INDEPENDENCE AND LÉVY PROCESSES IN QUANTUM PROBABILITY

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Abstract. This manuscript provides an introduction of the notion of independence in quantum probability and the theory of quantum stochastic processes with independent and stationary increments.

Preliminary Version

Contents

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1. Introduction

Quantum probability is a generalization of both classical probability theory and quantum mechanics that allows to describe the probabilistic aspects of quantum mechanics. This generalization is formulated in two steps. First the theory is reformulated in terms algebras of functions on probability spaces. So the notion of a probability space $(\Omega, \mathcal{R}, P)$ are replaced by the pair $(L^\infty(\Omega), E(\cdot) = \int_\Omega \cdot \mathrm{d}P)$ consisting of the commutative von Neumann algebra of bounded random variables and the expectation functional. Then the commutativity condition is dropped. In this way we arrive at the notion of a (von Neumann) algebraic probability space (N, Φ) consisting of a von Neumann algebra N and a normal (faithful tracial) state Φ. As we have seen this includes classical probability spaces in the form $(L^\infty(\Omega), E)$, it also includes quantum mechanical systems modelled by a Hilbert space H and a pure state $\psi \in H$ (or a mixed state $\rho \in S(H)$), if we take $N = B(H)$ and Φ the state defined by $\Phi(X) = \langle \psi, X\psi \rangle$ (or $\Phi(X) = \mathrm{tr}(\rho X)$ for $X \in B(H)$. Note that in this course we shall relax the conditions on N and Φ and work with involutive algebras and positive normalised functionals, i.e. *-algebraic probability spaces.

A stricting feature of quantum probability (also called noncommutative probability) is the existence of several notions of independence. This is the starting point of this course, which intends to give an introduction to the theory of quantum stochastic processes with independent increments.

2. Infinite Divisibility in Classical Probability

Let us first recall some definitions and facts about infinite divisibility and Lévy processes in classical probability. See also [Sko91, Ber98, Sat99, App04, App05, Kyp07].

2.1. Stochastic Independence.

Recall that two random variables $X_1 : (\Omega, \mathcal{F}, P) \to (E_1, \mathcal{E}_1)$ and $X_2 : (\Omega, \mathcal{F}, P) \to (E_2, \mathcal{E}_2)$ are called *independent* if their joint law $P_{(X_1,X_2)}$ is equal to the product of their marginal laws, i.e.

$$P_{(X_1,X_2)} = P_{X_1} \otimes P_{X_2}.$$

This means that

$$P(X_1 \in A_1, X_2 \in A_2) = P(X_1 \in A_1)P(X_2 \in A_2)$$

for all $A_1 \in \mathcal{E}_1$,$A_2 \in \mathcal{E}_2$.

2.2. Convolution.

Let G be a topological semigroup with neutral element element e and multiplication $m : G \times G \to G$. Then we can define the *convolution product* $\mu_1 \star \mu_2$ of two probability measures μ_1, μ_2 as the image measure $m_*(\mu_1 \otimes \mu_2)$ of their product $\mu_1 \otimes \mu_2$, i.e.

$$(\mu_1 \star \mu_2)(A) = (\mu_1 \otimes \mu_2)(\{(g_1, g_2) \in G \times G; g_1 g_2 \in A\})$$

for $A \in \mathcal{B}(G)$.

If $X_1, X_2 : (\Omega, \mathcal{F}, P) \to (G, \mathcal{B}(G))$ are two independent random variables with distributions $P_{X_1} = \mu_1$, $P_{X_2} = \mu_2$, then their product has distribution

$$P_{X_1 X_2} = m_*(P_{X_1} \otimes P_{X_2}) = P_{X_1} \star P_{X_2} = \mu_1 \star \mu_2.$$

2.3. Infinite divisibility, continuous convolutions semigroups, and Lévy processes.

Definition 2.1. A probability measure μ on a topological semigroup G is called *infinitely divisible*, if for every integer $n \geq 1$ there exists a probability measure μ_n such that

$$\mu = \underbrace{\mu_n \star \cdots \star \mu_n}_{n \text{ times}}.$$

Definition 2.2. A family $(\mu_t)_{t\geq 0}$ of probability measure on a topological semigroup is called a *continuous convolution semigroup (ccs)* if

(i) $\lim_{t \searrow 0} \mu_t = \delta_e$ weakly, i.e. $\lim_{t \searrow 0} \int_G f \mu_t = f(e)$ for all $f \in C_b(G)$.
(ii) $\mu_s \star \mu_t = \mu_{s+t}$ for all $s, t \geq 0$.

Definition 2.3. A probability measure μ is called *embeddable into a continuous convolution semigroup* if there exists a continuous convolution semigroup $(\mu_t)_{t\geq 0}$ such that $\mu = \mu_1$.

Clearly, a probability measure that is embeddable into a continuous convolution semigroup is also infinitely divisible. On many groups, e.g., $(\mathbb{R}^d,+)$ the converse is also true, but there exist also groups where the converse does not hold.

Definition 2.4. A stochastic process $(X_{st})_{0\leq s\leq t}$ with values in a topological semigroup is called a (right) *Lévy process*, if

(i) (increment property) $X_{ss}=e$ and $X_{st}X_{tu}=X_{su}$ a.s. for all $0\leq s\leq t\leq u$;
(ii) (independence) the increments $X_{s_1t_1},\ldots,X_{s_nt_n}$ are independent for all $n\geq 1$ and all $s_1\leq t_1\leq s_2\leq\cdots\leq t_n$;
(iii) (stationarity) $P_{X_{st}}=P_{X_{s+h,t+h}}$ for all $h>0$ and all $0\leq s\leq t$, i.e. the law of X_{st} depends only on $t-s$;
(iv) (weak continuity) $(X_{st})_{0\leq s\leq t}$ is stochastically in probability, i.e. $X_{st}\overset{t\searrow s}{\longrightarrow} X_{ss}$ in probability.

We define $X_t=X_{0t}$. If G is a group, then the increments can be recovered from $(X_t)_{t\geq 0}$ by $X_{st}=X_{0s}^{-1}X_{0t}$.

A stochastic process $(X_t)_{t\geq 0}$ indexed by $\mathbb{R}_+$ and with values in a group is called a Lévy processes, if its increment processes $(X_{st})_{0\leq s\leq t}$ with $X_{st}=X_s^{-1}X_t$ is Lévy process in the sense of Definition 2.4.

Proposition 2.5. *If (X_{st}) is a Lévy process with values in a topological semigroup G, then its marginal distributions $\mu_t=P_{X_{0t}}$ form a continuous convolution semigroup.*

Exercise 2.6. Prove this Proposition.

Conversely, given a continuous convolutions semigroup $(\mu_t)_{t\geq 0}$ of probability measures on a topological semigroup G, one can construct a Lévy process with values in G whose marginals are equal to the convolution semigroup $(\mu_t)_{t\geq 0}$.

2.4. The De Finetti-Lévy-Khintchine formula on $(\mathbb{R}_+,+)$.

Let us start with a description of infinitely divisible probability measures on the semigroup $(\mathbb{R}_+,+)$.

Theorem 2.7. *A probability measure μ on $\mathbb{R}_+$ is infinitely divisible if and only if there exist $b\geq 0$ and ν a measure on $\mathbb{R}_+$ with $\int_0^\infty 1\wedge x\,\mathrm{d}\nu(x)<\infty$ such that the Laplace transform of μ has the form*

$$\psi_\mu(\lambda)=\int_0^\infty e^{-\lambda x}\mathrm{d}\mu(x)=\Phi(\lambda)$$

for all $\lambda\geq 0$, with

$$\Phi(\lambda)=b\lambda+\int_0^\infty(1-e^{-\lambda x})\mathrm{d}\nu(x).$$

The pair (b,ν) is uniquely determined by μ.

Proof. See [Ber98]. □

The pair (b,ν) is called the *characteristics* or the *characteristic pair of* μ.

Corollary 2.8. *Every infinitely divisible probability measure on* $\mathbb{R}_+$ *is embeddable into a ccs.*

2.5. **Lévy-Khintchine formulas on cones.** We also have the following generalisation for proper closed cones in finite-dimensional vector spaces.

Recall that a non-empty subset K in a real or complex vector space is called a cone if the following two conditions are satisfied,

(i) $x_1, x_2 \in K$ implies $x_1 + x_2 \in K$,
(ii) $\lambda \geq 0$, $x \in K$ implies $\lambda x \in K$.

A cone is called proper if $K \neq \{0\}$ and $K \cap (-K) = \{0\}$, i.e. K does not contain a straight line.

Theorem 2.9. [Sko91] *Let* $K \subset \mathbb{R}^d$ *be a proper closed cone and* μ *a probability measure on* K*. Then* μ *is infinitely divisible if and only if there exist* $b \in K$ *and* ν *a measure on* K *such that* $\int_K 1 \wedge ||x||\, \mathrm{d}\nu(x) < \infty$ *such that the Fourier transform of* μ *has the form*

$$\hat{\mu}(y) = \int_K e^{i\langle y,x\rangle}\mathrm{d}\mu(x) = \exp\left(i\langle y,b\rangle + \int_K (e^{i\langle y,x\rangle} - 1)\mathrm{d}\nu(x)\right)$$

for $y \in \mathbb{R}^d$.

In this case the Laplace transform is well-defined on the dual cone

$$K' = \{y \in \mathbb{R}^d : \langle y,x\rangle \geq 0 \forall x \in K\}$$

and has the form

$$\psi_\mu(y) = \exp\left(-\langle y,b\rangle - \int_K (1 - e^{-\langle y,x\rangle})\mathrm{d}\nu(x)\right)$$

for $y \in K'$.

2.6. **The Lévy-Khintchine formula on** $(\mathbb{R}^d,+)$**.**

Theorem 2.10. [Sat99, App04] *A probability measure* μ *on* $\mathbb{R}^d$ *is infinitely divisible if and only if its Fourier transform is of the form*

$$\hat{\mu}(u) = \int_{\mathbb{R}^d} e^{i\langle x,u\rangle}\mathrm{d}\mu(x) \tag{2.1}$$

$$\exp\left(i\langle b,u\rangle - \frac{1}{2}\langle u,Au\rangle + \int_{\mathbb{R}^d-\{0\}} (e^{i\langle u,y\rangle} - 1 - i\langle u,y\rangle \mathbf{1}_{||y||<1})\mathrm{d}\nu(y)\right),$$

for all $u \in \mathbb{R}^d$.

(b,A,ν) *are called the characteristics of* μ*, they are uniquely determined by* μ.

Corollary 2.11. *Any infinitely divisible probability measure on* $\mathbb{R}^d$ *is embeddable into a ccs.*

2.7. **The Markov semigroup of a Lévy processes.** Recall that Markov processes indexed by $\mathbb{R}_+$ can be characterized — intuitively — as stochastic processes $(X_t)_{t\geq 0}$ for which for any t the past and the future w.r.t. t are independent conditionally on X_t. Lévy processes are Markov processes, they are even Feller processes, i.e. their Markov semigroup maps $C_0(G)$ to itself. The independence of the increments allows for a simple description of their Markov semigroup $(T_t)_{t\geq 0}$,

$$(T_t f)(g) = E\big(f(gX_t)\big) = \int_G f(gg')\mathrm{d}P_{X_t}(g'), \qquad g \in G,$$

for $t \geq 0$, $f \in C_0(G)$.

2.8. **Hunt's formula.** Let G be a Lie group with Lie algebra $\mathfrak{g}$.

To a Lévy process in G we have its Markov semigroup $(T_t)_{t\geq 0}$ and its infinitesimal generator L. Hunt's theorem describes Lévy processes in G in terms of their generators.

