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Strong solidity of free q.g. factors

Def: Let M be a vN algebra and $A \subset M$. A is Cartan if (i), $\exists E_A : M \to A$ finite normal cond. exp.; (ii), $A' \cap M = A$; (iii), $\mathcal{N}_{M}(A)'' = M$, where

$$\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$$

Ex: $\Gamma \curvearrowright (X,\mu)$ gives $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$. Then (i), (iii) are true, and (ii) $\Leftrightarrow$ the action $\Gamma \curvearrowright X$ free.

This is in fact the motivation for Cartan.

If M has no Cartan subalgebra, then $M \ncong L^\infty(X) \rtimes \Gamma$ for every free action.

$\to$ situation very far away from amenability.

Def: M is strongly solid if for all $A \subset M$ diffuse (having no minimal projection) amenable, $\mathcal{N}_{M(A)}''$ is amenable.

This implies that every $N \subset M$ nonamenable and diffuse has no Cartan

Th: Ozawa Popa 07: The $L\mathbb{F}_n$ are strongly solid.

Th A Popa-Vaes '12: If Γ is ICC nonamenable, not essential, weakly amenable and biexact, then $L\Gamma$ is strongly solid

Th A (Isono 2012) $\mathbb{G}$ CQG, $L^\infty(\mathbb{G})$ nonamenable II_1-factor, $\hat{\mathbb{G}}$ weakly amenable, biexact, then $L^\infty(\mathbb{G})$ is strongly solid.

Cor: $L^\infty(A_o(1_n))$ is strongly solid.
$A_u(1_n)$

Def: Weak amenability: Recall Γ is amenable iff $\exists \varphi_i : \Gamma \to \mathbb{C}$ of finite support and positive definite with $\varphi_i(g) \to 1$ for every $g \in \Gamma$, which yields

$$m_{\varphi_i} : \underset{\cap}{\overset{L\Gamma}{\mathbb{C}[\Gamma]}} \to \underset{\cap}{\overset{L\Gamma}{\mathbb{C}[\Gamma]}} \text{ c.b.} \quad \text{s.t. } m_{\varphi_i} \to \mathrm{id}_{L\Gamma} \text{ pointwise weakly}$$
$$\lambda_s \mapsto \varphi_i(s)\lambda_s$$

Weak amenability removes positive definite and adds $\|m_{\varphi_i}\|_{cb}$ uniformly bounded.

This is also called w^*CBAP for $L\Gamma$.

Biexactness: Γ is biexact if (i) Γ is exact (i.e. $C^*_\lambda\Gamma$ is exact) and

(ii) $\exists\, \vartheta : C^*_\lambda\Gamma \otimes_{min} C^*_\rho(\Gamma) \xrightarrow[ucp]{} B(\ell^2\Gamma)$ s.t.

$\vartheta(a\otimes b) - ab \in \mathbb{K}(\ell^2\Gamma)$ (for all $a\otimes b \in \otimes_{min}$)

$\to$ c.f. condition AO

For QG, biexact is easy to translate as the concepts are purely C^*-algebraic.

take $\Gamma = \widehat{\mathbb{G}}$ $\quad C^*_\lambda\Gamma = C_{red}(\mathbb{G})$, $C^*_\rho\Gamma = U C_{red}(\mathbb{G}) U$ (right GNS construction)

(in the non-Kac type, we would need Tomita-Takesaki theory here).

Proof of Theorem A (by Popa-Vaes). Setting: $M = L^\infty(\mathbb{G}) \supset C_{red}(\mathbb{G}) = A$, τ = Haar state.

$B \subset M$ diffuse amenable

$P := \mathcal{N}_M(B)''$, $H = L^2(\mathbb{G})$

Goal: P is amenable, that is, $\exists\, \varphi : B(H) \to \mathbb{C}$ state s.t. $\varphi(ax) = \varphi(xa)$ and $\varphi|_P = \tau$.

We say φ is a P-central state.

Let $\odot$ be the algebraic $\otimes$ and $D = M \odot M^{op} \odot P^{op} \odot P$

$\cup$

$D_0 := A \odot A^{op} \odot P^{op} \odot P$.

