golf
Kenny 1

Let us write the induction : $H \subseteq G$ closed subgroup, π a (projective) unitary irr. rep. Then $\mathrm{Ind}_G(B(\mathcal{H}_\pi)) = \{z \in C(G) \otimes B(\mathcal{H}) \mid (\rho_h \otimes \mathrm{id})z = (\mathrm{id} \otimes \mathrm{Ad}\,\pi(h))(z)\}$

right translation

Now do it with $G = SU(2)$: the S^1-invariant f^{ns} are $C(S^2)$.

- $D_\infty = \mathbb{Z}_2 \ltimes S_1$.
- there also are the exceptional groups $\tilde{\mathfrak{S}}_4, \tilde{\mathfrak{S}}_5, \tilde{\mathfrak{A}}_5$.

By an equivariant Hilbert C^*-module for a CQG $\mathbb{G}$ and an action $\mathbb{G} \curvearrowright A$ I mean a fin. generated projective right Hilbert A-module $\mathcal{E}$ equipped with a linear map $\alpha_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{E} \otimes C(\mathbb{G})$ s.t.

1) $[(1 \otimes C(\mathbb{G}))\cdot \alpha_{\mathcal{E}}(A)] = \mathcal{E} \otimes C(\mathbb{G})$

2) $\alpha_{\mathcal{E}}(\xi \cdot x) = \alpha_{\mathcal{E}}(\xi)\cdot \alpha(a)$, $\xi \in \mathcal{E}, x \in A$.

3) $\langle \alpha_{\mathcal{E}}(\xi), \alpha_{\mathcal{E}}(\eta)\rangle_{\mathcal{E} \otimes C(\mathbb{G})} = \alpha(\langle \xi, \eta\rangle_A)$

4) $(\alpha_{\mathcal{E}} \otimes \mathrm{id})\circ \alpha_{\mathcal{E}} = (\mathrm{id} \otimes \Delta)\circ \alpha_{\mathcal{E}}$.

We get a C^*-category $\mathcal{D}_A$ with as objects these $\mathcal{E}$ and $\mathrm{Mor}(\mathcal{E}, \mathcal{F}) =$
$$\{T \in \mathcal{L}(\mathcal{E}, \mathcal{F}) : \alpha_{\mathcal{F}}(T\xi) = (T \otimes \mathrm{id})\alpha_{\mathcal{E}}(\xi) \text{ for } \xi \in \mathcal{E}\}$$

"f. g. projective" implies that $\mathrm{Mor}(\mathcal{E}, \mathcal{F})$ are f.d., equipped with a Banach space structure and an involution $* : \mathrm{Mor}(\mathcal{E}, \mathcal{F}) \longrightarrow \mathrm{Mor}(\mathcal{F}, \mathcal{E})$ satisfying the C^*-axiom

N.B.: If $A = \mathbb{C}$ with $\alpha(1) = 1 \otimes 1$, then $\mathcal{D}_A$ is just the category of f.d. unitary corepresentations of $C(\mathbb{G})$

- The unitary corepresentations are $u \in B(\mathcal{H}) \otimes C(\mathbb{G})$ unitary s.t. $(1 \otimes \Delta)u = u_{12}u_{13} \in B(\mathcal{H}) \otimes C(\mathbb{G}) \otimes C(\mathbb{G})$ and the associated "equivariant Hilbert A-module" is just the Hilbert space $\mathcal{H}$ with $\delta_u : \mathcal{H} \longrightarrow \mathcal{H} \otimes C(\mathbb{G}) : \xi \longrightarrow u(\xi \otimes 1)$

The special element in $\mathcal{D}_A$ is A itself as a right A-module by $\langle a, b\rangle = a^* b$.

The module category structure is $\mathrm{Rep}(\mathbb{G}) \times \mathcal{D}_A \longrightarrow \mathcal{D}_A$

$$\underset{\in \mathrm{Rep}(\mathbb{G})}{(\mathcal{H}_u, u)} \times \underset{\in \mathcal{D}_A}{(\xi, \alpha_\xi)} \longrightarrow (\mathcal{H}_u \otimes \xi, u \otimes \alpha_\xi),$$

where $(u \otimes \alpha_\xi)(\zeta \otimes \eta) = u_{13}(\zeta \otimes \alpha_\xi(\eta))$

and ordinary tensor product for morphisms.