Fix a basis $(X_j, 1 \leq j \leq n)$ of $\mathfrak{g}$ and define the dense linear manifold $C_2^L(G)$ by

$$C_2^L(G) = \{f \in C_0(G); X_i^L(f) \in C_0(G),\ X_i^L X_j^L(f) \in C_0(G) \text{ for all } 1 \leq i,j \leq n\}.$$

$C_2^L(G)$ is a Banach space with respect to the norm

$$||f||_{2,L} = ||f|| + \sum_{i=1}^n ||X_i^L f|| + \sum_{j,k=1}^n ||X_j^L X_k^L f||.$$

The space $C_2^R(G)$ and the norms $||\cdot||_{2,R}$ are defined similarly. Note that the smooth functions of compact support $C_c^\infty(G) \subseteq C_2^L(G) \cap C_2^R(G)$.

There exist functions $x_i \in C_c^\infty(G), 1 \leq i \leq n$ so that $(x_1, \ldots, x_n)$ are a system of canonical co-ordinates for G at e.

Theorem 2.12 (Hunt's theorem)**.** *Let X be a Lévy process in G with infinitesimal generator L then*

(1) $C_2^L(G) \subseteq Dom(L)$.
(2) *For each $g \in G, f \in C_2^L(G)$,*

$$\begin{aligned} Lf(g) &= b^i X_i^L f(g) + a^{ij} X_i^L X_j^L f(g) \\ &\quad + \int_{G-\{e\}} (f(gh) - f(g) - y^i(g) X_i^L f(g))\nu(\mathrm{d}h), \end{aligned}$$

where $b = (b^1, \ldots b^n) \in \mathbb{R}^n, a = (a^{ij})$ is a non-negative-definite, symmetric $n \times n$ real-valued matrix and ν is a Lévy measure on $G - \{e\}$.

Conversely, for any linear operator with such a representation there exists a Lévy process (unique up to stochastic equivalence) with generator L.

Michael Skeide has given a C^*-algebraic proof of Hunt's Theorem for compact Lie groups in [Ske99].

Exercise 2.13. Show how we can recover the Lévy-Khintchine formular for $\mathbb{R}^d$ from Hunt's Theorem.

3. Lévy Processes on Involutive Bialgebras

In this section we will give the definition of Lévy processes on involutive bialgebras, cf. Subsection 3.1, and develop their general theory.

In Subsection 3.2 we will begin to develop their basic theory. We will see that the marginal distributions of a Lévy process form a convolution semigroup of states and that we can associate a generator with a Lévy process on an involutive bialgebra, that characterizes uniquely its distribution, like in classical probability. By a GNS-type construction we can get a so-called Schürmann triple from the generator.

This Schürmann triple can be used to obtain a realization of the process on a symmetric Fock space. This realization can be found as the (unique) solution of a quantum stochastic differential equation. It establishes the one-to-one correspondence between Lévy processes, convolution semigroups of states, generators, and Schürmann triples. We will not present the representation theorem here, but refer to [Sch93, Chapter 2].

Finally, in Subsection 3.3, we look at several examples.

For more information on Lévy processes on involutive bialgebras, see also [Sch93][Mey95, Chapter VII][FS99], [Fra06].

3.1. **Definition of Lévy processes on involutive bialgebras.** A *quantum probability space* in the purely algebraic sense is a pair $(\mathcal{A}, \Phi)$ consisting of a unital $*$-algebra $\mathcal{A}$ and a state (i.e. a normalized positive linear functional) Φ on $\mathcal{A}$. Positivity in this purely algebraic context simply means $\Phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$. A *quantum random variable* j over a quantum probability space $(\mathcal{A}, \Phi)$ on a $*$-algebra $\mathcal{B}$ is simply a $*$-algebra homomorphism $j : \mathcal{B} \to \mathcal{A}$. A *quantum stochastic process* is an indexed family of random variables $(j_t)_{t\in I}$. For a quantum random variable $j : \mathcal{B} \to \mathcal{A}$ we will call $\varphi_j = \Phi \circ j$ its *distribution* in the state Φ. For a quantum stochastic process $(j_t)_{t\in I}$ the functionals $\varphi_t = \Phi \circ j_t : \mathcal{B} \to \mathbb{C}$ are called *marginal distributions*. The *joint distribution* $\Phi \circ \left(\coprod_{t\in I} j_t\right)$ of a quantum stochastic process is a functional on the free product $\coprod_{t\in I} \mathcal{B}$, see Section 6.

Two quantum stochastic processes $\left(j_t^{(1)} : \mathcal{B} \to \mathcal{A}_1\right)_{t\in I}$ and $\left(j_t^{(2)} : \mathcal{B} \to \mathcal{A}_2\right)_{t\in I}$ on $\mathcal{B}$ over $(\mathcal{A}_1, \Phi_1)$ and $(\mathcal{A}_2, \Phi_2)$ are called *equivalent*, if there joint distributions coincide. This is the case, if and only if all their moments agree, i.e. if

$$\Phi_1\left(j_{t_1}^{(1)}(b_1)\cdots j_{t_n}^{(1)}(b_n)\right) = \Phi_2\left(j_{t_1}^{(2)}(b_1)\cdots j_{t_n}^{(2)}(b_n)\right)$$

holds for all $n \in \mathbb{N}$, $t_1, \ldots, t_n \in I$ and all $b_1, \ldots, b_n \in \mathcal{B}$.

The term 'quantum stochastic process' is sometimes also used for an indexed family $(X_t)_{t\in I}$ of operators on a Hilbert space or more generally of elements of a quantum probability space. We will reserve the name *operator process* for this. An operator process $(X_t)_{t\in I}\subseteq\mathcal{A}$ (where $\mathcal{A}$ is a $*$-algebra of operators) always defines a quantum stochastic process $(j_t:\mathbb{C}\langle a,a^*\rangle\to\mathcal{A})_{t\in I}$ on the free $*$-algebra with one generator, if we set $j_t(a)=X_t$ and extend j_t as a $*$-algebra homomorphism. On the other hand operator processes can be obtained from quantum stochastic processes $(j_t:\mathcal{B}\to\mathcal{A})_{t\in I}$ by choosing an element x of the algebra $\mathcal{B}$ and setting $X_t=j_t(x)$.

The notion of independence we use for Lévy processes on involutive bialgebras is the so-called tensor or boson independence. In Section 6 we will see that other interesting notions of independence exist.

Definition 3.1. Let $(\mathcal{A},\Phi)$ be a quantum probability space and $\mathcal{B}$ a $*$-algebra. The quantum random variables $j_1,\dots,j_n:\mathcal{B}\to\mathcal{A}$ are called *tensor or Bose independent* (w.r.t. the state Φ), if

(i) $\Phi\big(j_1(b_1)\cdots j_n(b_n)\big)=\Phi\big(j_1(b_1)\big)\cdots\Phi\big(j_n(b_n)\big)$ for all $b_1,\dots,b_n\in\mathcal{B}$, and
(ii) $[j_l(b_1),j_k(b_2)]=0$ for all $k\neq l$ and all $b_1,b_2\in\mathcal{B}$.

Recall that an *involutive bialgebra* $(\mathcal{B},\Delta,\varepsilon)$ is a unital $*$-algebra $\mathcal{B}$ with two unital $*$-homomorphisms $\Delta:\mathcal{B}\to\mathcal{B}\otimes\mathcal{B}$, $\varepsilon:\mathcal{B}\to\mathbb{C}$ called *coproduct* or *comultiplication* and *counit*, satisfying

$$(\mathrm{id}\otimes\Delta)\circ\Delta=(\Delta\otimes\mathrm{id})\circ\Delta \qquad \text{(coassociativity)}$$
$$(\mathrm{id}\otimes\varepsilon)\circ\Delta=\mathrm{id}=(\varepsilon\otimes\mathrm{id})\circ\Delta \qquad \text{(counit property).}$$

Let $j_1,j_2:\mathcal{B}\to\mathcal{A}$ be two linear maps with values in some algebra $\mathcal{A}$, then we define their *convolution* $j_1\star j_2$ by

$$j_1\star j_2=m_{\mathcal{A}}\circ(j_1\otimes j_2)\circ\Delta.$$

Here $m_{\mathcal{A}}:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}$ denotes the multiplication of $\mathcal{A}$, $m(a\otimes b)=ab$ for $a,b\in\mathcal{A}$.

Using *Sweedler's notation* $\Delta(b)=b_{(1)}\otimes b_{(2)}$, this becomes $(j_1\star j_2)(b)=j_1(b_{(1)}j_2(b_{(2)})$. If j_1 and j_2 are two independent quantum random variables, then $j_1\star j_2$ is again a quantum random variable, i.e. a $*$-homomorphism. The fact that we can compose quantum random variables allows us to define Lévy process, i.e. processes with independent and stationary increments.

Example 3.2. Let (G,e) be a semigroup with identity e. We call a function $f:G\to\mathbb{C}$ *representative*, if there exists a finite-dimensional representation $\pi:G\to M_n$ and vectors $u,v\in\mathbb{C}^n$ such that $f(g)=\langle v,\pi(g)u\rangle$ for all $g\in G$. The algebra

$$\mathcal{R}(G)=\{f:G\to\mathbb{C}\text{ a respresentative function}\}$$

is an involutive bialgebra with pointwise multiplication and conjugation, and the coproduct $\Delta:\mathcal{R}(G)\to\mathcal{R}(G)\otimes\mathcal{R}(G)\cong\mathcal{R}(G\times G)$ and counit $\varepsilon:\mathcal{R}(G)\to\mathbb{C}$ defined by

$$\Delta f(g_1,g_2)=f(g_1g_2),$$

pour $g_1,g_2\in G$ and $\varepsilon(f)=f(e)$.

Definition 3.3. Let $\mathcal{B}$ be an involutive bialgebra. A quantum stochastic process $(j_{st})_{0\leq s\leq t}$ on $\mathcal{B}$ over some quantum probability space $(\mathcal{A},\Phi)$ is called a *Lévy process*, if the following four conditions are satisfied.

(1) (Increment property) We have

$$\begin{array}{rcll} j_{rs}\star j_{st} &=& j_{rt} & \text{for all } 0\leq r\leq s\leq t, \\ j_{tt} &=& \varepsilon\mathbf{1} & \text{for all } 0\leq t, \end{array}$$

i.e. $j_{tt}(b)=\varepsilon(b)\mathbf{1}$ for all $b\in\mathcal{B}$, where $\mathbf{1}$ denotes the unit of $\mathcal{A}$.

(2) (Independence of increments) The family $(j_{st})_{0\leq s\leq t}$ is independent, i.e. the quantum random variables $j_{s_1,t_1},\ldots,j_{s_nt_n}$ are independent for all $n\in\mathbb{N}$ and all $0\leq s_1\leq t_1\leq s_2\leq\cdots\leq t_n$.

(3) (Stationarity of increments) The distribution $\varphi_{st}=\Phi\circ j_{st}$ of j_{st} depends only on the difference $t-s$.

(4) (Weak continuity) The quantum random variables j_{st} converge to j_{ss} in distribution for $t\searrow s$.

Exercise 3.4. Recall that an *(involutive) Hopf algebra* $(\mathcal{B},\Delta,\varepsilon,S)$ is an (involutive) bialgebra $(\mathcal{B},\Delta,\varepsilon)$ equipped with a linear map called *antipode* $S:\mathcal{B}\to\mathcal{B}$ satisfying

$$S\star\mathrm{id}=\mathbf{1}\circ\varepsilon=\mathrm{id}\star S. \tag{3.1}$$

The antipode is unique, if it exists. Furthermore, it is an algebra and coalgebra anti-homomorphism, i.e. it satisfies $S(ab)=S(b)S(a)$ for all $a,b\in\mathcal{B}$ and $(S\otimes S)\circ\Delta=\tau\circ\Delta\circ S$, where $\tau:\mathcal{B}\otimes\mathcal{B}\to\mathcal{B}\otimes\mathcal{B}$ is the *flip* $\tau(a\otimes b)=b\otimes a$. If $(\mathcal{B},\Delta,\varepsilon)$ is an involutive bialgebra and $S:\mathcal{B}\to\mathcal{B}$ a linear map satisfying (3.1), then S satisfies also the relation

$$S\circ *\circ S\circ *=\mathrm{id}.$$

In particular, it follows that the antipode S of an involutive Hopf algebra is invertible. This is not true for Hopf algebras in general.

