{1}$. (Hint: QONB for E^*!)*
- *$\exists\mathfrak{n}$ such that $\overline{E^{\mathfrak{n}}}^s \cong \overline{\mathcal{B}^{\mathfrak{n}}}^s$. (Hint: $E = \mathcal{B} \oplus p\mathcal{B} \rightsquigarrow \overline{E^{\infty}}^s \cong$ what?)*
- *Use this to show that:*
  - *$\exists\mathfrak{n}$ such that $\rho' \otimes \mathrm{id}_{\mathfrak{n}} \simeq \mathrm{id}_{\mathcal{B}'} \otimes \mathrm{id}_{\mathfrak{n}}$.*
  - *vN-algebras $\mathcal{A}$ and $\mathcal{B}$ are (vN-)Morita equivalent $\Longleftrightarrow$ $\exists$ faithful normal representations with isomorphic commutants $\Longleftrightarrow \exists\mathfrak{H}$ such that $\mathcal{A} \bar{\otimes}^s \mathcal{B}(\mathfrak{H}) \cong \mathcal{B} \bar{\otimes}^s \mathcal{B}(\mathfrak{H})$.*

Thank you!

Bibliography I

- G. Abbaspour and M. Skeide, *Generators of dynamical systems on Hilbert modules*, Commun. Stoch. Anal. **1** (2007), 193–207, (arXiv: math.OA/0611097).
- D.P. Blecher, *A new approach to Hilbert C^*–modules*, Math. Ann. **307** (1997), 253–290.
- P. Bikram, K. Mukherjee, R. Srinivasan, and V.S. Sunder, *Hilbert von Neumann Modules*, Commun. Stoch. Anal. **6** (2012), 49–64, (arXiv: 1102.4663).
- B.V.R. Bhat and M. Skeide, *Tensor product systems of Hilbert modules and dilations of completely positive semigroups*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **3** (2000), 519–575, (Rome, Volterra-Preprint 1999/0370).
- G.G. Kasparov, *Hilbert C^*–modules, theorems of Stinespring & Voiculescu*, J. Operator Theory **4** (1980), 133–150.

Bibliography II

- E.C. Lance, *Hilbert C^*–modules*, Cambridge University Press, 1995.
- P.S. Muhly and B. Solel, *Quantum Markov processes (correspondences and dilations)*, Int. J. Math. **51** (2002), 863–906, (arXiv: math.OA/0203193).
- ______, *Duality of W^*-correspondences and applications*, Quantum Probability and Infinite Dimensional Analysis — From Foundations to Applications (M. Schürmann and U. Franz, eds.), Quantum Probability and White Noise Analysis, no. XVIII, World Scientific, 2005, pp. 396–414.
- P.S. Muhly, M. Skeide, and B. Solel, *Representations of $\mathcal{B}^a(E)$*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9** (2006), 47–66, (arXiv: math.OA/0410607).
- G.J. Murphy, *Positive definite kernels and Hilbert C^*–modules*, Proc. Edinburgh Math. Soc. **40** (1997), 367–374.

Bibliography III

- W.L. Paschke, *Inner product modules over B^*–algebras*, Trans. Amer. Math. Soc. **182** (1973), 443–468.
- M.A. Rieffel, *Induced representations of C^*-algebras*, Adv. Math. **13** (1974), 176–257.
- ______, *Morita equivalence for C^*–algebras and W^*–algebras*, J. Pure Appl. Algebra **5** (1974), 51–96.
- M. Skeide, *Generalized matrix C^*–algebras and representations of Hilbert modules*, Mathematical Proceedings of the Royal Irish Academy **100A** (2000), 11–38, (Cottbus, Reihe Mathematik 1997/M-13).
- ______, *Hilbert modules and applications in quantum probability*, Habilitationsschrift, Cottbus, 2001, Available at `http://web.unimol.it/skeide/`.
- ______, *Dilations, product systems and weak dilations*, Math. Notes **71** (2002), 914–923.

Bibliography IV

- ______, *Commutants of von Neumann modules, representations of $\mathcal{B}^a(E)$ and other topics related to product systems of Hilbert modules*, Advances in quantum dynamics (G.L. Price, B .M. Baker, P.E.T. Jorgensen, and P.S. Muhly, eds.), Contemporary Mathematics, no. 335, American Mathematical Society, 2003, (Preprint, Cottbus 2002, arXiv: math.OA/0308231), pp. 253–262.
- ______, *Three ways to representations of $\mathcal{B}^a(E)$*, Quantum Probability and Infinite Dimensional Analysis — From Foundations to Applications (M. Schürmann and U. Franz, eds.), Quantum Probability and White Noise Analysis, no. XVIII, World Scientific, 2005, (arXiv: math.OA/0404557), pp. 504–517.
- ______, *Von Neumann modules, intertwiners and self-duality*, J. Operator Theory **54** (2005), 119–124, (arXiv: math.OA/0308230).

Bibliography V

- ———, *Commutants of von Neumann correspondences and duality of Eilenberg-Watts theorems by Rieffel and by Blecher*, Banach Center Publications **73** (2006), 391–408, (arXiv: math.OA/0502241).
- ———, *Classification of E_0–semigroups by product systems*, Preprint, arXiv: 0901.1798v3, 2009.
- ———, *Unit vectors, Morita equivalence and endomorphisms*, Publ. Res. Inst. Math. Sci. **45** (2009), 475–518, (arXiv: math.OA/0412231v5 (Version 5)).
- M. Skeide and K. Sumesh, *CP-H-Extendable maps between Hilbert modules and CPH-semigroups*, Preprint, arXiv: 1210.7491v1, 2012.