Michael Skeide
4/3/2013

Hilbert and von Neumann modules

Def.: A complex vector space H with $\langle \cdot , \cdot \rangle : H \times H \longrightarrow \mathbb{C}$ sesquilinear is

- a semi-Hilbert-space if $\langle x, x \rangle \geq 0$;
- a pre-Hilbert space if $\langle x, x \rangle = 0 \Rightarrow x = 0$;
- a Hilbert space if it is complete wrt $\|x\| = \sqrt{\langle x, x \rangle}$.

Exercise: prove that $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (what about real Hilbert space?)

Def.: $a : G \longrightarrow H$ is adjointable if there's $a^* : H \longrightarrow G$ s.t. $\langle ag, h \rangle = \langle g, a^* h \rangle$

We will write $\mathcal{L}(G,H)$, $\mathcal{L}^a(G,H)$, $\mathcal{B}(G,H)$, $\mathcal{B}^a(G,H)$ for the space of linear, l. adjointable, bounded, b. a. maps. (Here G, H are pre-Hilbert; ⚠ if they are only semi-Hilbert, linearity is no more automatic)

Exercise. If H is a non-complete, are the 4 spaces pairwise different?

Cauchy-Schwarz inequality: $\langle x, y \rangle \langle y, x \rangle \leq \langle y, y \rangle \langle x, x \rangle$.
generalizes to Hilbert modules

- $\| \cdot \|$ is a seminorm (so that we can take quotients)
- $\|x\| = \sup_{\|y\| \leq 1} |\langle y, x \rangle|$, so that $\mathcal{B}^a(H)$ is a pre-C^*-algebra.
- $\langle \cdot , \cdot \rangle$ is continuous (so that we can complete a pre-HS)

Self-duality (HS): If $f \in \mathcal{B}(H, \mathbb{C})$, there is $y \in H$ so that $f = \langle y, \cdot \rangle$.
(doesn't generalize!)

Consequences:
- $\mathcal{B}(G,H) = \mathcal{B}^a(G,H)$ $(= \mathcal{L}^a(G,H))$
- If G is a Hilbert subspace of H, there is $p = p^* \in \mathcal{B}(H)$ s.t. $G = pH$.
- If G is a subspace of a Hilbert space H, then $\overline{G} = G^{\perp\perp}$ and $H = G^{\perp\perp} \oplus G^{\perp}$.
- Hilbert space have orthonormal bases.

Def: If B is a C^*-algebra, a right B-module E_B [one requires $(x,b)\mapsto xb$ to be bilinear] with $\langle\cdot,\cdot\rangle : E\times E\to \mathbb{C}$ sesquilinear is

- a semi-H-module if $\langle x,x\rangle\geq 0$
- a pre-H-module if $\langle x,x\rangle=0\Rightarrow x=0$
- a Hilbert-module if it is complete wrt $\|x\|:=\sqrt{\|\langle x,x\rangle\|}$

if furthermore we have compatibility : $\langle x, yb\rangle=\langle x,y\rangle b$

Exercise : prove that $\langle x,y\rangle=\langle y,x\rangle^*$ ~~(what about real Hilbert space?)~~

N.B : $\langle xb,y\rangle=\langle y,xb\rangle^*=(\langle y,x\rangle b)^*=b^*\langle y,x\rangle^*=b^*\langle x,y\rangle$.

Def : Adjointability is defined similarly. Then we define $\mathcal{L}^r(G,H)$ the right linear maps, $\mathcal{L}^a(G,H)$ the adjointable maps (they are automatically right linear), $B^r(G,H)$, $B^a(G,H)$

Exercise.
- $\langle x,y\rangle=\langle x,y'\rangle$ for all $x \Rightarrow y=y'$
- $a\in\mathcal{L}^a(E,F)$ is right linear and closeable

N.B.: suppose $E_B\supset S$ with span $\underset{\text{the set of products}}{\widetilde{SB}}=E$ and $a: S\to F$, $a^*: T\to E$
$F_B\supset T$ with span $TB=F$ s.t. $\langle as,t\rangle=\langle s,a^*t\rangle$ for $s\in S$ and $t\in T$,

then there's $a\in\mathcal{L}^a(E,F)$ with $a^*\in\mathcal{L}^a(F,E)$, extending S to E and T to F.

Corollary : action of a C^*-algebra A on S such that $\langle a\cdot s,s'\rangle=\langle s,a^*\cdot s'\rangle$.
This yields $\pi: A\to B^a(\bar{E})$ with $\pi(a)s=a\cdot s$. Is it faithful? surjective?
Then $a\in\mathcal{L}^a(E)$, while $A=$ span $\mathcal{U}(A)$...

Cauchy-Schwarz inequality. $\langle x,y\rangle\langle y,x\rangle\leq\|\langle y,y\rangle\|\langle x,x\rangle$.

Proof: Let $z=\|\langle y,y\rangle\|x-y\langle y,x\rangle$. Then

$$0\leq\langle z,z\rangle=\|\langle y,y\rangle\|^2\langle x,x\rangle-2\|\langle y,y\rangle\|\langle x,y\rangle\langle y,x\rangle+\underbrace{\langle x,y\rangle\langle y,y\rangle\langle y,x\rangle}_{\text{this is of form } a^*\overset{}{\underset{b}{}}a\leq\|b\|a^*a;\ \text{thus}, \leq\langle x,y\rangle\|\langle y,y\rangle\|\langle y,x\rangle}$$

We get $0\leq\|\langle y,y\rangle\|^2\langle x,x\rangle-\|\langle y,y\rangle\|\langle x,y\rangle\langle y,x\rangle$.

if $\|\langle y,y\rangle\|\neq 0$, ✓. If $\|\langle x,x\rangle\|\neq 0$, ✓ by symmetry.

What if $\|\langle x,x\rangle\|=\|\langle y,y\rangle\|=0$? ... !