Show that if $(k_t)_{t\geq 0}$ is any quantum stochastic process on an involutive Hopf algebra, then the quantum stochastic process defined by

$$j_{st}=m_{\mathcal{A}}\circ\big((k_s\circ S)\otimes k_t\big)\circ\Delta,$$

for $0\leq s\leq t$, satisfies the increment property (1) in Definition 3.3. A one-parameter stochastic process $(k_t)_{t\geq 0}$ on a Hopf $*$-algebra H is called a *Lévy process on H*, if its increment process $(j_{st})_{0\leq s\leq t}$ with $j_{st}=\big(k_s\circ S)\otimes k_t\big)\circ\Delta$ is a Lévy process on H in the sense of Definition 3.3.

Let $(j_{st})_{0\leq s\leq t}$ be a Lévy process on some involutive bialgebra. We will denote the marginal distributions of $(j_{st})_{0\leq s\leq t}$ by $\varphi_{t-s}=\Phi\circ j_{st}$. Due to the stationarity of the increments this is well defined.

Lemma 3.5. *The marginal distributions $(\varphi_t)_{t\geq 0}$ of a Lévy process on an involutive bialgebra $\mathcal{B}$ form a* convolution semigroup of states *on $\mathcal{B}$, i.e. they satisfy*

(1) $\varphi_0 = \varepsilon$, $\varphi_s \star \varphi_t = \varphi_{s+t}$ *for all* $s, t \geq 0$, *and* $\lim_{t\searrow 0} \varphi_t(b) = \varepsilon(b)$ *for all* $b \in \mathcal{B}$, *and*
(2) $\varphi_t(\mathbf{1}) = 1$, *and* $\varphi_t(b^*b) \geq 0$ *for all* $t \geq 0$ *and all* $b \in \mathcal{B}$.

Proof. $\varphi_t = \Phi \circ j_{0t}$ is clearly a state, since j_{0t} is a $*$-homomorphism and Φ a state.

From the first condition in Definition 3.3 we get

$$\varphi_0 = \Phi \circ j_{00} = \Phi(1)\varepsilon = \varepsilon,$$

and

$$\varphi_{s+t}(b) = \Phi\big(j_{0,s+t}(b)\big) = \Phi\left(\sum j_{0s}(b_{(1)}) j_{s,s+t}(b_{(2)})\right),$$

for $b \in \mathcal{B}$, $\Delta(b) = \sum b_{(1)} \otimes b_{(2)}$. Using the independence of increments, we can factorize this and get

$$\begin{aligned}\varphi_{s+t}(b) &= \sum \Phi\big(j_{0s}(b_{(1)})\big)\Phi\big(j_{s,s+t}(b_{(2)})\big) = \sum \varphi_s(b_{(1)})\varphi_t(b_{(2)}) \\ &= \varphi_s \otimes \varphi_t\big(\Delta(b)\big) = \varphi_s \star \varphi_t(b)\end{aligned}$$

for all $\in \mathcal{B}$.

The continuity is an immediate consequence of the last condition in Definition 3.3. □ □

Lemma 3.6. *The convolution semigroup of states characterizes a Lévy process on an involutive bialgebra up to equivalence.*

Proof. This follows from the fact that the increment property and the independence of increments allow to express all joint moments in terms of the marginals. E.g., for $0 \leq s \leq t \leq u \leq v$ and $a, b, c \in \mathcal{B}$, the moment $\Phi\big(j_{su}(a)j_{st}(b)j_{sv}(c)\big)$ becomes

$$\begin{aligned}\Phi\big(j_{su}(a)j_{st}(b)j_{sv}(c)\big) &= \Phi\big((j_{st} \star j_{tu})(a)j_{st}(b)(j_{st} \star j_{tu} \star j_{uv})(c)\big) \\ &= \Phi\big(j_{st}(a_{(1)})j_{tu}(a_{(2)})j_{st}(b)j_{st}(c_{(1)})j_{tu}(c_{(2)})j_{uv}(c_{(3)})\big) \\ &= \Phi\big(j_{st}(a_{(1)}bc_{(1)})j_{tu}(a_{(2)}c_{(2)})j_{uv}(c_{(3)})\big) \\ &= \varphi_{t-s}(a_{(1)}bc_{(1)})\varphi_{u-t}(a_{(2)}c_{(2)})\varphi_{v-u}(c_{(3)}).\end{aligned}$$

□ □

It is possible to reconstruct process $(j_{st})_{0\leq s\leq t}$ from its convolution semigroup, see [Sch93, Section 1.9] or [FS99, Section 4.5]. Therefore, we even have a one-to-one correspondence between equivalence classes of Lévy processes on $\mathcal{B}$ and convolution semigroups of states on $\mathcal{B}$.

3.2. **The generator and the Schürmann triple of a Lévy process.** In this subsection we will meet two more objects that classify Lévy processes, namely their generator and their triple (called Schürmann triple by P.-A. Meyer, see [Mey95, Section VII.1.6]).

We begin with a technical lemma.

Lemma 3.7. **(a):** *Let $\psi : \mathcal{C} \to \mathbb{C}$ be a linear functional on some coalgebra $\mathcal{C}$. Then the series*

$$\exp_\star \psi(b) \stackrel{\text{def}}{=} \sum_{n=0} \frac{\psi^{\star n}}{n!}(b) = \varepsilon(b) + \psi(b) + \frac{1}{2}\psi \star \psi(b) + \cdots$$

converges for all $b \in \mathcal{C}$.

(b): *Let $(\varphi_t)_{t\geq 0}$ be a convolution semigroup on some coalgebra $\mathcal{C}$. Then the limit*

$$L(b) = \lim_{t \searrow 0} \frac{1}{t}\big(\varphi_t(b) - \varepsilon(b)\big)$$

exists for all $b \in \mathcal{C}$. Furthermore we have $\varphi_t = \exp_\star tL$ for all $t \geq 0$.

The proof of this lemma relies on the fundamental theorem of coalgebras, see [ASW88, Sch93].

Proposition 3.8. (Schoenberg correspondence) *Let $\mathcal{B}$ be an involutive bialgebra, $(\varphi_t)_{t\geq 0}$ a convolution semigroup of linear functionals on $\mathcal{B}$ and*

$$L = \lim_{t \searrow 0} \frac{1}{t}\big(\varphi_t - \varepsilon\big).$$

Then the following are equivalent.

(i): *$(\varphi_t)_{t\geq 0}$ is a convolution semigroup of states.*

(ii): *$L : \mathcal{B} \to \mathbb{C}$ satisfies $L(\mathbf{1}) = 0$, and it is hermitian and* conditionally positive*, i.e.*

$$L(b^*) = \overline{L(b)}$$

for all $b \in \mathcal{B}$, and

$$L(b^* b) \geq 0$$

for all $b \in \mathcal{B}$ with $\varepsilon(b) = 0$.

Proof. We prove only the (easy) direction (i)⇒(ii), the converse will follow from the representation theorem, which can be found in [Sch93, Chapter 2].

The first property follows by differentiating $\varphi_t(\mathbf{1}) = 1$ w.r.t. t.

Let $b \in \mathcal{B}$, $\varepsilon(b) = 0$. If all φ_t are states, then we have $\varphi_t(b^*b) \geq 0$ for all $t \geq 0$ and therefore

$$L(b^*b) = \lim_{t \searrow 0} \frac{1}{t}\big(\varphi_t(b^*b) - \varepsilon(b^*b)\big) = \lim_{t \searrow 0} \frac{\varphi_t(b^*b)}{t} \geq 0.$$

Similarly, L is hermitian, since all φ_t are hermitian. □ □

We will call a linear functional satisfying condition (ii) of the preceding Proposition a *generator*. Lemma 3.7 and Proposition 3.8 show that Lévy processes can also be characterized by their generator $L = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \varphi_t$.

Let D be a pre-Hilbert space. Then we denote by $\mathcal{L}(D)$ the set of all linear operators on D that have an adjoint defined everywhere on D, i.e.

$$\mathcal{L}(D) = \left\{ X : D \to D \text{ linear} \;\middle|\; \begin{array}{l} \text{there exists } X^* : D \to D \text{ linear s.t.} \\ \langle u, Xv\rangle = \langle X^*u, v\rangle \text{ for all } u, v \in D \end{array} \right\}.$$

$\mathcal{L}(D)$ is clearly a unital $*$-algebra.

Definition 3.9. Let $\mathcal{B}$ be a unital $*$-algebra equipped with a unital hermitian character $\varepsilon : \mathcal{B} \to \mathbb{C}$ (i.e. $\varepsilon(\mathbf{1}) = 1$, $\varepsilon(b^*) = \overline{\varepsilon(b)}$, and $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ for all $a, b \in \mathcal{B}$). A *Schürmann triple on* $(\mathcal{B}, \varepsilon)$ is a triple (ρ, η, L) consisting of

- a unital $*$-representation $\rho : \mathcal{B} \to \mathcal{L}(D)$ of $\mathcal{B}$ on some pre-Hilbert space D,
- a ρ-ε-1-*cocycle* $\eta : \mathcal{B} \to D$, i.e. a linear map $\eta : \mathcal{B} \to D$ such that

$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\varepsilon(b) \tag{3.2}$$

  for all $a, b \in \mathcal{B}$, and
- a hermitian linear functional $L : \mathcal{B} \to \mathbb{C}$ that has the bilinear map $\mathcal{B} \times \mathcal{B} \ni (a, b) \mapsto -\langle \eta(a^*), \eta(b)\rangle$ as a ε-ε-2-*coboundary*, i.e. that satisfies

$$-\langle \eta(a^*), \eta(b)\rangle = \partial L(a, b) = \varepsilon(a)L(b) - L(ab) + L(a)\varepsilon(b) \tag{3.3}$$

  for all $a, b \in \mathcal{B}$.

We will call a Schürmann triple *surjective*, if the cocycle $\eta : \mathcal{B} \to D$ is surjective.

Theorem 3.10. *Let $\mathcal{B}$ be an involutive bialgebra. We have one-to-one correspondences between Lévy processes on $\mathcal{B}$ (modulo equivalence), convolution semigroups of states on $\mathcal{B}$, generators on $\mathcal{B}$, and surjective Schürmann triples on $\mathcal{B}$ (modulo unitary equivalence).*

Proof. It only remains to establish the one-to-one correspondence between generators and Schürmann triples.

Let (ρ, η, L) be a Schürmann triple, then we can show that L is a generator, i.e. a hermitian, conditionally positive linear functional with $L(\mathbf{1}) = 0$.

The cocycle has to vanish on the unit element $\mathbf{1}$, since

$$\eta(\mathbf{1}) = \eta(\mathbf{1} \cdot \mathbf{1}) = \rho(\mathbf{1})\eta(\mathbf{1}) + \eta(\mathbf{1})\varepsilon(\mathbf{1}) = 2\eta(\mathbf{1}).$$

This implies

$$L(\mathbf{1}) = L(\mathbf{1} \cdot \mathbf{1}) = \varepsilon(\mathbf{1})L(\mathbf{1}) + \langle \eta(\mathbf{1}), \eta(\mathbf{1})\rangle + L(\mathbf{1})\varepsilon(\mathbf{1}) = 2L(\mathbf{1}) = 0.$$

Furthermore, L is hermitian by definition and conditionally positive, since by (3.3) we get

$$L(b^*b) = \langle \eta(b), \eta(b)\rangle = ||\eta(b)||^2 \geq 0$$

for $b \in \ker \varepsilon$.

