Fusion rules (cont'd)

François
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We continue the proof of the theorem begun on April, 9th.

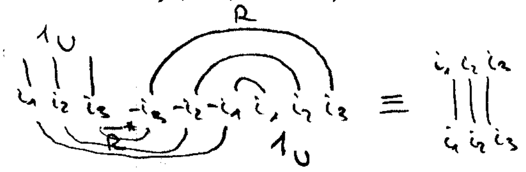
Consider the collection of spaces span $\{T_p : p \in NC_s(i_1, \dots, i_k; j_1, \dots, j_l)\}$.
They form a C^*-tensor category with duals:

- objects: $(i_1, \dots, i_k) \in \mathbb{Z}_s^k$, where $(i_1,\dots,i_k)\otimes(j_1,\dots,j_l) = (i_1,\dots,i_k,j_1,\dots,j_l)$
- morphisms: $T_p : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes l}$ with $p \in NC_s(i_1,\dots,i_k; j_1,\dots,j_l)$.
- conjugate:  $\in NC_s(i_1,\dots,i_k; (-i_k),\dots,(-i_1))$

$\overline{R} : 1 \longmapsto U \otimes \overline{U} \rightsquigarrow \overline{R} : \emptyset \longmapsto (i_1,\dots,i_k; (-i_k),\dots,(-i_1))$

$R : 1 \longmapsto \overline{U} \otimes U \rightsquigarrow R : \emptyset \longmapsto ((-i_k),\dots,(-i_1), i_1,\dots,i_k)$

$(\overline{R}^* \otimes 1_U)\circ(1_U \otimes R) = 1_U$:
$(1_U \otimes R^*)\circ(\overline{R}\otimes 1_U) = 1_U$



Frobenius reciprocity: $\mathrm{Hom}(U_{i_1}\otimes\dots\otimes U_{i_k}; U_{j_1}\otimes\dots\otimes U_{j_l}) \simeq \mathrm{Hom}(1; \overline{U_{i_k}\otimes\dots\otimes U_{i_1}}\otimes(U_{j_1}\otimes\dots\otimes U_{j_l}))$
$\simeq \mathrm{Hom}(1; U_{-i_k}\otimes\dots\otimes U_{-i_1}\otimes U_{j_1}\otimes\dots\otimes U_{j_l})$

$$\left\{\begin{matrix} i_1 \dots i_k \\ j_1,\dots,j_l\end{matrix}\right\} \in NC_s(i_1,\dots,i_k; j_1,\dots,j_l)$$

$$\sim \left\{\begin{matrix} \emptyset \\ (-i_k),\dots,(-i_1), j_1,\dots,j_l\end{matrix}\right\} \in NC_s(\emptyset; (-i_k),\dots(-i_1), j_1,\dots,j_l)$$

$\boxed{2}$: span$\{T_p : p \in NC_s(\,;\,)\} \subseteq \mathrm{Hom}(U_{i_1}\otimes\dots\otimes U_{i_k}; U_{j_1}\otimes\dots\otimes U_{j_l})$:
Frobenius reciprocity: we only prove the result for $k=0$; we can even restrict to the case of one block partitions $p = \{\overset{\emptyset}{\sqcap\!\sqcap\!\sqcap\!\sqcap}\}$. We have to prove that T_p; $p \in NC_s(\emptyset; j_1,\dots,j_l)$, intertwins 1 and $U_{j_1}\otimes\dots\otimes U_{j_l}$.

$$T_p : \mathbb{C} \longrightarrow (\mathbb{C}^N)^{\otimes l}$$
$$1 \longmapsto \sum_a{}' e_a^{\otimes l}$$

(recall that if $p \in NC(i_1,\dots,i_k; j_1,\dots,j_l)$,
$T_p : (\mathbb{C}^N)^{\otimes k} \longrightarrow (\mathbb{C}^N)^{\otimes l}$
$e_{a_1}\otimes\dots\otimes e_{a_k} \longmapsto \sum_{b_1\dots b_l} p(\underline{a},\underline{b})\, e_{b_1}\otimes\dots\otimes e_{b_l}$

where $p(\underline{a},\underline{b}) = \begin{cases} 1 \text{ if all the strings of } p \text{ link equal indices} \\ 0 \text{ otherwise}\end{cases}$