Paschke 1973 ... Rieffel 1974 ... !

$\hookrightarrow$ in his study of group representations...

Corollary: • $\|\cdot\|$ is a semi-norm: $\|x+y\|^2 = \|\langle x+y, x+y\rangle\| \leq \|x\|^2 + 2\|\langle x,y\rangle\| + \|y\|^2$ $\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$ $\leq (\|x\| + \|y\|)^2$

Take $\mathcal{N} = \{x : \langle x, x\rangle = 0\}$: $\langle x+\mathcal{N}, y+\mathcal{N}\rangle = \{\langle x,y\rangle\}$

- $\|x\| \overset{\leq}{=} \sup_{\|y\|\leq 1} \|\langle y,x\rangle\| \geq \frac{\|\langle x,x\rangle\|}{\|x\|} = \|x\|$

- $\|a^*a\| = \|a\|^2 = \sup_{\|x\|\leq 1} \|\langle x, a^*a x\rangle\| \leq \sup_{\|x\|,\|y\|\leq 1} \|\langle y, a^*a x\rangle\| = \|a^*a\|$

Note that $\|a\| = \|a^*\|$.

Note also $\|\langle y, ax\rangle\| = \|\langle x, a^*y\rangle\|$

Example: • a HS is a Hilbert $\mathbb{C}$-module

• B is an Hilbert-B-module with inner product $\langle b,b'\rangle = \langle b^*b'\rangle$,

• more generally, a closed right ideal R in B is an H-submodule

• Direct sum: $\underset{\substack{i\in I \\ \text{closed direct sum}}}{\bigoplus} E_i = \overline{\underset{i\in I}{\bigoplus} E_i}$ $\leftarrow$ algebraic direct sum $\quad \langle x,y\rangle = \sum_i' \langle x_i, y_i\rangle$

Exercise • $\bigoplus E_i = \{(x_i) : \sum_i' \langle x_i, x_i\rangle \text{ exists}\}$

• $B^r(\bigoplus_{i\in\mathbb{N}} B, B) = ?$

$\hookrightarrow \bigoplus_{i\in I} E = \underset{(n=\#I)}{\mathbb{C}^n} \otimes E$ with $\langle f\otimes x, g\otimes y\rangle = \langle f,g\rangle\langle x,y\rangle$

and $L^2(\Omega)\otimes E =: \underset{\substack{\uparrow \\ \text{completion wrt} \\ \|f\| = \sqrt{\|\int |f(\omega)|^2 \mu(d\omega)\|}}}{L^2(\Omega;E)} \overset{\uparrow}{\not\supseteq} L^2_{\text{Bochner integrable}}(\Omega;E)$ i.e., strongly measurable

$\hookrightarrow$ which is the completion wrt $\|f\| = \sqrt{\int \|f(\omega)\|^2 \mu(d\omega)}$ (it is only a pre-Hilbert module)

hint: consider $\Omega = \mathbb{N}$

N.B.: not every element of $L^2(\Omega;E)$ can be represented as a function.

• (closed) Left ideals L in a C^*-algebra $\mathcal{A}$: they are in 1-1 correspondence with

[hereditary sub C^*algebras by $L \mapsto \overline{\text{sp}}\, L^*L$

$\hookrightarrow$ Def: If $b\in B^+$ and $a\leq b$, $a\in\mathcal{A}^+$, then $a\in B$.

$\Leftrightarrow$ $B\mathcal{A}B \subset B$]

Example: $p\mathcal{A}p$, $p = 1_B$.

• $E^* = \{x^* : x\in E\} \subset B^a(E,B)$ by $x^*(y) := \langle x,y\rangle$.

The adjoint of x^* is then $x : b \longmapsto xb$.

E^* is then a Hilbert $\mathcal{K}(E)$-module, where $\mathcal{K}(E) = \overline{\text{span}}\{xy^* : x,y\in E\}$ where $xy^* : z \longmapsto x\langle y,z\rangle$

is the compact operators on E.

From now on, E is complete.

Here, the inner product in E^* is $\langle x^*, y^*\rangle = xy^*$.

N.B: $xx^* \geq 0$ in $\mathcal{K}(E)$ has a square root; let us embed E into a C^*-algebra in the following way: $B\begin{pmatrix}\mathbb{C}\\ H\end{pmatrix} = \begin{pmatrix} B(\mathbb{C}), & B(H,\mathbb{C}) \\ B(\mathbb{C}, H), & B(H)\end{pmatrix}$

$$B^a\begin{pmatrix}B\\ E\end{pmatrix} = \begin{pmatrix} B^a(B), & B^a(E,B) \\ B^a(B,E), & B^a(E)\end{pmatrix}$$

Thus $xx^* \geq 0$ takes sense in $B^a\begin{pmatrix}B\\ A\end{pmatrix}$ easily

$$\supset \begin{pmatrix} B & E^* \\ E & B^a(E)\end{pmatrix} \supset \begin{pmatrix} B & E^* \\ E & K(E)\end{pmatrix} \supset \begin{pmatrix} B_E & E^* \\ E & \mathcal{K}(E)\end{pmatrix}$$

reduced linking algebra of E

Here appears the linking algebra of E : $B_E := \overline{\text{span}}\langle E, E\rangle \subset B$.
(extended)

If B_E, we say E is <u>full</u>.

If A contains B_E as an ideal, then E_A as Hilbert A-module.

Take an approximate unit (u_λ) for B_E (not bounded, self-adjoint, etc)

[This enables for example that if you have $(u_\lambda) \subset \mathcal{F}$, you get $(v_\lambda) \subset A$.]

Then $\lim xu_\lambda = x$ and $\lim_\lambda xu_\lambda a$ still exists: it is xa and you have defined the action $A \curvearrowright E$. If $A \supset \mathcal{K}(E)$ as ideal, then E^* is a Hilbert A-module...