Let now L be a generator. The sesqui-linear form $\langle \cdot, \cdot \rangle_L : \mathcal{B} \times \mathcal{B} \to \mathbb{C}$ defined by

$$\langle a, b\rangle_L = L\Big(\big(a - \varepsilon(a)\mathbf{1}\big)^*\big(b - \varepsilon(b)\mathbf{1}\big)\Big)$$

for $a, b \in \mathcal{B}$ is positive, since L is conditionally positive. Dividing $\mathcal{B}$ by the null-space

$$\mathcal{N}_L = \{a \in \mathcal{B} | \langle a, a\rangle_L = 0\}$$

we obtain a pre-Hilbert space $D = \mathcal{B}/\mathcal{N}_L$ with a positive definite inner product $\langle\cdot,\cdot\rangle$ induced by $\langle\cdot,\cdot\rangle_L$. For the cocycle $\eta : \mathcal{B} \to D$ we take the canonical projection, this is clearly surjective and satisfies Equation (3.3).

The $*$-representation ρ is induced from the left multiplication on $\mathcal{B}$ on $\ker\varepsilon$, i.e.

$$\rho(a)\eta\big(b-\varepsilon(b)\mathbf{1}\big) = \eta\Big(a\big(b-\varepsilon(b)\mathbf{1}\big)\Big) \quad \text{or} \quad \rho(a)\eta(b) = \eta(ab) - \eta(a)\varepsilon(b)$$

for $a,b \in \mathcal{B}$. To show that this is well-defined, we have to verify that left multiplication by elements of $\mathcal{B}$ leaves the null-space invariant. Let therefore $a,b \in \mathcal{B}$, $b \in \mathcal{N}_L$, then we have

$$\begin{aligned}
\Big|\Big|\Big(a\big(b-\varepsilon(b)\mathbf{1}\big)\Big)\Big|\Big|^2 &= L\Big(\big(ab-a\varepsilon(b)\mathbf{1}\big)^*\big(ab-a\varepsilon(b)\mathbf{1}\big)\Big) \\
&= L\Big(\big(b-\varepsilon(b)\mathbf{1}\big)^* a^*\big(ab-a\varepsilon(b)\mathbf{1}\big)\Big) \\
&= \big\langle b-\varepsilon(b)\mathbf{1}, a^*a\big(b-\varepsilon(b)\mathbf{1}\big)\big\rangle_L \\
&\leq ||b-\varepsilon(b)\mathbf{1}||^2 \; \big|\big|a^*a\big(b-\varepsilon(b)\mathbf{1}\big|\big|^2 = 0,
\end{aligned}$$

with Schwarz' inequality.

That the Schürmann triple (ρ,η,L) obtained in this way is unique up to unitary equivalence follows similarly as for the usual GNS construction. □ □

Exercise 3.11. Let $(X_t)_{t\geq 0}$ be a classical real-valued Lévy process with all moments finite (on some probability space $(\Omega,\mathcal{F},P)$). Define a Lévy process on the free unital algebra $\mathbb{C}[x]$ generated by one symmetric element $x = x^*$ with the coproduct and counit determined by $\Delta(x) = x\otimes\mathbf{1} + \mathbf{1}\otimes x$ and $\varepsilon(x) = 0$, whose moments agree with those of $(X_t)_{t\geq 0}$. More precisely, such that

$$\Phi\big(j_{st}(x^k)\big) = \mathbb{E}\left((X_t - X_s)^k\right)$$

holds for all $k \in \mathbb{N}$ and all $0 \leq s \leq t$.

Construct the Schürmann triple for Brownian motion and for a compound Poisson process (with finite moments).

For the classification of Gaussian and drift generators on an involutive bialgebra $\mathcal{B}$ with counit ε, we need the ideals

$$\begin{aligned}
K &= \ker\varepsilon, \\
K^2 &= \operatorname{span}\{ab | a,b \in K\}, \\
K^3 &= \operatorname{span}\{abc | a,b,c \in K\}.
\end{aligned}$$

Proposition 3.12. *Let L be a conditionally positive, hermitian linear functional on $\mathcal{B}$. Then the following are equivalent.*

(i): $\eta = 0$*,*
(ii): $L|_{K^2} = 0$*,*
(iii): *L is an ε-derivation, i.e. $L(ab) = \varepsilon(a)L(b) + L(a)\varepsilon(b)$ for all $a,b \in \mathcal{B}$,*

(iv): *The states* φ_t *are homomorphisms, i.e.* $\varphi_t(ab) = \varphi_t(a)\varphi_t(b)$ *for all* $a, b \in \mathcal{B}$ *and* $t \geq 0$.

If a conditionally positive, hermitian linear functional L satisfies one of these conditions, then we call it and the associated Lévy process a *drift*.

Proposition 3.13. *Let* L *be a conditionally positive, hermitian linear functional on* $\mathcal{B}$.

Then the following are equivalent.

(i): $L|_{K^3} = 0$,

(ii): $L(b^*b) = 0$ *for all* $b \in K^2$,

(iii): $L(abc) = L(ab)\varepsilon(c) + L(ac)\varepsilon(b) + L(bc)\varepsilon(a) - \varepsilon(ab)L(c) - \varepsilon(ac)L(b) - \varepsilon(bc)L(a)$ *for all* $a, b, c \in \mathcal{B}$,

(iv): $\rho|_K = 0$ *for the representation* ρ *in the surjective Schürmann triple* (ρ, η, L) *associated to* L *by the GNS-type construction presented in the proof of Theorem 3.10,*

(v): $\rho = \varepsilon\mathbf{1}$, *for the representation* ρ *in the surjective Schürmann triple* (ρ, η, L) *associated to* L *by the GNS-type construction presented in the proof of Theorem 3.10,*

(vi): $\eta|_{K^2} = 0$ *for the cocycle* η *in any Schürmann triple* (ρ, η, L) *containing* L,

(vii): $\eta(ab) = \varepsilon(a)\eta(b) + \eta(a)\varepsilon(b)$ *for all* $a, b \in \mathcal{B}$ *and the cocycle* η *in any Schürmann triple* (ρ, η, L) *containing* L.

If a conditionally positive, hermitian linear functional L satisfies one of these conditions, then we call it and also the associated Lévy process *quadratic* or *Gaussian*.

The proofs of the preceding two propositions can be carried out as an exercise or found in [Sch93, Section 5.1].

Proposition 3.14. *Let* L *be a conditionally positive, hermitian linear functional on* $\mathcal{B}$. *Then the following are equivalent.*

(i): *There exists a state* $\varphi : \mathcal{B} \to \mathbb{C}$ *and a real number* $\lambda > 0$ *such that*

$$L(b) = \lambda\big(\varphi(b) - \varepsilon(b)\big)$$

for all $b \in \mathcal{B}$.

(ii): *There exists a Schürmann triple* (ρ, η, L) *containing* L, *in which the cocycle* η *is trivial, i.e. of the form*

$$\eta(b) = \big(\rho(b) - \varepsilon(b)\big)\omega, \qquad \text{for all } b \in \mathcal{B},$$

for some non-zero vector $\omega \in D$. *In this case we will also call* η *the* coboundary *of the vector* ω.

If a conditionally positive, hermitian linear functional L satisfies one of these conditions, then we call it a *Poisson generator* and the associated Lévy process a *compound Poisson process*.

Proof. To show that (ii) implies (i), set $\varphi(b) = \frac{\langle\omega,\rho(b)\omega\rangle}{\langle\omega,\omega\rangle}$ and $\lambda = ||\omega||^2$.

For the converse, let (D,ρ,ω) be the GNS triple for $(\mathcal{B},\varphi)$ and check that (ρ,η,L) with $\eta(b) = \big(\rho(b)-\varepsilon(b)\big)\omega$, $b\in\mathcal{B}$ defines a Schürmann triple. □ □

Remark 3.15. The Schürmann triple for a Poisson generator $L = \lambda(\varphi-\varepsilon)$ obtained by the GNS construction for φ is not necessarily surjective. Consider, e.g., a classical additive $\mathbb{R}$-valued compound Poisson process, whose Lévy measure μ is not supported on a finite set. Then the construction of a surjective Schürmann triple in the proof of Theorem 3.10 gives the pre-Hilbert space $D_0 = \operatorname{span}\{x^k|k = 1,2,\ldots\} \subseteq L^2(\mathbb{R},\mu)$. On the other hand, the GNS-construction for φ leads to the pre-Hilbert space $D = \operatorname{span}\{x^k|k = 0,1,2,\ldots\} \subseteq L^2(\mathbb{R},\mu)$. The cocycle η is the coboundary of the constant function, which is not contained in D_0.

3.3. **Examples.**

3.3.1. *Lévy processes on the circle* $\mathbb{T}$ *and on the real line* $\mathbb{R}$. Consider the involutive bialgebra

$$\mathcal{B} = \operatorname{span}\{e_\lambda;\lambda\in\mathbb{R}\}$$

with the multiplication

$$e_\lambda\cdot e_\mu = e_{\lambda+\mu}, \qquad \lambda,\mu\in\mathbb{R},$$

involution $e_\lambda^* = e_{-\lambda}$ for $\lambda\in\mathbb{R}$, coproduct

$$\Delta(e_\lambda) = e_\lambda\otimes e_\lambda$$

and counit $\varepsilon(e_\lambda) = 1$ for all $\lambda,\mu\in\mathbb{R}$. Consider also the subalgebras

$$\mathcal{B}_\alpha = \operatorname{span}\{e_{k\alpha};k\in\mathbb{Z}\}$$

for $\alpha > 0$.

The basis elements e_λ can be represented as exponential functions $e_\lambda:\mathbb{R}\ni x\to e^{i\lambda x}\in\mathbb{C}$, since this representation of $\mathcal{B}$ is faithful, we can view $\mathcal{B}$ as a subalgebra of $\mathcal{R}(\mathbb{R})$.

The subalgebra $\mathcal{B}_\alpha$ is generated as a *-algebra by one unitary element e_α, it is therefore commutative and isomphic to the algebra of polynomials on the unit circle $\mathbb{T}$.

3.3.2. *Additive Lévy processes.* For a vector space V the *tensor algebra* $\mathcal{T}(V)$ is the vector space

$$\mathcal{T}(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n},$$

where $V^{\otimes n}$ denotes the n-fold tensor product of V with itself, $V^{\otimes 0} = \mathbb{C}$, with the multiplication given by

$$(v_1\otimes\cdots\otimes v_n)(w_1\otimes\cdots\otimes w_m) = v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_m,$$

for $n,m\in\mathbb{N}$, $v_1,\ldots,v_n,w_1,\ldots,w_m\in V$. The elements of $\bigcup_{n\in\mathbb{N}} V^{\otimes n}$ are called homogeneous, and the degree of a homogeneous element $a\neq 0$ is n if $a\in V^{\otimes n}$.

If $\{v_i | i \in I\}$ is a basis of V, then the tensor algebra $\mathcal{T}(V)$ can be viewed as the free algebra generated by v_i, $i \in I$. The tensor algebra can be characterized by the following universal property.

There exists an embedding $\imath : V \to \mathcal{T}(V)$ of V into $\mathcal{T}(V)$ such that for any linear mapping $R : V \to \mathcal{A}$ from V into an algebra there exists a unique algebra homomorphism $\mathcal{T}(R) : \mathcal{T}(V) \to \mathcal{A}$ such that the following diagram commutes,

$$\begin{tikzcd} V \arrow[r, "R"] \arrow[d, "\imath"'] & \mathcal{A} \\ \mathcal{T}(V) \arrow[ur, "\mathcal{T}(R)"'] & \end{tikzcd}$$

i.e. $\mathcal{T}(R) \circ \imath = R$.

Conversely, any algebra homomorphism $Q : \mathcal{T}(V) \to \mathcal{A}$ is uniquely determined by its restriction to V.