Then $\psi : D \longrightarrow B(H \otimes H \otimes L^2(P^{op}))$

$a\otimes b\otimes c\otimes d \longmapsto a\otimes b\otimes cd$

$\Theta : D \longrightarrow B(H \otimes L^2(P^{op}))$

$a\otimes b\otimes c\otimes d \longmapsto ab \otimes cd$

Step 1: Weak amenability: $\exists\, \xi_i \in H \otimes L^2(P^{op})$ "nice vectors"

Th. (Ozawa-Popa '07, Ozawa '10). For B amenable $\subset M$ weakly amenable,

there are $(\xi_i)_i \subset L^2(A\otimes A^{op})^+$ unit vectors s.t. (i) $\langle (x\otimes 1)\xi_i \mid \xi_i\rangle \to \tau(x)$ for $x \in M$,

(ii) $\|\xi_i - (a\otimes\bar{a})\xi_i\|_{2,\tau} \to 0$ for $a \in \mathcal{U}(B)$, (iii) $\|\xi_i - (u\otimes\bar{u})\xi_i(u\otimes\bar{u})^*\|_{2,\tau} \to 0$ for $u \in \mathcal{N}_M(B)$

Define $\Omega_1 : B(H\otimes L^2(P^{op})) \xrightarrow{state} \mathbb{C}$ s.t. $\Omega_1 := \lim_i \langle \cdot\, \xi_i \mid \xi_i\rangle$ (here $\bar{a} = (a^{op})^*$)

(i) and (iii) imply $\begin{cases} \text{(iv)}\ \Omega_1(x\otimes 1) = \tau(x) \text{ for } x\in M \\ \text{(v)}\ \Omega_1(\Theta(u\otimes\bar u\otimes\bar u\otimes u)) = 1 \text{ for } u\in \mathcal{U}_M(B). \end{cases}$

Step 2 are Popa's techniques, but we don't need them today.

Lemma: $\Omega_1(P\otimes 1) = 0$ for all $P\in K(H)$.

(∵) (ii) implies $\Omega_1(a\otimes\bar a) = 1 = \Omega_1(a^*\otimes\bar a^*)$ for $a\in\mathcal{U}(B)$ and

$\Omega_1((a\otimes\bar a)x) = \Omega_1(x) = \Omega_1(x(a\otimes\bar a)^*)$ for $a\in\mathcal{U}(B)$.

Take $u_n\in\mathcal{U}(B)$ s.t. $u_n\to 0$ strongly.

Then $\Omega_1(P\otimes 1) = \Omega_1((u_n\otimes\bar u_n)(P\otimes 1)(u_n\otimes\bar u_n)^*)$

$= \Omega_1(u_n P u_n^*\otimes 1)$

$= \Omega_1\Big(\underbrace{\frac{1}{k}\sum_{n=1}^{k} u_n P u_n^*}_{\to 0 \text{ in norm!}}\otimes 1\Big)$ [P is compact and $u_n\to 0$ strongly]

By the lemma: there are $P_j\in K(H)$ projections increasing to 1 with $\Omega_1(P_j\otimes 1) = 0$. (vi)

$\boxed{\text{Step 3}}$ Use biexactness: lemma: $\lim_i \|\Theta(S)(P_j^\perp\otimes 1)\| \le \|\psi(S)\|$ for $S\in D_0$.

(∵) Take $\vartheta: A\otimes A^{op}\xrightarrow{ucp} B(H)$ ($S = a\otimes b\otimes c\otimes d$) in biexactness.

$(\vartheta\otimes id)\psi(S) - \Theta(S) = (\underbrace{\vartheta(a\otimes b) - ab}_{K(H)})\otimes cd = x$.

Since $P_j^\perp\downarrow 0$ strongly, $X(P_j^\perp\otimes 1)\to 0$ in norm.

$\|\Theta(S)(P_j^\perp\otimes 1)\| \le \underbrace{\|X\|}_{\to 0} + \underbrace{\|(\vartheta\otimes id)\psi(S)(P_j^\perp\otimes 1)\|}_{\le \|\psi(S)\|}$

$\boxed{\text{Step 4}}$ Construction of a P-central state.