Let us apply this: $M_n(B) \subset B^a(B^n)$ (equality if B is unital)

$$\left(\left(E^*_{\mathcal{K}(E)}\right)^n_{\mathcal{K}(E)}\right)^* =: E_n{}_{M_n(B_E)} \subset M_n(B).$$

n.b.: $\mathcal{K}((E^*)^n) = M_n(B_E)$ n.b.: $E_n = \left((E^*)^n\right)^*_{M_n(B)}$

you can write the elements as row vectors
$x = (x_1, \dots, x_n)$, $\langle x, y\rangle = (\langle x_i, y_j\rangle)_{ij} \subset M_n(B)$

Then $(E_n)^m = M_{mn}(E) \ni x = (x_{ij})$, $(\langle x, y\rangle)_{ij} = \sum_i \langle x_{ni}, y_{nj}\rangle$
$= (E^m)_n$.

Special class of operators: <u>projections</u>: $p = p^*p \in B^a(E)$; $v : E_B \xrightarrow{\text{isometry}} F_B$.
$\langle vx, vy\rangle = \langle x, y\rangle$

<u>unitary</u> if surjective: then $u^* = u^{-1}$.

$v \subset B^a(E, vF)$

<u>Prop</u>: $v \in B^a(E, F)$ iff $\exists\, p \in B^a(F)$ onto vF
iff vF is complemented in F, that is, $F = vF \oplus (vF)^\perp$

Proof: There is v^* suppose let $p = vv^*$ with $pF = vE$. then $v^{-1}p = v^* \in B^a(F, E)$.

We interpret H^* as $B(H, \mathbb{C})$: similarly,
$$E^* = \mathcal{K}(E, B) \subset B^a(E, B) \subset B^r(E, B);$$
right-linear
E is <u>self-dual</u> if $E^* = B^r(E, B)$.

Typically, right ideals are not self-dual in $\mathcal{B}$. When are they?

Ex.: $B = C[0,1]$, $I = C_0\,]0,1]$: there is no $i \in I$ s.t. $\mathrm{id}_I(j) = i^* j$ for $j \in I$.

If E is not self-dual, then $\mathcal{B}^r(\frac{B}{E}) \neq \mathcal{B}^a(\frac{B}{E})$.

If E is self-dual, does this imply $\mathcal{B}^r(E) = \mathcal{B}^a(E)$.

A double centralizer for a C^*-algebra A is a pair (L,R) of linear maps on A st $a L(b) = R(a) b$. Then $M(A) = \{(L,R)\}$ is a C^*-algebra with multiplication $(L,R)(L',R') = (LL', R'R)$ and involution $(L,R) = (R^*, L^*)$, where $L^*(a) = L(a^*)^*$.

It contains A as an ideal: $L_a : b \mapsto ab$, $R_a : b \mapsto ba$, $\begin{array}{l} A \to M(A) \\ a \mapsto (L_a, R_a) \end{array}$.

A separates elements of $M(A)$. It is an "essential ideal", being maximal for this property.

$\mathcal{B}^a(E) = M(\mathcal{K}(E))$, $\underset{\text{for } C^*\text{-algebras}}{\mathcal{B}^a(B) = M(B)}$, $\mathcal{B}^a(E) = \mathcal{B}^a(\mathcal{K}(E))$

cf. strict topology....

① $\underline{\mathcal{B}^a(B) = M(B)}$: If $L = a$, $R = * \circ a^* \circ *$, $b_1 L(b_2) = b_1 a b_2$ and $R(b_1) b_2 = (a^* b_1^*)^* b_2$

If you have (L,R), you set $a = L$, $a^* = * \circ R \circ *$

② $\underline{\mathcal{B}^a(E) = \mathcal{B}^a(K(E))}$: If $a \in \mathcal{B}^a(E)$, $a(xy^*) = (ax)y^*$. This proves one direction

For the other direction, let $a \in \mathcal{B}^a(\mathcal{K}(E))$, $a(x\langle y,z\rangle) = (a(xy^*))z$. Then you get a representation of one C^*-algebra onto the other.

This also follows from Morita equivalence and tensor products.

this proves ③ $M(K(E)) = \mathcal{B}^a(E)$.

Def: the strict topology is defined by the seminorms $\|\cdot k\|$, $\|k \cdot\|$, $k \in K(E)$.

the $*$-strong topology ——————— $\|\cdot x\|$, $\|x^* \cdot\|$, $x \in E$

Ex: what's the strong, the $*$-strong completion of $B(H)$?

Exercise: • strict is equivalent to $*$-strong on bounded subsets.

• approximate units and topologies: how do they converge?

Def.: A correspondence from A to B (or Hilbert A-B-bimodule) ${}_AE_B$ is

- a Hilbert B-module E
- an A-B-bimodule (i.e., $(ax)b=a(xb)$) s.t. $\begin{matrix} A \longrightarrow B^a(E) \\ a \longmapsto (x \longmapsto ax) \end{matrix}$ defines a nondegenerate *-homomorphism (i.e. $\overline{\text{span}}\, AE=E$)

⌜N.B.: ① $a: E_B \longrightarrow F_B$ is a right module map: $a \in B^r(E,F)$, if and only if $\langle ax, ax\rangle \leq M\langle x,x\rangle$; this has so far no "algebraic proof⌟

② $(M \bar{\otimes} N)' = M' \bar{\otimes} N'$ is a hard problem.

Example: Paschke's GNS construction for CP-maps (1973)

Let $T: A \longrightarrow B$ be a c.p. map: $\sum_{ij} b_i^* T(a_i^* a_j) b_j \geq 0$ for all finite choices.

Let us define an inner product on $(\underset{A}{A} \otimes \underset{\text{vector space tensor product}}{B})_B$: $\langle a\otimes b, a'\otimes b'\rangle := b^* T(a^* a') b'$

We get a semi-Hilbert module.