In a similar way, an involution on V gives rise to a unique extension as an involution on $\mathcal{T}(V)$. Thus for a $*$-vector space V we can form the tensor $*$-algebra $\mathcal{T}(V)$. The tensor $*$-algebra $\mathcal{T}(V)$ becomes a $*$-bialgebra, if we extend the linear $*$-maps

$$\begin{array}{rclrcl} \varepsilon & : & V \to \mathbb{C}, & \varepsilon(v) & = & 0, \\ \Delta & : & V \to \mathcal{T}(V) \otimes \mathcal{T}(V), & \Delta(v) & = & v \otimes \mathbf{1} + \mathbf{1} \otimes v, \end{array}$$

as $*$-homomorphisms to $\mathcal{T}(V)$. We will denote the coproduct $\mathcal{T}(\Delta)$ and the counit $\mathcal{T}(\varepsilon)$ again by Δ and ε. The tensor $*$-algebra is even a Hopf $*$-algebra with the antipode defined by $S(v) = -v$ on the generators and extended as an anti-homomorphism.

We will now study Lévy processes on $\mathcal{T}(V)$. Let D be a pre-Hilbert space and suppose we are given

(1) a linear $*$-map $R : V \to \mathcal{L}(D)$,
(2) a linear map $N : V \to D$, and
(3) a linear $*$-map $\psi : V \to \mathbb{C}$ (i.e. a hermitian linear functional),

then

$$J_t(v) = \Lambda_t\big(R(v)\big) + A_t^*\big(N(v)\big) + A_t\big(N(v^*)\big) + t\psi(v) \tag{3.4}$$

for $v \in V$ extends to a Lévy process $(j_t)_{t\geq 0}$, $j_t = \mathcal{T}(J_t)$, on $\mathcal{T}(V)$ (w.r.t. the vacuum state).

In fact, and as we shall prove in the following two exercises, all Lévy processes on $\mathcal{T}(V)$ are of this form, cf. [Sch91b].

Exercise 3.16. Show that (R, N, ψ) can be extended to a Schürmann triple on $\mathcal{T}(V)$ as follows

(1) Set $\rho = \mathcal{T}(R)$.
(2) Define $\eta : \mathcal{T}(V) \to D$ by $\eta(\mathbf{1}) = 0$, $\eta(v) = N(v)$ for $v \in V$, and

$$\eta(v_1 \otimes \cdots \otimes v_n) = R(v_1) \cdots R(v_{n-1}) N(v_n)$$

for homogeneous elements $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$, $n \geq 2$.

(3) Finally, define $L : \mathcal{T}(V) \to \mathbb{C}$ by $L(\mathbf{1}) = 0$, $L(v) = \psi(v)$ for $v \in V$, and

$$L(v_1 \otimes \cdots \otimes v_n) = \begin{cases} \big\langle N(v_1^*), N(v_2)\big\rangle & \text{if } n = 2, \\ \big\langle N(v_1^*), R(v_2) \cdots R(v_{n-1}) N(v_n)\big\rangle & \text{if } n \geq 3, \end{cases}$$

for homogeneous elements $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$, $n \geq 2$.

Prove furthermore that all Schürmann triples of $\mathcal{T}(V)$ are of this form.

Exercise 3.17. Let (ρ, η, L) be a Schürmann triple on $\mathcal{T}(V)$. Write down the corresponding quantum stochastic differential equation for homogeneous elements $v \in V$ of degree 1 and show that its solution is given by (3.4).

3.3.3. *Lévy processes on finite semigroups.*

Exercise 3.18. Let $(G, \cdot, e)$ be a finite semigroup with unit element e. Then the complex-valued functions $\mathcal{F}(G)$ on G form an involutive bialgebra. The algebra structure and the involution are given by pointwise multiplication and complex conjugation. The coproduct and counit are defined by

$$\begin{aligned} \Delta(f)(g_1, g_2) &= f(g_1 \cdot g_2) \qquad \text{for } g_1, g_2 \in G, \\ \varepsilon(f) &= f(e), \end{aligned}$$

for $f \in \mathcal{F}(G)$.

Show that the classical Lévy processes in G are in one-to-one correspondence to the Lévy processes on the $*$-bialgebra $\mathcal{F}(G)$.

4. Lévy Processes on Compact Quantum Groups and their Markov Semigroups

See [CFK12]

4.1. **Compact Quantum Groups.** The notion of compact quantum groups has been introduced in [Wor87a]. Here we adopt the definition from [Wor98] (Definition 1.1 of that paper).

Definition 4.1. A *C*-bialgebra* (a compact quantum semigroup) is a pair (A, Δ), where A is a unital C*-algebra, $\Delta : \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ is a unital, $*$-homomorphic map which is coassociative, i.e.

$$(\Delta \otimes \mathrm{id}_{\mathsf{A}}) \circ \Delta = (\mathrm{id}_{\mathsf{A}} \otimes \Delta) \circ \Delta.$$

If the quantum cancellation properties

$$\overline{\mathrm{Lin}}((1 \otimes \mathsf{A})\Delta(\mathsf{A})) = \overline{\mathrm{Lin}}((\mathsf{A} \otimes 1)\Delta(\mathsf{A})) = \mathsf{A} \otimes \mathsf{A},$$

are satisfied, then the pair (A, Δ) is called a *compact quantum group* (CQG).

If the algebra A of a compact quantum group is commutative, then A is isomorphic to the algebra $C(G)$ of continuous functions on a compact group G. To emphasise that for an arbitrary (i.e. not necessarily non-commutative) compact quantum group (A,Δ) the algebra A replaces the algebra of continuous functions on an (abstract) quantum analog of a group, the notation $\mathbb{G}=(\mathsf{A},\Delta)$ and $\mathsf{A}=C(\mathbb{G})$ is also frequently used.

The map Δ is called the *coproduct* of A and it induces the convolution product of functionals

$$\lambda\star\mu:=(\lambda\otimes\mu)\circ\Delta,\quad \lambda,\mu\in\mathsf{A}'.$$

The following fact is of the fundamental importance, cf. [Wor98, Theorem 2.3].

Proposition 4.2. *Let* A *be a compact quantum group. There exists a unique state* $h\in\mathsf{A}'$ *(called the* Haar state *of* A*) such that for all* $a\in\mathsf{A}$

$$(h\otimes\mathrm{id}_{\mathsf{A}})\circ\Delta(a)=(\mathrm{id}_{\mathsf{A}}\otimes h)\circ\Delta(a)=h(a)1.$$

In general, the Haar state of a compact quantum group need not be faithful or tracial.

4.1.1. *Corepresentations.* An element $u=(u_{jk})_{1\leq j,k\leq n}\in M_n(\mathsf{A})$ is called an *n-dimensional corepresentation of* $\mathbb{G}=(\mathsf{A},\Delta)$ if for all $j,k=1,\ldots,n$ we have $\Delta(u_{jk})=\sum_{p=1}^n u_{jp}\otimes u_{pk}$. All corepresentations considered in this paper are supposed to be finite-dimensional. A corepresentation u is said to be *non-degenerate*, if u is invertible, *unitary*, if u is unitary, and *irreducible*, if the only matrices $T\in M_n(\mathbb{C})$ with $Tu=uT$ are multiples of the identity matrix. Two corepresentations $u,v\in M_n(\mathsf{A})$ are called *equivalent*, if there exists an invertible matrix $U\in M_n(\mathbb{C})$ such that $Uu=vU$.

An important feature of compact quantum groups is the existence of the dense $*$-subalgebra $\mathcal{A}$ (the algebra of the *polynomials* of A), which is in fact a Hopf $*$-algebra – so for example $\Delta:\mathcal{A}\to\mathcal{A}\odot\mathcal{A}$. With the notation $\mathbb{G}=(\mathsf{A},\Delta)$, we often refer to $\mathcal{A}$ as to $\mathrm{Pol}(\mathbb{G})$.

Fix a complete family $(u^{(s)})_{s\in\mathcal{I}}$ of mutually inequivalent irreducible unitary corepresentations of A, then $\{u^{(s)}_{k\ell};s\in\mathcal{I},1\leq k,\ell\leq n_s\}$ (where n_s denotes the dimension of $u^{(s)}$) is a linear basis of $\mathcal{A}$, cf. [Wor98, Proposition 5.1]. We shall reserve the index $s=0$ for the trivial corepresentation $u^{(0)}=\mathbf{1}$. The Hopf algebra structure on $\mathcal{A}$ is defined by

$$\varepsilon(u_{jk})=\delta_{jk},\quad S(u_{jk})=(u_{kj})^*\quad\text{for } j,k=1,\ldots,n_s,$$

where $\varepsilon:\mathcal{A}\to\mathbb{C}$ is the counit and $S:\mathcal{A}\to\mathcal{A}$ is the antipode. They satisfy

$$(\mathrm{id}\otimes\varepsilon)\circ\Delta = \mathrm{id}=(\varepsilon\otimes\mathrm{id})\circ\Delta, \tag{4.1}$$

$$m_{\mathcal{A}}\circ(\mathrm{id}\otimes S)\circ\Delta = \varepsilon(a)frm[o]--=m_{\mathcal{A}}\circ(S\otimes\mathrm{id})\circ\Delta, \tag{4.2}$$

$$\big(S(a^*)^*\big) = a \tag{4.3}$$

for all $a\in\mathcal{A}$. Let us also remind that the Haar state is always faithful on $\mathcal{A}$.

Set $V_s = \operatorname{span}\{u^{(s)}_{jk}; 1 \le j,k \le n_s\}$ for $s \in \mathcal{I}$. By [Wor98, Proposition 5.2], there exists an irreducible unitary corepresentation $u^{(s^c)}$, called the *contragredient* representation of $u^{(s)}$, such that $V_s^* = V_{s^c}$. Clearly $(s^c)^c = s$.

We shall frequently use *Sweedler notation* for the coproduct of an element $a \in \mathcal{A}$, i.e. omit the summation and the index in the formula $\Delta(a) = \sum_i a_{(1),i} \otimes a_{(2),i}$ and write simply $\Delta(a) = a_{(1)} \otimes a_{(2)}$.

4.1.2. *The dual discrete quantum group.* To every compact quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$ there exists a dual discrete quantum group $\hat{\mathbb{G}}$, cf. [PW90]. For our purposes it will be most convenient to introduce $\hat{\mathbb{G}}$ in the setting of van Daele's algebraic quantum groups, cf. [VD98, VD03]. However, the reader should be aware that we adopt a slightly different convention for the Fourier transform.

A pair (A, Δ), consisting of a $*$-algebra A (with or without identity) and a coassociative comultiplication $\Delta : A \to M(A \otimes A)$, is called an *algebraic quantum group* if the product is non-degenerated (i.e. $ab = 0$ for all a implies $b = 0$), if the two operators $T_1 : A \odot A \ni a \otimes b \mapsto \Delta(a)(b \otimes 1) \in A \otimes A$ and $T_1 : A \odot A \ni a \otimes b \mapsto \Delta(a)(1 \otimes b) \in A \otimes A$ are well-defined bijections and if there exists a nonzero left integrals positive functional on A. Here, $M(B)$ denotes the set of multipliers on B. We refer the reader to [VD98] for details.

If (A, Δ) is a compact quantum group then $(\mathcal{A}, \Delta|_{\mathcal{A}})$ is an algebraic quantum group (of compact type) and the Haar state is a faithful left and right integral.

For $a \in \mathcal{A}$ we can define $h_a \in \mathcal{A}'$ by the formula

$$h_a(b) = h(ab) \quad \text{for } b \in \mathcal{A},$$

where h is the Haar state, and we denote by $\hat{\mathcal{A}}$ the space of linear functionals on $\mathcal{A}$ of the form h_a for $a \in \mathcal{A}$.