Consider $(U_{j_1}\otimes\dots\otimes U_{j_l})(1\otimes\sum_a{}' e_a^{\otimes l}) = \sum_{\substack{c_1,\dots,c_l\\ d_1,\dots,d_l}} U^{j_1}_{c_1d_1}\cdots U^{j_l}_{c_ld_l}\otimes e_{c_1d_1}\otimes\dots\otimes e_{c_ld_l}(1\otimes\sum_a{}' e_a^{\otimes l})$

where $e_{c_1d_1}(e_f) = \delta_{d_1,f}\, e_{c_1}$
$$= \sum_{\substack{c_1,\dots,c_l\\ a}}{}' U^{j_1}_{c_1a}\cdots U^{j_l}_{c_la}\otimes e_{c_1}\otimes\dots\otimes e_{c_l}$$

$$= \sum_{a,c} U_{ca}^{j_1}\cdots U_{ca}^{j_l} \otimes e_c^{\otimes l} = \sum_{a,c}{}' U_{ca}^{j_1+\dots+j_l} \otimes e_c^{\otimes l} = \sum_{a,c}{}' \overset{U_{ca}U_{ca}^*}{p_{ca}} \otimes e_c^{\otimes l}$$

As $\left\{\begin{smallmatrix}\emptyset \\ \sqcap\sqcap\sqcap\sqcap\sqcap \\ j_1 \;\cdots\; j_l\end{smallmatrix}\right\} \in NC_s(\emptyset, j_1,\dots,j_l)$, we have $j_1+\dots+j_l \equiv 0\,[s]$, $= \sum_c{}' 1 \otimes e_c^{\otimes l}$.

$\boxed{\subseteq}$: more involved: $Hom(U_{i_1}\otimes\dots\otimes U_{i_k}; U_{j_1}\otimes\dots\otimes U_{j_l}) \subseteq span\{T_p : p \in NC_s(i_1,\dots,i_k; j_1,\dots,j_l)\}$

It is enough to prove this for indices $\begin{cases} i_t = \pm 1 \\ j_r = \pm 1\end{cases}$

"lemma": consider $H_N^{\infty+}$ and assume you proved the result for this QG:

you have V_{ij} s.t. $Hom(V_{i_1}\otimes\dots\otimes V_{i_k}; V_{j_1}\otimes\dots\otimes V_{j_l}) = span\{T_p : p \in NC_\infty(i_1,\dots,i_k; j_1,\dots,j_l)\}$ (i.e. in each block, $\Sigma i = \Sigma j$)

Consider $\pi : C(H_N^{\infty+}) \longrightarrow C(H_N^{s+})$, $V_{ij} \longmapsto U_{ij}$ where $\ker\pi = \langle V_{ij}^s = V_{ij}V_{ij}^* \rangle$ (here we used universality)

These latter relations can be described by diagrams:

$V_{ij}^s = V_{ij}V_{ij}^* \iff \sum_j V_{ij}^{s+1}V_{ij}^* = 1$:

$\Rightarrow$: $\sum_j V_{ij}^{s+1}V_{ij}^* = \sum_j V_{ij}V_{ij}^*V_{ij}V_{ij}^* = \sum' p_{ij} = 1$

$\Leftarrow$: $\sum_j V_{ij}^{s+1}V_{ij}^*V_{ik}V_{ik}^* = V_{ik}V_{ik}^*$ as $ab = 0 \Rightarrow ab^* = 0$

s.t. $V_{ik}^{s+1}\underbrace{V_{ik}^*V_{ik}}_{p_{ik}}V_{ik}^* = V_{ik}V_{ik}^*$ and $V_{ik}^s p_{ik}^2 = p_{ik}$

and $V_{ik}^s = p_{ik}$.