Now consider $\mathcal{N} = \{x : \langle x,x\rangle = 0\}$ and ${}_AE_B = \overline{A\otimes B/\mathcal{N}}$

If A and B are unital, then $\xi = 1\otimes 1 + \mathcal{N}$ in E satisfies $T(a) = \langle \xi, a\xi\rangle$ and $E = \overline{\text{span}}\, A\xi B$.

Example: $\vartheta: B \xrightarrow[\text{endomorphism}]{\text{unital}} B$, then ${}_\vartheta B$ is endowed with $b.x = \vartheta(b)x$.

Then you have a B-B-bimodule. You recover ϑ by $\vartheta(b) = b\cdot 1$.

Tensor products (over B). ${}_AE_B$, ${}_BF_C$ have as vector space tensor product ${}_AE\otimes F_C$ with inner product $\langle x\otimes y, x'\otimes y'\rangle = \langle y, \underbrace{\langle x,x'\rangle}_{\in B} y'\rangle$. It is right-linear.

We have $\langle x\otimes y, x\otimes y\rangle = \langle y, \underbrace{\langle x,x\rangle}_{b^*b} y\rangle = \langle by, by\rangle \geq 0$.

And $\sum x_i \otimes y_i = \underbrace{X_n \otimes Y^n}_{\text{tensor product over } M_n(B)}$ with $X_n = (x_1, \dots, x_n) \in E_n$ and $Y_n = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in F^n$,

so that $\langle \sum x_i \otimes y_i, \sum_j x_j \otimes y_j\rangle = \langle X_n \otimes Y^n, X_n \otimes Y^n\rangle \geq 0$

Then define $E \odot F = \overline{E\otimes F/\mathcal{N}}$: it is the unique A-C-correspondence generated by $x \odot y$, with $\langle x\odot y, x'\odot y'\rangle = \langle y, \langle x, x'\rangle y'\rangle$ and $a(x\odot y) = (ax)\odot y$
in particular, $xb\odot y = x\odot by$

correspondence : disturbed channel
: time-evolution by automorphism
: or, for open systems, only by c.p. maps!
→ we are doing conditional expectations

A small exercise: $M_{n,\ell}(E)\odot M_{\ell,m}(F) = M_{n,m}(E\odot F)$

$A\times B \supset K$ $\quad \mathbb{C}K \subset M_{n,m}$ $\quad \sum_k (x_{k,i}, \underset{\substack{\uparrow \\ \mathrm{diag}\, M_n}}{y_{k,j}}) = \mathrm{diag}\, M_m$
$\# A = n$
$\# B = m$

GNS-construction for $T: A \xrightarrow{cp} B$: $({}_A E_B, \xi)$, $E = \overline{\operatorname{span}}\, A\xi B$,
$T(a) = \langle \xi, a\xi\rangle$. If you have another $S: B \xrightarrow{cp} C$, you get $({}_B F_C, \zeta)$, $F = \overline{\operatorname{sp}}\, B\zeta C$,
$S(b) = \langle \zeta, b\zeta\rangle$. Then $\langle \xi\odot\zeta, a\,\xi\odot\zeta\rangle = \underset{S(T(a))}{\langle \zeta, \langle\xi, a\xi\rangle\zeta\rangle} = S\circ T(a)$
You have $\xi\odot\zeta \in {}_A E\odot F_C$, but $\xi\odot\zeta$ does not generate all of ${}_A E\odot F_C$:
$E\odot F = \overline{\operatorname{span}}\, A\xi\odot B\zeta C$. It contains $G = \overline{\operatorname{span}}\, A\xi\odot\zeta C$ (up to isometry)
$\supset (G, \eta)$ $\quad \eta = \xi\odot\zeta$ under the same isometry

$T_t: B \longrightarrow B$ $\quad$ a semigroup: $T_s\circ T_t = T_{s+t}$ and $T_0 = \mathrm{id}$
where $t\in\mathbb{R}_+$ or $\mathbb{N}_0$
Consider $(\mathcal{E}_t, \xi_t)$:

$$\begin{array}{ll} \xi_t & \\ \downarrow & \\ \xi_{t_n}\odot\ldots\odot\xi_{t_1}, \quad t_1+\ldots+t_n = t & \mathcal{E}_{\underline{t}} = \mathcal{E}_{t_1}\odot\ldots\odot\mathcal{E}_{t_n} \\ \text{or, split further} & \\ \downarrow & \\ (\xi_{s^n_{m_n}}\odot\ldots\odot\xi_{s^n_1})\odot\cdots\odot(\xi_{s^1_{m_1}}\odot\ldots\odot\xi_{s^1_1}) & \mathcal{E}_{\underline{s}^n\cup\ldots\cup\underline{s}^1} \end{array}$$

and $E_t := \underset{\underline{t}\in\mathbb{J}_t}{\text{inductive limit}}\ \mathcal{E}_{\underline{t}}$ $\quad$ (here $\mathcal{E}_{\underline{s}}\odot\mathcal{E}_{\underline{t}} = \mathcal{E}_{\underline{s}\cup\underline{t}}$,
so that $E_s\odot E_t = E_{s+t}$.

and $E^\odot = (E_t)_{t\in\mathbb{R}_+}$ is a product system of B,B-correspondences.

$u_{s,t}: E_s\odot E_t \xrightarrow[\text{unital}]{\text{bil}} E_{s+t}$ and $x_s y_t := u_{s,t}(x_s\odot y_t)$ is associative

There are marginal conditions at 0: $E_0 = B$, $u_{0,t}, u_{t,0}$ are canonical.
Then $\xi_s\odot\xi_t = \xi_{s+t}$ (GNS-construction) defines a <u>unit</u> $\xi^\odot$ for $E^\odot$

Thus $(\mathcal{E}_t, \xi_t)$ define $(E^{\odot}, \xi^{\odot})$, $T_t = \langle \xi_t, \cdot\, \xi_t \rangle$

"continuous time GNS construction"

Suppose you have a correspondence ${}_B F_B$, you get $L^2(I,F) =: E_I$, $I \subset \mathbb{R}$.