The set $\hat{\mathcal{A}}$ becomes an associative $*$-algebra with the convolution of functionals as the multiplication: $\lambda \star \mu = (\lambda \otimes \mu) \circ \Delta$, and the involution $\lambda^*(x) = \overline{\lambda(S(x)^*)}$ $(\lambda, \mu \in \hat{\mathcal{A}})$. The Hopf structure is given as follows: the coproduct $\hat{\Delta}$ is the dual of the product on $\mathcal{A}$, the antipode $\hat{S}$ is the dual to S and the counit $\hat{\varepsilon}$ is the evaluation in $frm[o]--$. In particular, we have $\hat{S}(\lambda)(x) = \lambda(Sx)$ for $\lambda \in \hat{\mathcal{A}}$, $x \in \mathcal{A}$ and if $\hat{\Delta}(\lambda) \in \hat{\mathcal{A}} \otimes \hat{\mathcal{A}}$ then

$$\hat{\Delta}(\lambda)(x \otimes y) = \lambda_{(1)}(x) \otimes \lambda_{(2)}(y) = \lambda(xy), \quad x, y \in \mathcal{A}.$$

The pair $\hat{\mathbb{G}} = (\hat{\mathcal{A}}, \hat{\Delta})$ is an algebraic quantum group, called the *dual* of $\mathbb{G}$.

The linear map which associates to $a \in \mathcal{A}$ the functional $h_a \in \hat{\mathcal{A}}$ is called the *Fourier transform* and its value on an element a is also denoted by $\hat{a}$. Let us note that, due to the faithfulness of the Haar state h, $\hat{\mathcal{A}}$ separates the points of $\mathcal{A}$.

4.1.3. *Woronowicz characters and modular automorphism group.* For $a \in \mathsf{A}$, $\lambda \in \mathsf{A}'$ we define

$$\begin{aligned} \lambda \star a &= (\mathrm{id} \otimes \lambda)\Delta(a), \\ a \star \lambda &= (\lambda \otimes \mathrm{id})\Delta(a). \end{aligned}$$

If $a \in \mathcal{A}$ and $\lambda \in \mathcal{A}'$, then $\lambda \star a, a \star \lambda \in \mathcal{A}$.

For a compact quantum group A with the dense $*$-Hopf algebra $\mathcal{A}$, there exists a unique family $(f_z)_{z\in\mathbb{C}}$ of linear multiplicative functionals on $\mathcal{A}$, called *Woronowicz characters* (cf. [Wor98, Theorem 1.4]), such that

(1) $f_z(frm[o]--) = 1$ for $z \in \mathbb{C}$,
(2) the mapping $\mathbb{C} \ni z \mapsto f_z(a) \in \mathbb{C}$ is an entire holomorphic function for all $a \in \mathcal{A}$,
(3) $f_0(z) = \varepsilon$ and $f_{z_1} \star f_{z_2} = f_{z_1+z_2}$ for any $z_1, z_2 \in \mathbb{C}$,
(4) $f_z(S(a)) = f_{-z}(a)$ and $f_{\bar{z}}(a^*) = \overline{f_{-z}(a)}$ for any $z \in \mathbb{C}$, $a \in \mathcal{A}$,
(5) $S^2(a) = f_{-1} \star a \star f_1$ for $a \in \mathcal{A}$,
(6) the Haar state h satisfies:

$$h(ab) = h(b(f_1 \star a \star f_1)), \quad a, b \in \mathcal{A}.$$

In this case the formula

$$\sigma_t(a) = f_{it} \star a \star f_{it}, \quad t \geq 0, \tag{4.4}$$

defines a one parameter group of modular automorphisms of $\mathcal{A}$ and h is the $(\sigma, -1)$-KMS state, which means that it satisfies

$$h(ab) = h(b\sigma_{-i}(a)), \quad a, b \in \mathcal{A}, \tag{4.5}$$

cf. [BR97, Definition 5.3.1]. Such a state is σ-invariant, i.e. $h(\sigma_t(a)) = h(a)$ for $a \in \mathcal{A}$ and $t \geq 0$ (see [BR97, Proposition 5.3.3]).

The matrix elements of the irreducible unitary corepresentations satisfy the famous generalized Peter-Weyl orthogonality relations

$$h\left(\left(u_{ij}^{(s)}\right)^* u_{k\ell}^{(t)}\right) = \frac{\delta_{st}\delta_{j\ell} f_{-1}\left(u_{ki}^{(s)}\right)}{D_s}, \qquad h\left(u_{ij}^{(s)}\left(u_{k\ell}^{(t)}\right)^*\right) = \frac{\delta_{st}\delta_{ik} f_1\left(u_{\ell j}^{(s)}\right)}{D_s}, \tag{4.6}$$

where $f_1 : \mathcal{A} \to \mathbb{C}$ is the Woronowicz character and

$$D_s = \sum_{\ell=1}^{n_s} f_1\left(u_{\ell\ell}^{(s)}\right)$$

is the quantum dimension of $u^{(s)}$, cf. [Wor87a, Theorem 5.7.4]. Note that unitarity implies that the matrix

$$\left(f_1\left((u_{jk}^{(s)})^*\right)\right)_{1\leq j,k\leq n_s} \in M_{n_s}(\mathbb{C})$$

is invertible, with inverse $\left(f_1(u_{jk}^{(s)})\right)_{jk} \in M_{n_s}(\mathbb{C})$, cf. [Wor87a, Equation (5.24)].

Remark 4.3. The Haar state on compact quantum groups is a trace if and only if the antipode is involutive, i.e. we have $S^2(a) = a$ for all $a \in \mathcal{A}$. In this case we say that (A, Δ) is of Kac type. This is also equivalent to the following conditions, cf. [Wor98, Theorem 1.5],

(1) $f_z = \varepsilon$ for all $z \in \mathbb{C}$,
(2) $\sigma_t = \mathrm{id}$ for all $t \in \mathbb{R}$.

The antipode $S : \mathcal{A} \to \mathcal{A}$ is a closable operator and its closure $\overline{S}$ admits a polar decomposition

$$\overline{S} = R \circ \tau_{\frac{i}{2}}, \tag{4.7}$$

where $\tau_{\frac{i}{2}}$ is the analytic generator of a one parameter group $(\tau_t)_{t\in\mathbb{R}}$ of $*$-automorphisms of the C*-algebra A and $R : \mathsf{A} \to \mathsf{A}$ is a linear antimultiplicative norm preserving involution that commutes with hermitian conjugation and with the semigroup (τ_t), i.e. $\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbb{R}$, see [Wor98, Theorem 1.6]. The operator R is called the *unitary antipode*.

Moreover, τ and R are related to Woronowicz characters through the following formulas

$$\tau_t(a) = f_{it} \star a \star f_{-it}, \tag{4.8}$$

$$R(a) = S(f_{\frac{1}{2}} \star a \star f_{-\frac{1}{2}}) \tag{4.9}$$

for $a \in \mathcal{A}$.

4.2. **Translation invariant Markov semigroups.** Our goal is to construct Markov semigroups on compact quantum groups that reflect the structure of the quantum group. In this section we show that it is exactly the *translation invariant* Markovian semigroups that can be obtained from Lévy processes on the algebra of smooth functions $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$ of the quantum group $\mathbb{G} = (\mathsf{A}, \Delta)$.

For this purpose we first prove that the Markov semigroup $(T_t)_{t\geq 0}$ of a Lévy process on $\mathcal{A}$ has a unique extension to a strongly continuous Markov semigroup on both its reduced and its universal C*-algebra. We then show that the characterisation of Lévy processes in topological groups as the Markov processes which are invariant under time and space translations extends to compact quantum groups.

If $(j_{st})_{0\leq s\leq t}$ is a Lévy process on an $*$-algebra $\mathcal{A}$ with the convolution semigroup of states $(\varphi_t)_{t\geq 0}$ on $\mathcal{A}$ and the Markov semigroup $(T_t)_{t\geq 0}$ on $\mathcal{A}$, then, by a result of Bédos, Murphy and Tuset [BMT01, Theorem 3.3], each φ_t a extends to a continuous functional on A_u, the universal C*-algebra generated by $\mathcal{A}$. Then the formula $T_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta$ makes sense on A_u (where $\Delta : \mathsf{A}_u \to \mathsf{A}_u \otimes \mathsf{A}_u$ denotes the unique unital $*$-homomorphism that extends $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$) and one easily shows (in the same way as in Proposition below) that $(T_t)_t$ becomes a strongly continuous Markov semigroup of contractions on A_u. This means that each T_t ($t \geq 0$) is a unital, completely positive contraction and (T_t) is a strongly continuous semigroup on A_u.

For us, however, it will be more natural to consider the reduced C*-algebra generated by $\mathcal{A}$. This is the C*-algebra A_r obtained by taking the norm closure of the GNS representation of $\mathcal{A}$ with respect to the Haar state h. The Haar state h is by construction faithful on A_r. The coproduct on $\mathcal{A}$ extends to a unique unital $*$-homomorphism $\Delta : \mathsf{A}_r \to \mathsf{A}_r \otimes \mathsf{A}_r$ which makes the pair (A_r, Δ) a compact quantum group. The following result shows that, even though $\varphi_t : \mathcal{A} \to \mathbb{C}$ can be unbounded with respect to the reduced C*-norm and therefore not extend to A_r, $(T_t)_t$ always extends to a strongly continuous Markov semigroup on A_r.

Michael Brannan states on any C^*-algebraic version $C(\mathbb{G})$ of $\mathbb{G}$ define a continuous convolution operator on the reduced version $C_r(\mathbb{G})$, cf. [Bra11, Lemma 3.4]. We will need a similar result for convolution semigroups o states on $\mathrm{Pol}(\mathbb{G})$.

Theorem 4.4. *Each Lévy process $(j_{st})_{0\leq s\leq t}$ on the Hopf $*$-algebra $\mathcal{A}$ gives rise to a unique strongly continuous Markov semigroup $(T_t)_{t\geq 0}$ on A_r, the reduced C*-algebra generated by $\mathcal{A}$.*

Proof. Let $\mathcal{H}$ be the Hilbert space of the GNS representation of $\mathcal{A}$ associated to the Haar state h and let ξ denotes the related (normalized) cyclic vector. Then for any $a \in \mathcal{A}$ we have $h(a) = \langle \xi, \lambda(a)\xi\rangle$, where λ is the left regular representation. We denote by $\|.\|_r$ the norm in A_r, that is $\|a\|_r = \|\lambda(a)\|$, where $\|.\|$ denotes the operator norm.

In a similar way, we associate the Hilbert space $\mathcal{H}_t$, the GNS representation ρ_t on $\mathcal{H}_t$ and the normalized cyclic vector ξ_t to each state φ_t ($\varphi_t = \Phi \circ j_{0t}$, cf. Definition **??**). We have $\varphi_t(a) = \langle \xi_t, \rho_t(a)\xi_t\rangle$ for $a \in \mathcal{A}$.

We define the operators

$$\begin{aligned} i_t &: \mathcal{H} \ni v \to v \otimes \xi_t \in \mathcal{H} \otimes \mathcal{H}_t \\ \pi_t &: \mathcal{H} \otimes \mathcal{H}_t \ni v \otimes w \to \langle \xi_t, w\rangle_{\mathcal{H}_t} v \in \mathcal{H} \\ E_t &: B(\mathcal{H} \otimes \mathcal{H}_t) \ni X \to \pi_t \circ X \circ i_t \in B(\mathcal{H}). \end{aligned}$$

Since for each t, i_t is an isometry and π_t is contractive, E_t is contractive too: $\|E_t(X)\| = \|\pi_t \circ X \circ i_t\| \leq \|X\|$.