Now note that $\sum_j{}' V_{ij}^{s+1}V_{ij}^* = 1$, this means $\left\{\begin{smallmatrix}\emptyset \\ \sqcap\sqcap\sqcap\sqcap\end{smallmatrix}\right\} \in Hom(1; V^{\otimes s+1}\otimes\bar V)$

In fact, $(V^{\otimes s+1}\otimes\bar V)(1\otimes\sum_a{}' e_a^{\otimes s+2}) = \sum_{\substack{i_1,\dots,i_{s+2}\\ j_1,\dots,j_{s+2}}}{}' V_{i_1j_1}\cdots V_{i_{s+1}j_{s+1}}^{s+2}V_{i_{s+2}j_{s+2}}^* \otimes e_{i_1j_1}\otimes\dots\otimes e_{i_{s+2}j_{s+2}}$ $(1\otimes\sum_a{}' e_a^{\otimes s+1})$

$$= \sum_{i_1,\dots,i_{s+2},a}{}' V_{i_1a}\cdots V_{i_{s+1}a}V_{i_{s+2}a}^* \otimes e_{i_1}\otimes\dots\otimes e_{i_{s+2}}$$

$$= \sum_{i,a}{}' V_{ia}^{s+1}V_{ia}^* \otimes e_i^{\otimes s+2} = \sum_i{}' \Big(\sum_a V_{ia}^{s+1}V_{ia}^*\Big) \otimes e_i^{\otimes s+2}$$

$= \sum_i{}' 1 \otimes e_i^{\otimes s+2}$ and this is equivalent to $\sum V_{ia}^{s+1}V_{ia}^* = 1$.

So the relations generating $\ker\pi$ are encoded by the fact that $p = \left\{\begin{smallmatrix}\emptyset \\ \sqcap\sqcap\sqcap\sqcap\sqcap \\ 1 \cdots 1\; -1\end{smallmatrix}\right\} \in NC_s(\emptyset, \underbrace{1,\dots,1}_{s+1}, -1)$

and $T_p \in Hom(1; V^{\otimes s+1}\otimes\bar V)$ and with what we know about the intertwiner space in $H_N^{\infty+}$ we get that the collection of spaces $Hom(U_{i_1}\otimes\dots\otimes U_{i_k}, U_{j_1}\otimes\dots\otimes U_{j_l})$ are generated by the diagrams in $NC_\infty(i_1,\dots,i_k; j_1,\dots,j_l)$ and p_0.

Q: Do we get in this way all the diagrams in $NC_s(i_1,\dots,i_k; j_1,\dots,j_l)$?

A: Yes, we have $Hom(U_{i_1}\otimes\dots\otimes U_{i_k}; U_{j_1}\otimes\dots\otimes U_{j_l}) = span\{T_p, p \in NC_s\binom{i_1,\dots,i_k}{j_1,\dots,j_l}\}$

- we have $NC_\infty \subset NC_s$
  $p_0 \in NC_s(\emptyset, \overbrace{1,\dots,1}^{s+1}, -1)$
- It remains to see that all diagrams in NC_s decompose as products of diagrams in NC_∞ and p_0.

$-(s=4)$ $\{\ \} \in NC_4(\emptyset; 1111)$

$\in NC_\infty(\emptyset; 1\,{-1}\,1\,{-1})$.

$-(s=3)$ $\{\ \} \in NC_3(0; 111)$

$p_0 = \{\ \}$ and $\{\ \} \in NC_\infty(1\,{-1}; 0)$.

We can check that $\{\ \} = \{\ \} = \{\ 1\} \circ p_0$

thus we get the result. But we still have to prove the lemma!
Where does the result on $H_N^{\infty+}$ come from?