The full Fock module $\mathcal{F}(E_{\mathbb{R}_+}) = \bigoplus_{n\in\mathbb{N}_0} E^{\odot n}_{\mathbb{R}_+}$, where $E^{\odot 0} = \omega B$ with $\langle \omega, \omega \rangle = 1$

$b\omega = \omega b$ for $b \in B$.

$L^2(\mathbb{R}_+^n, F^{\odot n})$ and consider elements f s.t. $f(\underline{t}) = 0$ unless $t_n \geq \dots \geq t_1$.

Consider $\Gamma(F)$. $\Gamma_t(F) = \chi_{[0,t]^n}\Gamma(F)$.

$$F_s \odot G_t(s_m, \dots, s_1, t_n, \dots, t_1) = F_s(s_m - t_1, \dots, s_1 - t_1) \odot G_t(t_n, \dots, t_1)$$

$s_1 > t_n$

If $\beta \in B$, $\zeta \in F$, $\xi_t^{(\beta,\zeta)} = \bigoplus \xi_t^n$, $\xi_t^n = e^{\beta(t-t_n)}\zeta \odot \dots \odot e^{\beta(t_2-t_1)}\zeta e^{t_1\beta}$

$\to$ units

$$T_t^{(\beta,\zeta),(\beta',\zeta')} = \langle \xi_t(\beta,\zeta), \cdot\, \xi_t(\beta',\zeta') \rangle = e^{t\mathcal{L}^{(\beta,\zeta),(\beta',\zeta')}}$$

with $\mathcal{L}^{(\beta,\zeta),(\beta',\zeta')}(b) = \langle \zeta, b\zeta' \rangle + \beta^* b + b\beta'$

cf difficult proof by Christensen and Evans … is there a direct proof?

indicates the presence of a unit.

$\vartheta_t : B \xrightarrow[\text{endomorphism}]{\text{unital}} B$ (E_0-semigroup)

${}_{\vartheta_t}B$: $b \cdot x_t = \vartheta_t(b) x_t$.

$x_s \odot y_t := \vartheta_t(x_s) y_t$

$\alpha_t = u_t \cdot u_t^*$

back to technicalities

E_B, ${}_B F_C$ give $E \odot F$

$\mathcal{B}^a(E) \ni a \longmapsto (x \odot y \xmapsto{a \odot id_F} ax \odot y)$ defines a representation of the C*-algebra.

right and left linear

$\mathcal{B}^{a,bil}(F) \ni a \underset{\text{(commutant)}}{\longmapsto} (x \odot y \xmapsto{id_E \odot a} x \odot ay) \subset \mathcal{B}^a(E \odot F)$.

$\underbrace{(\mathcal{B}^a(E) \odot id_F)'}_{\ni} = id_E \odot \mathcal{B}^{a,bil}(F)$ <u>if E_B is full!</u> ($\overline{\text{span}}\langle E, E \rangle = B$)

$a \in \mathcal{B}^a(E \odot F)$

$a(\alpha \odot id) = (\alpha \odot id)a$ and $v_0 : \langle x, x' \rangle y \longmapsto \langle x, ax' \rangle y$

and then you have to work to show that you really get every operator here.

This tensor product is $\simeq$ the Haagerup module product.

See the MS-Sumesh-exercise: for $x \in E$ and $0 < \alpha < 1$, there is a unique $x_\alpha \in x\underbrace{\overline{C^*(|x|)}}_{\text{C*-algebra generated by } |x|}$

[consider $xf \longmapsto |x|f$ …] s.t. $x_\alpha |x|^\alpha = x$

Example: E_B, ${}_B G_{\mathbb{C}}$. Define the Hilbert space $H = E \underset{B^a(E)}{\odot} G_{\mathbb{C}}$

Consider $x \odot id \in B(G, H)$, sending $g \longmapsto x \odot g$. Then

$(x \odot id)^*(y \odot g) = \langle x, y \rangle g$ and $(x \odot id)^*(y \odot id)\, g = \langle x, y \rangle g$.

The inner product $\langle x, y \rangle$ is represented by $(x \odot id)^*(y \odot id)$.

One also has $(xb \odot id) = (x \odot id)\, b$

Consider that $B \subset B(G)$. Then all maps become faithful!

$E \subset B(G, H)$

$EB \subset E$.

$E^* E \subset B$

$\overline{\mathrm{span}\, E\, G} = H$.

$B^a(E) \subset B(H)$

${}_A E_B$ GNS-construction on T

$\cup$

$\xi \in B(G, H)$ s.t. $T(a) = \langle \xi, a \xi \rangle$

$= \xi^* \rho(a) \xi$

with $\rho : A \to B^a(E) \subset B(H)$

$\xi \odot \xi_{C \subset B(L)}$

${}_B G = {}_B F_C \odot L$; knowing that $B \subset B(G')$ does not help!

$x \odot y \odot g : \langle g, \langle x, \langle y, y' \rangle x' \rangle g' \rangle$

$\langle (x \odot id)(y \odot id) g, (x' \odot id)(y' \odot id)\, g' \rangle$

Positivity is much easier to prove!

$\langle g_i, \langle x_i, x_j \rangle g_j \rangle$, $G = \oplus G_\alpha$, $G_\alpha = \overline{B g_\alpha}$

$H = E \odot G_\alpha = \overline{E \odot g_\alpha}$

About the tensor product with a Hilbert space:

${}_B G_C$: $h^\perp \overline{Bg}$ can be identified with $bh^\perp Bg$, so that

$$G = \overline{\bigoplus_{\alpha\in A} \underbrace{\overline{Bg_\alpha}}_{G_\alpha}} \quad \text{and} \quad E_B \odot G_C = \bigoplus_\alpha H_\alpha \text{ with } H_\alpha = E \odot_{G_\alpha} \overline{= E \odot G_\alpha}$$

Then $\langle \sum_i^\perp x^\alpha \odot g_\alpha , \sum_i x^\beta \odot y_\beta \rangle = \sum_\alpha \langle g_\alpha , \langle x^\alpha , x^\alpha \rangle g_\alpha' \rangle$.