This gives a module structure: $(u \otimes v) \otimes \xi \cong u \otimes (v \otimes \xi)$

$$1_{\mathbb{G}} \otimes \xi \cong \xi.$$

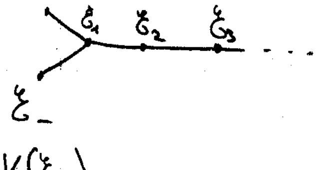
this is an abstract module C*-category for Rep $\mathbb{G}$.

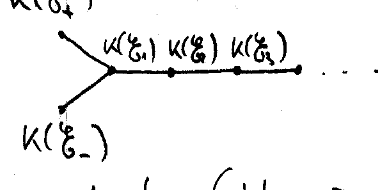
How can we construct a graph with this (this is already in Wassermann's article)

Having $SU_q(2) \curvearrowright A$, the vertices are the irreducible objects in $\mathcal{D}_A$: ξ_v s.t. $\mathrm{Mor}(\xi, \xi) = \mathbb{C}1$. (In fact, all objects in $\mathcal{D}_A$ are finite direct sums of irreducible objects.)

As edges, take ξ_v irreducible and consider the spin $\frac{1}{2}$ representation of $SU_q(2)$. Then $(u_{1/2} \otimes \xi_v) = \bigoplus_{w \in V} \underset{\in \mathbb{N}}{M_{vw}} \xi_w$: I put M_{vw} edges from v to w.

So I start with an action $SU_q(2) \curvearrowright A$.





Connectedness of the graph follows from ergodicity (Vergnioux):

Fact: If $\mathbb{G} \curvearrowright A$ ergodically, then for all $\xi, \mathcal{H}$ there is $u \in \mathrm{Rep}\,\mathbb{G}$ s.t. $\mathrm{Mor}(u \otimes \xi, \mathcal{H}) \neq \{0\}$.

How do we get weights on the edges?

A different setting: consider our $\mathcal{D}_A$ with the $\mathrm{Rep}\,\mathbb{G}$-module structure, and let again V be a parameterisation of a maximal set of irreducible objects in $\mathcal{D}_A$: ξ_v, ξ_w, etc.

Instead of just the number $\mathcal{N}_{vw}$, I consider the Hilbert space $\mathcal{H}_{vw} = \mathrm{Mor}(\xi_v, u \otimes \xi_w)$ of dimension $\mathcal{N}_{vw}$, where $\langle T, S \rangle = T^* S \in \mathrm{End}(\xi_v) = \mathbb{C}$.

In fact, we get a strong tensor <u>functor</u> from $\mathrm{Rep}(\mathbb{G}) \longrightarrow \mathrm{Hilb}_{V\times V}$, where $\mathrm{Hilb}_{V\times V}$ is the category of $V\times V$-graded Hilbert spaces $\mathcal{H} = \bigoplus_{v,w\in V} \mathcal{H}_{vw}$ and where $\dim \bigoplus_v \mathcal{H}_{vw} < \infty$ for all w. The <u>morphisms</u> are the grading-preserving bounded operators. The tensor structure is $(\mathcal{H} \boxtimes \mathcal{G})_{vw} = \bigoplus_r \mathcal{H}_{vr} \otimes \mathcal{G}_{rw}$

Then $\mathcal{F}_A\ \mathrm{Rep}\,\mathbb{G} \longrightarrow \mathrm{Hilb}_{V\times V}$ is really a $\otimes$-functor and

$$(\mathcal{F}_A(u))_{vw} = \mathrm{Mor}(\xi_v, u \otimes \xi_w). \text{ Then}$$

$$\mathcal{F}_A(u \otimes v)_{wz} = \mathrm{Mor}(\xi_w, u \otimes (v \otimes \xi_z)) \qquad v \otimes \xi_z \cong \text{''} \bigoplus_y \mathrm{Mor}(\xi_y, v \otimes \xi_z) \otimes \xi_y \text{''}$$

$$\cong \mathrm{Mor}(\xi_w, u \otimes \xi_y) \otimes \mathrm{Mor}(\xi_y, v \otimes \xi_z)$$

$$= \bigoplus_y (\mathcal{F}(u)_{wy} \otimes \mathcal{F}(v)_{yz}) = (\mathcal{F}(u) \boxtimes \mathcal{F}(v))_{wz}$$