Consider A_k: $X = (X_{ij})$, ${}^tX, X$ unitaries, $X_{ij}X_{ik}^* = X_{ji}X_{ki}^* = 0$ if $j \neq k$: this provides all the relations of $C(H_N^{\infty+})$ <u>except</u> the normality condition.

not a number

Thus $C(A_k) \xrightarrow{\pi_1} C(H_N^{\infty+})$ and $\ker \pi_1 = \langle X_{ij}X_{ij}^* = X_{ij}^* X_{ij} \rangle$

$X_{ij} \longmapsto V_{ij}$

You just have to assume that $\{H\} \in Hom(X \otimes \bar{X}; \bar{X} \otimes X)$

$C(U_N^+) \to C(A_k) \to C(H_N^{\infty+}) \to C(H_N^{s+})$

$Y, {}^tY$ unitaries. this may be an interesting QG, for Haagerup property for example.

<u>Corollary</u>: The basic corepresentations of H_N^{s+} are as follows:

(i) $U_1, \ldots, U_{s-1}$ are irreducible

(ii) $U_0 = (U_{ij} U_{ij}^*) = 1 \oplus \pi_0$, π_0 irreducible

(iii) $\pi_0, U_1, \ldots, U_{s-1}$ are all inequivalent.

<u>Proof</u>: $\underset{(t\neq 0)}{Hom}(U_t; U_t) = span\{T_p : p \in NC_s(t;t)\}$

$\{\ \}$ $NC_s(t,t')$ if $t=t'$ $\quad$ $\{\ \}$ $NC_s(t,t')$ if $t=t'=0$ $\quad$ and this implies that $\dim Hom(U_t, U_{t'}) = \# NC_s(t,t')$

$$= \begin{cases} 0 & \text{if } t \neq t' \quad (A) \\ 1 & \text{if } t = t' \neq 0 \quad (B) \\ 2 & \text{if } t = t' = 0 \quad (C) \end{cases}$$

Then $(A) \Rightarrow (iii)$

$(B) \Rightarrow (i)$ because $U_0 = (U_{ij}, U_{ij}^*) = 1 \oplus \pi_0$:

$(C) \Rightarrow (ii)$

$Hom(1, U_0) = span\{T_p : NC_s(\emptyset, 0)\}$

$\{\ \}$.

Now we want to take tensor products: fusion rules for H_N^{s+}

<u>Def</u>: The fusion semiring is $(R^+, \overline{}, \oplus, \otimes)$

(i) R^+ is the set of equivalent class of all corepresentations of H_N^{s+}

(ii) $\overline{}, \oplus, \otimes$ are the usual involution, direct sum, tensor product.

$$\underset{\text{irred. rep.}}{R_{irr}} \subset R^+ \subset \underset{\text{fusion ring.}}{R}$$

<u>Def</u>: Let $F = \langle \mathbb{Z}_s \rangle$ be the monoid of words over $\mathbb{Z}_s$

(i) involution : $\overline{(i_1, \dots, i_k)} = ((-i_k), \dots, (-i_1))$

(ii) fusion : $(i_1, \dots, i_k) \cdot (j_1, \dots, j_l) = i_1 \cdots i_{k-1} \underset{\text{mod. } s}{(i_k + j_1)} j_2 \cdots j_l$

<u>Theorem</u>: The irreducible corep of H_N^{s+} can be labeled r_x with $x \in \langle \mathbb{Z}_s \rangle$, involution $\overline{r_x} = r_{\bar{x}}$ and fusion rule : $r_x \otimes r_y = \sum'_{\substack{x = uz \\ y = \bar{z}v}} r_{uv} \oplus r_{u \cdot v}$

$\uparrow$ this term disappears if u or v is empty.

and $r_t = U_t \ominus \delta_{t,0} 1$ for $t = 0, \dots, s-1$

Ex.: $r_i \otimes r_j = \underset{\substack{x = i\emptyset \\ y = \emptyset j}}{r_{ij}} \oplus \overbrace{\underset{\substack{x = \emptyset i \\ y = i \text{ or } j\, \emptyset}}{r_{i+j}} \oplus 1}^{U_{i,j}}$

Proof: • consider $A = \{a_x : x \in \langle \mathbb{Z}_s \rangle\}$ abstract monoid with $a_x \cdot a_y = a_{xy}$

• $\mathbb{N}A$ = linear combinations of elements of A with coefficients in $\mathbb{N}$: this is a semiring : $a_x \otimes a_y = \sum'_{\substack{x = uz \\ y = \bar{z}w}} a_{xy} + a_{x \cdot y}$

• $\mathbb{Z}A$ a ring : the idea is to construct a morphism $\mathbb{Z}A$ and R, and prove that it restricts to an isomorphism at the level of A and R_{irr}.