If $\Phi \in B^r(\mathcal{E}, B)$, $B \subset B(G)$, then $\Phi \odot id_G \in \underline{\underline{B}}(E \underline{\odot} G, \mathbb{C})$

↑ difficult to prove!

$x \odot g \longmapsto \Phi(x) g$.

$$\underset{= {}_B G_C}{\underbrace{E_B \odot (F \odot_C L)}} \quad C \subset B(L)$$

: $\langle l, \langle x \odot y, x' \odot y' \rangle l' \rangle$

$= \langle l, \langle y, \langle x, x' \rangle y' \rangle l' \rangle$

$\langle (x \odot id_{F\odot L})(y \odot id_L) l, (x' \odot id_{F \odot L})(y' \odot id_L) l' \rangle$ a quadratic expression.

and $\langle l, \langle \sum_i x_i \odot y_i , \sum_j x_j \odot y_j \rangle l \rangle \geq 0$

this elementary proof works in more algebraic contexts.

Chapter III

${}_B E^*_{\mathcal{K}(E)} \odot E_B \cong {}_B (B_E)_B$ because $x^* \odot y \longmapsto \langle x, y \rangle$ is an isomorphism of correspondences.

${}_{\mathcal{K}(E)} E \odot_B E^*_{\mathcal{K}(E)} \cong {}_{\mathcal{K}(E)} \mathcal{K}(E)_{\mathcal{K}(E)}$

$x \odot y^* \longmapsto x y^*$.

More symmetric definition (Rieffel 1974): A and B are strongly Morita-equivalent if there exist correspondences ${}_A M_B$, ${}_B N_A$ such that ${}_A M \odot_B N_A \cong A$ and

${}_B N \odot_A M_B \cong B$

This is how Morita-equivalence occurs in algebra.

Morita equivalence is not convenient for C^* algebra.

Rieffel proves that Morita equivalence came up with a neutral element from the left.

that is how strong Morita equivalence (recall actions are nondegenerate)

st. Morita equivalence is an equivalence relation.

" 3 $1\frac{3}{2}$

M must be full. $A \curvearrowright A$ is faithful, so that $A \curvearrowright M$ also is.

thus M is faithful

If $B=A$, this is also meaningful: and $\{{}_BM_B\}$ is the so called <u>Picard group</u> of B.

Consider the case ${}_A({}_\vartheta B)_B$, $\vartheta : A \to B$, for example $B = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & M_2 \end{pmatrix} \subset M_3$,

${}_BM_B = \begin{pmatrix} & \mathbb{C}_2 \\ \mathbb{C}^2 & \end{pmatrix} \subset M_3$

then $M \odot M = B$: M is self inverse.

<u>Basic theorem</u>: $M \cong M \odot \underbrace{B \cong M \odot (N \odot M)}_{\text{noncanonical}} \text{"}=\text{"} \underbrace{(M \odot N) \odot M \cong A}_{\text{noncanonical}} \odot M \cong M$

<u>Prop</u>: You may choose the isomorphisms s.t. $(M \odot N) \odot M = M \odot (N \odot M)$
(and then $(N \odot M) \odot N = N \odot (M \odot N)$)

<u>theorem</u>: A full correspondence ${}_AM_B$ from A to B is a <u>Morita equivalence</u> iff the canonical $A \longrightarrow B^a(M)$ is an isomorphism onto $\mathcal{K}(M)$.

"$B^a(E)$ is almost as simple as $B(H)$" $\rho : B(G) \xrightarrow[\text{unital}]{\text{normal}} B(H)$ homomorphism

that is, $H = G \otimes \mathcal{H}$ ... all proofs want to reduce the question
$\rho(a) = a \otimes \mathrm{id}_{\mathcal{H}}$ to finite rank operators.

$\vartheta : B^a(E_B) \xrightarrow[\text{strict}]{\text{unital}} B^a(F_{\mathcal{C}})$, i.e., strictly continuous on bounded subsets;
recall strict means $\overline{\mathrm{span}}\, \vartheta(\mathcal{K}(E))F = F$.

Start with the correspondence ${}_{B^a(E)}F_{\mathcal{C}} = {}_{\mathcal{K}(E)}F_{\mathcal{C}} = \mathcal{K}(E) \odot F = (E \odot E^*) \odot F = E \odot \underbrace{(E^* \odot F)}_{{}_B(F_\vartheta)_{\mathcal{C}}}$

$= E \odot F_\vartheta$

Look how E acts: $\vartheta(a) = a \odot \mathrm{id}_{F_\vartheta}$

theorem: $u : x' \odot (x^* \odot y) \longmapsto \vartheta(x'x^*)y$ is unitary such that $\vartheta(a) = u(a \odot \mathrm{id}_{F_\vartheta})u^*$

I won't speak at Eilenberg-Watts thm [Blecher!]

A functor from one category of modules, sufficiently regular, $\gamma : \mathcal{C}^*_B \longrightarrow \mathcal{C}^*_{\mathcal{C}}$
is equivalent to $\bullet \odot \mathrm{id}_{{}_BF_{\mathcal{C}}}$.

" 3 B_3

Another consequence: $\vartheta = (\vartheta_t)$, $\vartheta_t : B^a(E) \xrightarrow[\text{endomorphism}]{\text{unital strict}} B^a(E)$ [E_0-semigroup]

yields ${}_{B}(E_t)_{B} := E^* \underset{\substack{\uparrow \\ \text{short for } \vartheta_t}}{\odot_t} E$ with $\vartheta_t : E \odot E_t \longrightarrow E$

$x \odot (y^* \odot_t z) \longmapsto \vartheta_t(xy^*)z$

with $u_{s,t} : E_s \odot E_t \longrightarrow E_{s+t}$

$(x^* \odot y) \odot (x'^* \odot_t y') \mapsto x^* \odot_{s+t} \underbrace{\vartheta_t(yx'^*)y'}_{\text{total subset of } E}$. It is an isometry. Check it!