Recapitulation: to classify ergodic actions of $SU_q(2)$ is to classify indecomposable module C^*-categories for $\mathrm{Rep}(SU_q(2))$ is to classify indecomposable strong tensor C^*-functors for any set V: $\mathcal{F} : \mathrm{Rep}(SU_q(2)) \longrightarrow \mathrm{Hilb}_{V\times V}$ [indecomposable: $\forall v, w \exists u$ $\mathcal{F}(u)_{vw} \neq 0$]

To get a graph from $\mathcal{F}$: $\dim(\mathcal{H}_{vw})$ edges from v to w.
To get weights on the edges: I use that there is a special intertwiner

$$R_{1/2} : \underset{\mathbb{C}}{\overset{\shortparallel}{\mathbb{1}_{\mathbb{G}}}} \longrightarrow u_{1/2} \otimes u_{1/2}.$$

Properties of $R_{1/2}$: $\begin{cases} (R_{1/2}^* \otimes 1)(1 \otimes R_{1/2}) = -\operatorname{sgn} q \\ R^* R = |q + q^{-1}| \end{cases}$ (choose a proper scaling) and I have corresponding properties for $\mathcal{Q}$.

Now apply the functor: consider $\mathcal{F}(R_{1/2}) : \mathcal{F}(\mathbb{1}_{\mathbb{G}}) \longrightarrow \mathcal{F}(u_{1/2} \otimes u_{1/2}) \cong \mathcal{F}(u_{1/2}) \boxtimes \mathcal{F}(u_{1/2})$.

If I take $(\ell^2(V) \otimes \mathcal{G})_{vw} = \bigoplus_z \ell^2(V)_{vz} \otimes \mathcal{G}_{zw} = \bigoplus_z \delta_{vz} \mathbb{C} \otimes \mathcal{G}_{zw} = \mathcal{G}_{vw}$ $\ell^2(V)$.

Thus I get a morphism $R = \mathcal{F}(R_{1/2}) : \ell^2 V \to \mathcal{F}(u_{1/2}) \otimes \mathcal{F}(u_{1/2})$
and a collection of maps $R_{vw} : \underset{\mathbb{C}\delta_v}{\overset{\shortparallel}{\mathbb{C}}} \longrightarrow \mathcal{F}(u_{1/2})_{vw} \otimes \mathcal{F}(u_{1/2})_{wv}$

Let me write $\mathcal{F}(u_{1/2})_{vw} = \mathcal{H}_{vw}$

Let us consider the antilinear operators $J_{vw} : \mathcal{H}_{vw} \to \mathcal{H}_{wv}$
$$\mathfrak{J} \longmapsto (\mathfrak{J}^* \otimes 1)(R_{vw}(1))$$

By the properties for $R_{1/2}$ and thus for $R = \mathcal{F}(R_{1/2})$, the J_{vw} satisfy

$(*)$ $\left\{ \begin{array}{l} J_{wv} J_{vw} = -\mathrm{sgn}(q) 1 \\ \text{for all } v \in V. \end{array} \right.$ and $\displaystyle\sum_{w \in V} \mathrm{Tr}(J_{vw}^* J_{vw}) = |q + q^{-1}|$ for $v \in V.$ $\}$

Let $\{\lambda_{vw}^{(1)}, \dots, \lambda_{vw}^{(n)}\}$ be the eigenvalues of $J_{vw}^* J_{vw}$.

Then the conditions imply: - reciprocality: $[\lambda_{vw}^{(1)}, \dots, \lambda_{vw}^{(n)}] = \left[\dfrac{1}{\lambda_{wv}^{(1)}}, \dots, \dfrac{1}{\lambda_{wv}^{(n)}}\right]$
- constant weight: $\displaystyle\sum_{w,k} \lambda_{vw}^{(k)} = |q + q^{-1}|$ $\ldots, \lambda_{vw}^{(a)} \lambda_{wv}^{(a)} = 1.$

say the edges from v to w are labelled $e_1, \dots, e_n$ ($n = \dim \mathcal{H}_{vw}$); I put $p(e_k) = \dfrac{\lambda_{vw}^{(k)}}{|q+q^{-1}|}$. We also have $p(e_k) p(\overline{e_k}) = |q+q^{-1}|^{-2}$: a $-|q+q^{-1}|$-random walk.