By (vi), $\Omega_1(x) = \Omega_1(x(P_j^\perp\otimes 1))$, $|\Omega_1(\Theta(S))| = \lim_i |\Omega_1(\Theta(S)(P_j^\perp\otimes 1))|$

$\le \|\psi(S)\|$ for $S\in D_0$

Claim: $|\Omega_1(\Theta(S))| \le C\|\psi(S)\|$ for all $S\in D$, for some C.

(∵) $\exists\, \varphi_i: M\to \underset{\text{not } M!}{A}$ c.b. normal, finite rank $\begin{cases}\varphi_i\to id_M \text{ point-strongly}\\ \sup_i \|\varphi_i\|\le k\end{cases}$ $A = C^*_{red}(G)\subset M = L^\infty(G)$.

Observe $\Omega_1(\Theta(S)) = \lim \Omega_1(\Theta(\varphi_i(a)\otimes\varphi_i^{op}(b)\otimes c\otimes d))$, ($S = a\otimes b\otimes c\otimes d\in D$)

[in fact, $|\Omega_1(x(\xi\otimes 1))|^2 \le \underbrace{\Omega_1(xx^*)}_{\|x\|^2}\underbrace{\Omega_1(\xi^*\xi\otimes 1)}_{\|\xi\|_{2,\tau}^2}$ for $x\in M$]

Then $|\Omega_1(\Theta(S))| = \lim_i |\Omega_1(\Theta(\underbrace{\varphi_i \otimes \varphi_i^{op} \otimes id \otimes id)(S)}_{Y_i \in D_0})|$

$\leq \overline{\lim_i} \|\underbrace{\psi(Y_i)}_{\overset{\shortparallel}{(\varphi_i \otimes \varphi_i^{op} \otimes id) \circ \psi(S)}}\|$

$\leq \overline{\lim_i} \underbrace{\|\varphi_i \otimes \varphi_i^{op} \otimes id\|_{cb}}_{\leq k^2} \cdot \|\psi(S)\|$.

Now let $\Omega_2 \circ \psi(S) := \Omega_1 \circ \Theta(S))$ for $S \in D$.

By the claim, Ω_2 is well-defined, bounded, positive, unital: it is a state!

Extend Ω_2 on $B H \otimes H \otimes L^2(P^{op})$ by the Hahn-Banach theorem.

(iv) and (v) become $\Omega_2(x \otimes 1 \otimes 1) = \tau(x)$ for $x \in M$,
$\Omega_2 \circ \psi(u \otimes \bar{u} \otimes \bar{u} \otimes u) = 1$ for $u \in \mathcal{N}_M(B)$.

<u>Claim</u> $\Omega_2|_{B(H) \otimes \mathbb{C} \otimes \mathbb{C}}$ is P-central.

(::) Let $U := \psi(u \otimes \bar{u} \otimes \bar{u} \otimes u)$ for $u \in \mathcal{N}_{M}(B)$, $\Omega_2(U) = \Omega_2(U^*) = 1$,
so that $\Omega_2(xU^*) = \Omega_2(x) = \Omega_2(Ux)$.

Let $x := \psi(1 \otimes u \otimes \bar{u} \otimes u)^*(\xi \otimes 1 \otimes 1)$ for $\xi \in B(H)$.

Then $\Omega_2(xU) = \Omega_2(Ux) = \boxed{\Omega_2(u\xi \otimes 1 \otimes 1)}$.
$\overset{\shortparallel}{\boxed{\Omega_2(\xi u \otimes 1 \otimes 1)}}$

N.B.: . Γ exact $\Leftarrow$ $\exists$ $B_\infty \subset \ell^\infty\Gamma / c_0\Gamma$ $\Gamma \curvearrowright B_\infty$ amenable and $B_\infty \cap \Gamma$ trivial

hyperbolic groups are a motivation of this definition.

. All $A_o(F)$, $A_u(F)$ are exact.

. $L^\infty(A_u(F))$ is always a factor.

$L^\infty(A_o(F))$: difficult to know whether it is a factor depending on F!

Classical case: $L^\infty(\Gamma)$ factor $\Leftarrow$ Γ is ICC. Then, if Γ is infinite, $L^\infty(\Gamma)$ is II_1! if Γ is amenable.