And $E_0 = E^* \odot E = B_E$ [here we need to require E be full]

cf. 2009: Classification of E_0-semigroups by product systems.
It does so up to cocycle conjugacy!

Def: If you have ϑ, ϑ' on $A \in 1$, a left unitary cocycle (u_t) for ϑ is s.t. $u_{s+t} = u_s \vartheta_s(u_t)$, and then define $\vartheta^u_t := u_t \vartheta_t(\cdot) u_t^*$

If $\vartheta' = \vartheta^u$ for some u, then ϑ and ϑ' are cocycle-equivalent.

If ϑ on A, ϑ' on A', $\alpha : A \longrightarrow A'$ an isomorphism, then define $\vartheta^\alpha_t := \alpha \circ \vartheta_t \circ \alpha^{-1}$

They are cocycle conjugate if $\vartheta^\alpha = \vartheta'^u$ for some α and (u_t).

$v_t : \Pi({}_{B}F_{B}) \odot \Gamma_t(F) \longrightarrow \Gamma(F)$

product system over $\begin{pmatrix} 0 & 0 \\ \mathbb{C}^2 & 0 \end{pmatrix}$ product system over $\begin{pmatrix} 0 & F_2 \\ 0 & 0 \end{pmatrix}$

<u>Classification result</u>: Theorem: ϑ on $B^a(E)$, ϑ' on $B^a(E')$, i.e. on full Hilbert bimodules are stably cocycle equivalent iff $E^\odot \cong E'^\odot$

$\vartheta \otimes \mathrm{id}$ and $\vartheta' \otimes \mathrm{id}$: $E \longrightarrow E^n = E \otimes \mathbb{C}^n$

on $B^a(E \otimes \mathbb{C}^n)$, $\vartheta(a) \otimes M_n$.

ϑ on $B^a(E)$ and ϑ' on $B^a(F_C)$ are stably cocycle conjugate iff

$E^\odot$ and $F^\odot$ are ME: ${}_{R}M_{C}$: ${}_{C}(M^* \odot E_t \odot M_{C}) = M^* \odot E^\odot \odot M \cong F^\odot$

Product systems have been invented by Arveson.

ϑ on $B(H)$ yields $H^\otimes = (H_t)$. In the other direction, does $H^\otimes \overset{?}{\rightsquigarrow} \vartheta$

three huge papers of 1989 to prove this. Skeide 2006 gives an easier proof.

11 3 13₄

It's easier if you have a unital unit $u^{\otimes} = (u_t)$ and a discrete $(H_n)_{n \in \mathbb{N}_0}$

defines by $H_t \longrightarrow u_n \otimes H_t \subset H_{n+t}$ $H_\infty = H_\infty \otimes H_t$

$\longleftarrow$: v_t the unitary and

You have a unit vector $u_1 \in H_1$ and

$\vartheta_t = v_t (\cdot \otimes id_t) v_t^*$

$H_n = H^{\otimes n} \ni u_1^{\otimes n} =: u_n$

$\int_0^\infty H_\lambda d\lambda \otimes H_t$, $\int_t^\infty H_\lambda d\lambda$: pb: isometries, not unital.

If you define $\int_t^{1+t} H_\lambda d\lambda$, $\int_t^\infty H_\lambda d\lambda$

Arveson provided immediately another proof: sections:

$u_1 \in H_1$: $u_1 \otimes f_t = f_{t+1}$ for $t > T_0$: $\int_T^{T+1} \langle f_t, g_t \rangle dt$ does not depend on T.

But this is unitarily equivalent to Skeide 2006

Riesz-Nagy: beautiful proof of Stone's theorem. → aperiodic part: discrete spectrum, op. valued Fourier series

→ a bounded part.

V. Liebscher 2003 has another complicated proof.

12/3/2013

Chapter IV. Von Neumann algebras.

- strong topology: Skeide's preferred
- normality: interesting as an algebraic concept.

$\Big\}\ B(H)$.

$\underset{\mathrm{Id}_H}{\underset{\shortparallel}{1}} \in A \subset B(H)$, $A = \overline{A}^s$, where the strong topology is generated by the $\|\cdot h\|$, $h \in H$.

Good property: If you have $a_\lambda \longrightarrow a$, $b_\mu \longrightarrow b$ and $\|a_\lambda\| \leq M$, then $a_\lambda b_\mu \longrightarrow ab$

as $\|(a_\lambda b_\mu - ab)h\| \leq \|a_\lambda(b_\mu - b)h\| + \|(a_\lambda - a)bh\| \leq M\|(b_\lambda - b)h\| + \|(a_\lambda - a)(bh)\|$.

- If $I = \{a \in B(H) : a^2 = 0\}$, show that $\overline{I}^s \in B(H)$.

Von Neumann's double commutant theorem: A is a vN algebra iff $A'' = A$.

If A isn't necessary equal to its strong closure (but a $*$-algebra), then $\overline{A}^s = A''$

- There are more topologies: weak topology, $*$-strong topology (for which $B(H)$ is complete).
- If you have an increasing sequence that is bounded, then $s\text{-}\lim a_\lambda = \mathrm{lub}(a_\lambda)$.

and recall that $\varphi \geq 0$ is normal iff $\mathrm{lub}\,\varphi(a_\lambda) = \varphi(\mathrm{lub}(a_\lambda))^2$

theorem: $\varphi \geq 0$ is normal iff φ is σ-weak. (depends upon axiom of choice)

and span $\{\text{normal } \varphi \geq 0\} = B(H)_* = L^1(H)$ and $B(H) = (B(H)_*)^*$

Now do the same for module!