Next we define

$$U : \lambda(\mathcal{A})\xi \otimes \rho_t(\mathcal{A})\xi_t \ni \lambda(a)\xi \otimes \rho_t(b)\xi_t \mapsto \lambda(a_{(1)})\xi \otimes \rho_t(a_{(2)}b)\xi_t \in \mathcal{H} \otimes \mathcal{H}_t$$

and we check that it is an isometry with adjoint given by

$$U^*(\lambda(a)\xi \otimes \rho_t(b)\xi_t) = \lambda(a_{(1)})\xi \otimes \rho_t(S(a_{(2)})b)\xi_t.$$

Indeed, using the invariance of the Haar measure, we show that U is isometric

$$\begin{aligned} &\|U(\lambda(a)\xi \otimes \rho_t(b)\xi_t)\|^2 = \|\lambda(a_{(1)})\xi \otimes \rho_t(a_{(2)}b)\xi_t\|^2 \\ &= h(a^*_{(1)}a_{(1)})\varphi_t(b^*a^*_{(2)}a_{(2)}b) = (h \otimes \varphi^b_t)(a^*_{(1)}a_{(1)} \otimes a^*_{(2)}a_{(2)}) = (h \star \varphi^b_t)(a^*a) \\ &= h(a^*a)\varphi^b_t(frm[o]--) = h(a^*a)\varphi_t(b^*b) = \|\lambda(a)\xi \otimes \rho_t(b)\xi_t\|^2, \end{aligned}$$

where $\varphi_t^b(x) := \varphi_t(b^*xb)$. Moreover, by the antipode property (4.2) we have

$$\begin{aligned} UU^*(\lambda(a)\xi \otimes \rho_t(b)\xi_t) &= U(\lambda(a_{(1)})\xi \otimes \rho_t(S(a_{(2)})b)\xi_t) \\ = \lambda(a_{(1)})\xi \otimes \rho_t(a_{(2)}S(a_{(3)})b)\xi_t &= \lambda(a_{(1)}\varepsilon(a_{(2)}))\xi \otimes \rho_t(b)\xi_t = \lambda(a)\xi \otimes \rho_t(b)\xi_t, \end{aligned}$$

which implies that U is an isometry with dense image and therefore extends to a unique unitary operator denoted again by U.

Now the fact that the Markov semigroup $(T_t)_t$ is bounded on A_r, i.e.

$$\|T_t(a)\|_r = \|\lambda(T_t(a))\|_{B(\mathcal{H})} \leq \|\lambda(a)\|_{B(\mathcal{H})} = \|a\|_r,$$

follows immediately from the relation

$$\lambda(T_t(a)) = E_t(U(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t})U^*), \tag{4.10}$$

since

$$\begin{aligned} \|\lambda(T_t(a))\| &= \|E_t(U(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t})U^*)\| \leq \|U(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t})U^*\| \\ &= \|\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}\| = \|\lambda(a)\|. \end{aligned}$$

To see that (4.10) holds, let us fix $v \in \mathcal{H}$ and $b \in \mathcal{A}$ such that $v = \lambda(b)\xi$. Then

$$\begin{aligned} E_t(U(\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t})U^*)v &= (\pi_t \circ U \circ (\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}) \circ U^* \circ i_t)(\lambda(b)\xi) \\ &= (\pi_t \circ U \circ (\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t}) \circ U^*)\,(\lambda(b)\xi \otimes \xi_t) \\ &= \pi_t \circ U \circ (\lambda(a) \otimes \mathrm{id}_{\mathcal{H}_t})\,\big(\lambda(b_{(1)})\xi \otimes \rho_t(S(b_{(2)}))\xi_t\big) \\ &= \pi_t \circ U\,\big(\lambda(ab_{(1)})\xi \otimes \rho_t(S(b_{(2)}))\xi_t\big) \\ &= \pi_t\,\big(\lambda(a_{(1)}b_{(1)})\xi \otimes \rho_t(a_{(2)}b_{(2)}S(b_{(3)}))\xi_t\big) \\ &= \pi_t\,\big(\lambda(a_{(1)}b)\xi \otimes \rho_t(a_{(2)})\xi_t\big) \\ &= \langle \xi_t, \rho_t(a_{(2)})\xi_t\rangle\,\lambda(a_{(1)}b)\xi \\ &= \lambda(a_{(1)}\varphi_t(a_{(2)}))\lambda(b)\xi = \lambda(T_t(a))v. \end{aligned}$$

This way we showed that each T_t extends to a contraction on A_r. The extensions form again a semigroup and since both Δ and φ_t are completely positive, T_t is completely positive and contractive. Let us now check that $(T_t)_t$ forms a strongly continuous semigroup on A_r.

For a given $a \in \mathsf{A}_r$ we chose by density $b \in \mathcal{A}$ such $\|a-b\|_r < \epsilon$. Recall that for $b \in \mathcal{A}$, $T_t(b) = \varphi_t \star b = (\mathrm{id} \otimes \varphi_t) \circ \Delta(b)$, where $(\varphi_t)_t$ is the convolution semigroup of states on $\mathcal{A}$ (cf. Section **??**). Then

$$\begin{aligned} \|T_t(a) - a\|_r &\leq \|T_t(a) - T_t(b)\|_r + \|T_t(b) - b\|_r + \|b - a\|_r \\ &\leq 2\|a-b\|_r + \|(\varphi_t \star b) - b\|_r \leq 2\epsilon + \sum \|b_{(1)}\varphi_t(b_{(2)}) - b_{(1)}\varepsilon(b_{(2)})\|_r \\ &= 2\epsilon + \sum |\varphi_t(b_{(2)}) - \varepsilon(b_{(2)})|\|b_{(1)}\|_r. \end{aligned}$$

Since $\lim_{t\to 0+} \varphi_t(b) = \varepsilon(b)$ for any $b \in \mathcal{A}$ and the sum is finite, we conclude that

$$\lim_{t\to 0+} \|T_t(a) - a\|_r = 0 \quad \text{for each } a \in \mathsf{A}_r.$$

□

The next results give the characterisation of Markov semigroup which are related to Lévy processes on compact quantum groups.

Lemma 4.5. *Let* (A,Δ) *be a compact quantum group and* $T:\mathsf{A}\to\mathsf{A}$ *be a completely bounded linear map.*

If T *is translation invariant, i.e. satisfies*

$$\Delta\circ T=(\mathrm{id}\otimes T)\circ\Delta$$

then $T(V_s)\subseteq V_s$ *for all* $s\in\mathcal{I}$ *and therefore* T *also leaves the* $*$*-Hopf algebra* $\mathcal{A}$ *invariant.*

Proof. Let $s,s'\in\mathcal{I}$, $s\neq s'$, and $1\leq j,k\leq n_s$, $1\leq p,q\leq n_{s'}$. Since the Haar state is idempotent, we have

$$\begin{aligned}
h\left(\left(u_{pq}^{(s')}\right)^*T\left(u_{jk}^{(s)}\right)\right) &= (h\star h)\left(\left(u_{pq}^{(s')}\right)^*T\left(u_{jk}^{(s)}\right)\right)\\
&= \sum_{r=1}^{n_{s'}}(h\otimes h)\left(\left(\left(u_{pr}^{(s')}\right)^*\otimes\left(u_{rq}^{(s')}\right)^*\right)\Delta\left(T\left(u_{jk}^{(s)}\right)\right)\right)\\
&= \sum_{r=1}^{n_{s'}}\sum_{\ell=1}^{n_s}(h\otimes h)\left(\left(u_{pr}^{(s')}\right)^*\otimes\left(u_{rq}^{(s')}\right)^*\left(u_{j\ell}^{(s)}\otimes T\left(u_{\ell k}^{(s)}\right)\right)\right)\\
&= \sum_{\ell=1}^{n_s}\delta_{ss'}\overline{\frac{f_1((u_{jp}^{(s)})^*)}{D_s}}h\left((u_{\ell q}^{(s')})^*T(u_{\ell k}^{(s)})\right),
\end{aligned}$$

i.e. $h\left((u_{pq}^{(s')})^*T(u_{jk}^{(s)})\right)=0$ for all $s,s'\in\mathcal{I}$, with $s\neq s'$, and all $1\leq j,k\leq n_s$, $1\leq p,q\leq n_{s'}$. Therefore $T(u_{jk}^{(s)})\in V_s$. □

Theorem 4.6. *Let* (A,Δ) *be a compact quantum group and* $(T_t)_{t\geq0}$ *a quantum Markov semigroup on* (A,Δ)*.*

Then $(T_t)_{t\geq0}$ *is the quantum Markov semigroup of a (uniquely determined) Lévy process on* $\mathcal{A}$ *if and only if* T_t *is translation invariant for all* $t\geq0$*.*

Proof. If $(T_t)_{t\geq0}$ comes from a Lévy process on $\mathcal{A}$, then there exists a generating functional ϕ on $\mathcal{A}$ such that the generator of the semigroup is $L_\phi(a)=\phi\star a$ ($a\in\mathcal{A}$) and $T_t=\exp(-tL_\phi)$ on $\mathcal{A}$. Then L_ϕ is translation invariant on $\mathcal{A}$:

$$(\mathrm{id}\otimes L_\phi)\circ\Delta(a)=a_{(1)}\otimes(\phi\star a_{(2)})=a_{(1)}\otimes a_{(2)}\phi(a_{(3)})=\Delta(a_{(1)})\phi(a_{(2)})=\Delta\circ L_\phi(a).$$

Next we observe that the powers of a translation invariant operator are again translation invariant: if $(\mathrm{id}\otimes L)\circ\Delta=\Delta\circ L$, then , by induction, we have

$$(\mathrm{id}\otimes L^{\circ n})\circ\Delta=(\mathrm{id}\otimes L)\circ\Delta\circ L^{\circ(n-1)}=\Delta\circ L^{\circ n}.$$

So $T_t=\exp(-tL_\phi)=\sum_{n\geq0}\frac{(-t)^n}{n!}L^{\circ n}$ is also translation invariant on $\mathcal{A}$ for each $t\geq0$. By continuity T_t is translation invariant on A.

Reciprocally, if T_t is translation invariant, then (by the previous Lemma) it maps $\mathcal{A}$ to itself, and so $\varphi_t := \varepsilon \circ T_t$ is well-defined on $\mathcal{A}$ (ε is defined on $\mathcal{A}$, but may not extend to A). From the Markov semigroup properties of (T_t) we deduce that $(\varphi_t)_t$ is a convolution semigroup of states on $\mathcal{A}$. The generating functional of this semigroup defines uniquely a Lévy process on $\mathcal{A}$.

□

A similar result was proved by Lindsay and Skalski (cf. [LS11, Proposition 3.2]) in case of C^*-bialgebras with the counit, satisfying the residual vanishing at infinity condition. This covers for example the case of coamenable compact quantum groups, but their proof is considerably more technical than the simple algebraic argument we presented here.

5. Independences and Convolutions in Noncommutative Probability

Until now we only considered tensor independence, which is the natural generalization of the notion of stochastic independence used in classical probability and also corresponds to the notion of independent observables used in quantum mechanics. But in quantum probability there exist also other notions of independence. In this section we shall study the most prominent examples, freeness, monotone independence, and monotone independence, and the convolutions for probability measures on $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{T}$ derived from them.

5.1. **Nevanlinna theory and Cauchy transforms.** Denote by $\mathbb{C}^+ = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ and $\mathbb{C}^- = \{z \in \mathbb{C}; \operatorname{Im} z < 0\}$ the upper and lower half plane. For μ a probability measure on $\mathbb{R}$ and $z \in \mathbb{C}^+$, we define its *Cauchy transform* G_μ by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \mathrm{d}\mu(x)$$

and its *reciprocal Cauchy transform* F_μ by

$$F_\mu(z) = \frac{1}{G_\mu(z)}.$$

Denote by $\mathcal{F}$ the following class of holomorphic self-maps,

$$\mathcal{F} = \left\{ F : \mathbb{C}^+ \to \mathbb{C}^+; F \text{ holomorphic and } \inf_{z \in \mathbb{C}^+} \frac{\operatorname{Im} F(z)}{\operatorname{Im} z} = 1 \right\}$$

The map $\mu \mapsto F_\mu$ defines a bijection between the class $\mathcal{M}_1(\mathbb{R})$ of probability measures on $\mathbb{R}$ and $\mathcal{F}$, as follows from the following theorem.