In fact, it is a $(q+q^{-1})$-random walk: the multiplicity of 1 in $J_{vv}^* J_{vv}$ is even because $J_{vv}^2 = -\mathrm{sgn}(q) 1$.

Conversely, if we have a $(q+q^{-1})$-random walk, we can construct from it solutions of $(*)$, so we have at our disposal a candidate for $\mathcal{F}(u_{1/2})_{vw} = \mathcal{H}_{vw}$ and $\mathcal{F}(R_{1/2}) = R$.

The point is that because of my compatibility conditions on the J_{vw} and a universal property of $\mathrm{Rep}(SU_q(2))$, this is sufficient to construct a strong tensor functor for $\mathrm{Rep}(SU_q(2)) \to \mathrm{Hilb}_{V \times V}$.

It remains to check that the two constructions are really inverse one of the other.

If $\mathcal{C}$ is a tensor-C^*-category and x an object in $\mathcal{C}$, R a morphism $1_\mathcal{C} \to x \otimes x$ s.t. $\left\{ \begin{array}{l} (R^* \otimes \mathrm{id})(\mathrm{id} \otimes R) = -(\mathrm{sgn}\, q)\, \mathrm{id} \\ R^* R = |q + q^{-1}| \end{array} \right.$, then there exists a (structured) tensor C^*-functor $\mathcal{F}$ such that $\begin{array}{l} \mathcal{F}(u_{1/2}) = x \\ \mathcal{F}(R_{1/2}) = R \end{array}$ and $\mathcal{F}$ is uniquely det. by this.

Proposition: Let Γ be a symmetric graph equipped with a T-reciprocal random walk, then $\|T\| \leq |T|$.

Proof: Consider $B: \ell^2(V) \longrightarrow \ell^2(E)$

$$\delta_v \longmapsto \sum_{s(e)=v} w(e)^{1/2} \delta_e \quad \text{where } w(e) = |T| \cdot p(e)$$

Also write $U: \ell^2 E \longrightarrow \ell^2 E$, $\delta_e \longmapsto \delta_{\bar{e}}$. Then

- $B^* B = |T| \, \mathrm{Id}$ (by random walk stuff)
- $B^* U B = M(\Gamma)$, adjacency matrix for Γ.

thus $\|T\| \leq |T|$.

If you are given a graph, and a weight, it is quite difficult to recover explicitly (A, α). There however is a C^*-algebra B with a coaction β by $C(SU_q(2))$ (not ergodic, and B not unital) but such that β is G-equivariantly Morita-equivalent with A (they have the same rep. theory) and such that B can be explicitly described in terms of generators and relations which are very similar to those for the $A_o(F)$-qg's.
[In fact, $B = K(\oplus_v \mathcal{E}_v)$, where we sum over all irreducible $SU_q(2)$-equivalent Hilbert moduls]
B is the universal C^*-algebra generated by a copy of the $*$-algebra of functions with finite support on V and elements $z_{e,j}$, $e \in E$, $j \in \{1,2\}$, and such that

(1) $\quad \underset{\substack{\downarrow \\ \text{Dirac in } v}}{\delta_v} z_{e,j} \delta_w = \delta_{v,s(e)} \delta_{w,t(e)} z_{e,j}$

(2) $\quad \sum'_{e,\, t(e)=w} z_{e,j}^* = \delta_{j,k} \delta_w$ for all $w \in V$

$$z_{e,1} z_{f,1}^* + z_{e,2} z_{f,2}^* = \delta_{e,f} \delta_{s(e)}$$

(3) Put $E_{e,f} = -\mathrm{sgn}(q) < J_{t(e),s(e)} \xi_j, \xi_i >$
(wrt chosen J for the graph and orthogonal base for the Hilbert spaces H_{vw})

then $\quad z_{e,j}^* = \sum'_{\substack{f \\ (s(f), t(f)) \\ (t(e), s(e))}} E_{ef} \left(F_{1j} z_{f1} + F_{2j} z_{f2} \right)$ where $F = \begin{pmatrix} 0 & \sqrt{|q|} \\ -\frac{\mathrm{sgn} q}{\sqrt{|q|}} & 0 \end{pmatrix}$. Then

$SU_q(2) \cong A_o(F)$. "a multivalued version of the $A_o(F)$ groups"