Recall $E \subset \begin{pmatrix} B & E^* \\ E & \underset{\underset{B^a(E)}{\shortparallel}}{B^a(E)} \end{pmatrix}$. $B = B'' \subset B(G)$. If $H = E \odot G$ and x is identified with $x : G \longrightarrow E \odot G$, $g \longmapsto x \odot g$

then $xb = xb$ and $x^* y = \langle x, y \rangle = x^* y$ for $x \in E$,

Definition: [2000] E is a vN-B-module if $B^a\binom{B}{E}$ is a vN-algebra (on $\binom{G}{H}$)

(i.e., in $\begin{pmatrix} B(G) & B(H,G) \\ B(G,H) & B(H) \end{pmatrix}$)

Prop. This holds iff $E = \overline{E}^s \subset B(G,H)$

I.e., $\underset{E}{\underset{\cap}{a_\lambda x}} \xrightarrow{s} \underset{E}{\underset{\cap}{a x}}$

Def $\frac{2003}{2006}$ The linear subspace $E \subset B(G,H)$ is a concrete vN module if $EB \subset E$, $E^*E \subset B$, $\overline{\mathrm{span}}\, EG = H$, $E = \overline{E}^s$

of Rieffel 1974/2

12/3/2013

If (E_B, H) is a concrete vN-module, then $E \odot G \cong H$
$x \odot g \longmapsto xg$

$B^{a,bil}({}_B G_{\mathbb{C}}) = B' \subset B(G)$.

Consider $\rho'(b') := id_E \odot b'$ | ad hoc def^{n} $\rho'(b') xg = xb'g$.

$$\begin{pmatrix} B & E^* \\ E & B^a(E) \end{pmatrix}'' = \left\{ \begin{pmatrix} b' & \\ & \rho'(b') \end{pmatrix} : b' \in B' \right\}' : \begin{pmatrix} B & C_{B'}(B(H,G)) \\ C_{B'}(B(G,H)) & \rho'(B')' \end{pmatrix} = \mathcal{A}$$

Here ρ' is the commutant lifting and $C_B(E) = \{ x \in E : bx = xb \text{ for } b \in B \}$

of Muhly-Solel: deal w. W*-algebras: you have to choose your rep all the time...
It is better to have sthg unique!

Lemma (Muhly Solel 2002): Let us start with H a normal nondegenerate rep. ρ' of B'; define $E := C_{B'}(B(G,H))^{B', \rho'}$. It fulfills all properties of a concrete vN module, but $\overline{sp}\, EG = H$. But we also have this!

Proof: Let $H_0 = \overline{sp}\, \mathcal{A} G$. There is a projection p in $\mathcal{A}'$...
there is $p' \in B'$ such that $p = \begin{pmatrix} p' & \\ & \rho'(p') \end{pmatrix}$
Then $H_0 \supset G$ and $p' = 1_G$ and $\rho(p') = 1_H$.
Where does normality occur: In the fact that $\left\{ \begin{pmatrix} b' & \\ & \rho'(b') \end{pmatrix} \right\}$ is a vN-algebra!

We just proved that: **Th**: the categories cvN_B and ${}_{B'}cvN$ are isomorphic.
The morphisms of cvN_B are $B^a(E)$; the morphisms of ${}_{B'}cvN$ are $B^{bil}(H)$.

Bimodule version: **Def**: A von Neumann A-B-bimodule is a von Neumann B-module and a von Neumann A-B-correspondence s.th. $\langle x, \bullet x \rangle$ is normal for all $x \in E$.
A concrete vN A-B-bimodule is s.t. the Stinespring representation
$\rho : A \longrightarrow B^a(E) \subset B(H)$ is normal.

Theorem: $({}_A E_B, H) \longleftrightarrow (\underbrace{\rho', \rho}, H) \equiv (\rho, \rho', H) \longleftrightarrow ({}_{B'} E'_{A'}, H)$
$(B \subset B(G), A \subset B(K))$
commutant lifting of A'
Stinespring of B'

$E \longleftrightarrow E'$ commutant: here $E' = C_A(B(K,H))$
→ very similar to Connes correspondence.

${}_A H_B$, $B \subset L^2(B)$ (noncommutative), $B' \cong B^{op}$ via JbJ^{-1}

What can we do? ${}_{\mathcal{A}}E_{\mathcal{B}} \odot {}_{\mathcal{B}}F_{\mathcal{C}}$, where $\mathcal{C} \subset \mathcal{B}(L)$:

$${}_{\mathcal{A}}E_{\mathcal{B}} \odot {}_{\mathcal{B}}F_{\mathcal{C}} \odot L \quad \Omega \quad (x \odot id_{F \odot L})(y \odot id_L) \subset \mathcal{B}(L, E \odot (F \odot L))$$

To this amounts the complicated tensor product of Connes correspondences.

<u>Self-duality.</u> The quickest proof.

$E = C_{\mathcal{B}'}(\mathcal{B}(G,H))$. If $\varphi \in \mathcal{B}^r(E, \underset{\uparrow}{\mathcal{B}})$, $x^* = \varphi \odot id_G \in \mathcal{B}(G, E \odot G)'$

↳ this is already in Rieffel 1974 "scalar ring" "bdd linear functionals"

intertwines the actions of $\mathcal{B}'$:

$$\underbrace{x^*\underbrace{(id \odot b')(y \odot g)}_{y \odot b'g}}_{\varphi(y)b'g} = b'\varphi(y)g.$$

Quasi-Orthonormal bases: $\langle e_\beta, e_\alpha \rangle = \delta_{\alpha\beta} p_\alpha = \delta_{\alpha\beta} p_\alpha^* p_\alpha$

there is a QONB and $\sum e_\beta e_\beta^*$ strongly converges to 1 in $\mathcal{B}^a(E)$

$$E_{\mathcal{B}} \curvearrowright (\rho', H).$$