Then $(\mathbb{Z}A, +, \cdot)$ is a free ring ; actually, $(\mathbb{Z}A, +, \otimes)$ also is a free ring.

There is a conversion formula : $a_{i_1} \otimes \cdots \otimes a_{i_k} = \sum_l \sum_{j_1, \dots, j_l} C_{i_1, \dots, i_k}^{j_1, \dots, j_l} a_{j_1, \dots, j_l}$

with coefficients $C \in \mathbb{N}$.

$a_{i_1 \cdots i_k} = \sum_l \sum'_{j_1, \dots, j_l} D_{i_1 \cdots i_k}^{j_1 \cdots j_l} a_{j_1} \otimes \cdots \otimes a_{j_l}$ with $D \in \mathbb{Z}$.

Th: We can define $\phi : \mathbb{Z}A \longrightarrow R$, $a_i \longmapsto r_i$ s.t. $\phi(a_{i_1 \dots i_n}) = r_{i_1 \dots i_n} = \sum_t \sum D^{j_1 \dots j_t}_{i_1 \dots i_n} r_{j_1} \otimes \dots \otimes r_{j_t}$

We prove that (i) ϕ is injective
(ii) $\phi(A) \subset R_{irr}$ (irreducible coreps of H_N^{s+})
(iii) $\phi : A \longrightarrow R_{irr}$ is surjective.

(i) You prove that ϕ "commutes" with the linear form that counts the numbers of 1 in x ($\in A$, $x \in F$): $\#(1 \in a_{i_1} \otimes \dots \otimes a_{i_n}) = \#(1 \in r_{i_1} \otimes \dots \otimes r_{i_n})$

iff $\#(1 \in (a_{i_1} + \delta_{i_1,0} 1) \otimes \dots \otimes (a_{i_n} + \delta_{i_n,0} 1)) = \#(1 \in U_{i_1} \otimes \dots \otimes U_{i_n})$
$= \# NC_s(\emptyset, i_1, \dots, i_n)$

by induction: $(a_i + \delta_{i,0}) \otimes (a_j + \delta_{j,0}) = a_i \otimes a_j + \dots$

Let $p = (a_i + \delta_{i,0} 1) \otimes (a_j + \delta_{j,0} 1) = a_{ij} + a_{i+j} + \delta_{i+j,0} 1 + \delta_{j,0} a_i + \delta_{i,0} a_j + \delta_{i,0} \delta_{j,0} 1$.

We get two copies of the trivial corepresentation in p, corresponding to $\{ \emptyset, | \, | \}$, $\{ \emptyset, \sqcap \}$

And $\phi(\alpha) = 0$ implies that $\phi(\alpha \otimes \alpha^*) = 0$, so that $\#(1 \in \phi(\alpha \otimes \alpha^*)) = 0$

thus $a_i \otimes a_j^* = a_{ij} + a_{i+j} + \delta_{i+j,0} 1$
and $\#(1 \in a_i \otimes a_j^*) = \delta_{i+j,0}$

$\#(1 \in \alpha \otimes \alpha^*) = 0$
$\alpha \otimes \alpha^* = 0$
$\alpha = 0$.

(ii) $\phi(A) \subset R_{irr}$
$\phi(r_t) \in R_{irr}$, $t = 0, \dots, s-1$: we prove that $\phi(a_{i_1 \dots i_n}) \in R_{irr}$ by induction.

(iii) $\phi : A \longrightarrow R_{irr}$ is surjective: $\phi(\mathbb{N}A)$ contains all the tensor products between U and $\overline{U}$: we can deduce that such morphism must be surjective.