Theorem 5.1. [Maa92] *Let $F : \mathbb{C}^+ \to \mathbb{C}^+$ be holomorphic, then the following are equivalent.*

- **(i):** $\inf_{z \in \mathbb{C}^+} \frac{\operatorname{Im} F(z)}{\operatorname{Im} z} = 1$;
- **(ii):** *there exists a $\mu \in \mathcal{M}_1(\mathbb{R})$ such that $F = F_\mu$.*

Furthermore, μ is uniquely determined by F.

Similarly, for μ a probability measure on the unit circle $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$ or on the positive half-line $\mathbb{R}_+ = \{x \in \mathbb{R}; x \geq 0\}$, we define

$$\psi_\mu(z) = \int \frac{xz}{1 - xz} \mathrm{d}\mu$$

and

$$K_\mu(z) = \frac{\psi_\mu(z)}{1 + \psi_\mu(z)}$$

for $z \in \mathbb{C}\backslash \operatorname{supp} \mu$.

The map $\mu \mapsto K_\mu$ defines bijections between the class $\mathcal{M}_1(\mathbb{T})$ of probability measures on $\mathbb{T}$ and the class

$$\mathcal{S} = \{K : \mathbb{D} \to \mathbb{D}; K \text{ holomorphic and } K(0) = 0\},$$

where $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$, and between the class $\mathcal{M}_1(\mathbb{R}_+)$ of probability measures on $\mathbb{R}_+$ and the class

$$\mathcal{P} = \left\{ K : \mathbb{C}\backslash\mathbb{R}_+ \to \mathbb{C}\backslash\mathbb{R}_+; \begin{array}{c} K \text{ holomorphic}, \lim_{t \nearrow 0} K(t) = 0, K(\overline{z}) = \overline{K(z)}, \\ \pi \geq \arg K(z) \geq \arg z \text{ for all } z \in \mathbb{C}^+ \end{array} \right\},$$

cf. [BB05] and the references therein.

In the following, if X is an operator with distribution $\mu = \mathcal{L}(X, \Omega)$ w.r.t. Ω, then we will write G_X, F_X, Ψ_X or K_X instead of $G_{\mathcal{L}(X,\Omega)}$, $F_{\mathcal{L}(X,\Omega)}$, $\psi_{\mathcal{L}(X,\Omega)}$, or $K_{\mathcal{L}(X,\Omega)}$ for the transforms of the distribution of X.

5.2. **Free convolutions.** By $\mathbb{A}_k$ we call denote the set of alternating k-tuples of 1's and 2's, i.e.

$$\mathbb{A}_k = \big\{(\varepsilon_1, \ldots, \varepsilon_k) \in \{1, 2\}^k; \varepsilon_1 \neq \varepsilon_2 \neq \ldots \neq \varepsilon_k\big\}.$$

Definition 5.2. [Voi86] Let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{B}(H)$ be two $*$-algebras of bounded operators on a Hilbert space and assume $\mathbf{1} \in \mathcal{A}_i$, $i = 1, 2$. Let Ω be a unit vector in H and denote by Φ the vector state associated to Ω. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *free*, if we have

$$\Phi(X_1 \cdots X_k) = 0$$

for all $k \geq 1$, $\varepsilon \in \mathbb{A}_k$, $X_1 \in \mathcal{A}_{\varepsilon_1}, \ldots, X_k \in \mathcal{A}_{\varepsilon_k}$ such that

$$\Phi(X_1) = \cdots = \Phi(X_k) = 0.$$

Two normal operators X and Y are called *free*, if the algebras $\mathrm{alg}(X) = \{h(X); h \in C_b(\mathbb{C})\}$ and $\mathrm{alg}(Y) = \{h(Y); h \in C_b(\mathbb{C})\}$ they generate are free.

Theorem 5.3. [Maa92, CG05, CG06] *Let μ and ν be two probability measures on the real line, with reciprocal Cauchy transforms F_μ and F_ν. Then there exist unique functions $Z_1, Z_2 \in \mathcal{F}$ such that*

$$F_\mu\big(Z_1(z)\big) = F_\nu\big(Z_2(z)\big) = Z_1(z) + Z_2(z) - z$$

for all $z \in \mathbb{C}^+$.

The function $F = F_\mu \circ Z_1 = F_\nu \circ Z_2$ also belongs to $\mathcal{F}$ and is therefore the the reciprocal Cauchy transform of some probability measure λ. One defines the additive free convolution of μ and ν as this unique probability measure and writes $\mu \boxplus \nu = \lambda$. This is justified by the following theorem.

Theorem 5.4. [Maa92, BV93] *Let X and Y be two self-adjoint operators on some Hilbert space H that are free w.r.t. some unit vector $\Omega \in H$. If Ω is cyclic, i.e. if*

$$\overline{\mathrm{alg}\{h(X), h(Y); h \in C_b(\mathbb{R})\}\Omega} = H.$$

then $X + Y$ is essentially self-adjoint and the distribution w.r.t. Ω of its closure is equal to the additive free convolution of the distributions of X and Y w.r.t. to Ω, i.e.

$$\mathcal{L}(X + Y, \Omega) = \mathcal{L}(X, \Omega) \boxplus \mathcal{L}(Y, \Omega).$$

There exist analogous results for the multiplicative convolutions of probability measures on the unit circle and the positive half-line, cf. [Maa92, BV93, CG05, CG06]

Theorem 5.5.
(i) *Let μ and ν be two probability measures on the unit circle with transforms K_μ and K_ν and whose first moments do not vanish, $\int_{\mathbb{T}} x \mathrm{d}\mu(x) \neq 0$, $\int_{\mathbb{T}} x \mathrm{d}\nu(x) \neq 0$. Then there exist unique functions $Z_1, Z_2 \in \mathcal{S}$ such that*

$$K_\mu\big(Z_1(z)\big) = K_\nu\big(Z_2(z)\big) = \frac{Z_1(z)Z_2(z)}{z}$$

for all $z \in \mathbb{D}\backslash\{0\}$. The multiplicative free convolution $\lambda = \mu \boxtimes \nu$ is defined as the unique probability measure λ with transform $K_\lambda = K_\mu \circ Z_1 = K_\nu \circ Z_2$.
(ii) *Let U and V be two unitary operators on some Hilbert space H that are free w.r.t. some unit vector $\Omega \in H$. Then the products UV and VU are also unitary and their distributions w.r.t. to Ω are equal to the free convolution of the distributions of U and V w.r.t. Ω, i.e. i.e.*

$$\mathcal{L}(UV, \Omega) = \mathcal{L}(VU, \Omega) = \mathcal{L}(U, \Omega) \boxtimes \mathcal{L}(V, \Omega).$$

Theorem 5.6.
(i) *Let μ and ν be two probability measures on the positive half-line such that $\mu \neq \delta_0$, $\nu \neq \delta_0$ and denote their transforms by K_μ and K_ν. Then there exist unique functions $Z_1, Z_2 \in \mathcal{P}$ such that*

$$K_\mu\big(Z_1(z)\big) = K_\nu\big(Z_2(z)\big) = \frac{Z_1(z)Z_2(z)}{z}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}_+$. The multiplicative free convolution $\lambda = \mu \boxtimes \nu$ is defined as the unique probability measure λ with transform $K_\lambda = K_\mu \circ Z_1 = K_\nu \circ Z_2$.
(ii) *Let X and Y be two positive operators on some Hilbert space H that are free w.r.t. some unit vector $\Omega \in H$. Assume furthermore that Ω is cyclic, i.e. that*

$$\overline{\mathrm{alg}\{h(X), h(Y); h \in C_b(\mathbb{R})\}\Omega} = H.$$

Then the products $\sqrt{X}Y\sqrt{X}$ and $\sqrt{Y}X\sqrt{Y}$ are essentially self-adjoint and positive, and their distributions w.r.t. to Ω are equal to the free convolution of the distributions of X and Y w.r.t. Ω, i.e. i.e.

$$\mathcal{L}(\sqrt{X}Y\sqrt{X},\Omega) = \mathcal{L}(\sqrt{Y}X\sqrt{Y},\Omega) = \mathcal{L}(X,\Omega) \boxtimes \mathcal{L}(Y,\Omega).$$

5.3. **Monotone Convolutions.**

Definition 5.7. [Mur00] Let $\mathcal{A}_1,\mathcal{A}_2 \subset \mathcal{B}(H)$ be two $*$-algebras of bounded operators on a Hilbert space H, and let $\Omega \in H$ be a unit vector. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are *monotonically independent* w.r.t. Ω, if we have

$$\langle \Omega, X_1X_2\cdots X_k\Omega\rangle = \left\langle \Omega, \prod_{\kappa:\varepsilon_\kappa=1} X_\kappa\Omega\right\rangle \prod_{\kappa:\varepsilon_\kappa=2}\langle\Omega, X_\kappa\Omega\rangle$$

for all $k\in\mathbb{N}$, $\varepsilon\in\mathbb{A}_k$, $X_1\in\mathcal{A}_{\varepsilon_1},\ldots,X_k\in\mathcal{A}_{\varepsilon_k}$.

Remark 5.8.

(a) Note that this notion depends on the order, i.e. if $\mathcal{A}_1$ and $\mathcal{A}_2$ are monotonically independent, then this does *not* imply that $\mathcal{A}_2$ and $\mathcal{A}_1$ are monotonically independent. In fact, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are monotonically independent and $\mathcal{A}_2$ and $\mathcal{A}_1$ are also monotonically independent, and $\Phi(\cdot)=\langle\Omega,\cdot\,\Omega\rangle$ does not vanish on one of the algebras, then restrictions of Φ to $\mathcal{A}_1$ and $\mathcal{A}_2$ have to be homomorphisms. To prove this for the restriction to, e.g., $\mathcal{A}_1$, take an element $Y\in\mathcal{A}_2$ such that $\Phi(Y)\neq 0$, then

$$\Phi(X_1X_2) = \frac{\Phi(X_1YX_2)}{\Phi(Y)} = \Phi(X_1)\Phi(X_2)$$

for all $X_1,X_2\in\mathcal{A}_1$.

(b) The algebras are not required to be unital. If $\mathcal{A}_1$ is unital, then the restriction of $\Phi(\cdot)=\langle\Omega,\cdot\,\Omega\rangle$ to $\mathcal{A}_2$ has to be a homomorphism, since monotone independence implies

$$\langle\Omega, XY\Omega\rangle = \langle\Omega, X\mathbf{1}Y\Omega\rangle = \langle\Omega,X\Omega\rangle\langle\Omega,Y\Omega\rangle$$

for $X,Y\in\mathcal{A}_2$.

(c) In the definition of monotone independence the condition

$$XYZ = \langle\Omega, Y\Omega\rangle XZ$$

for all $X,Z\in\mathcal{A}_1$, $Y\in\mathcal{A}_2$ is often also imposed. If the state vector Ω is cyclic for the algebra generated by $\mathcal{A}_1$ and $\mathcal{A}_2$, then this is automatically satisfied. Let $X_1,X_3,\ldots,Z_1,Z_3,\ldots\in\mathcal{A}_1$ and $Y,X_2,X_4,\ldots,Z_2,Z_4,\ldots\in\mathcal{A}_2$, then

$$\begin{aligned}
&\langle X_1\cdots X_n\Omega, YZ_1\cdots Z_m\Omega\rangle = \langle\Omega, X_n^*\cdots X_1^*YZ_1\cdots Z_m\Omega\rangle\\
=\ &\langle\Omega,Y\Omega\rangle\prod_{k\text{ even}}\langle\Omega,X_k^*\Omega\rangle\prod_{\ell\text{ even}}\langle\Omega,Z_\ell\Omega\rangle\langle X_1X_3\cdots\Omega, Z_1Z_3\cdots\Omega\\
=\ &\langle\Omega,Y\Omega\rangle\langle X_1\cdots X_n\Omega, Z_1\cdots Z_m\Omega\rangle,
\end{aligned}$$

for all $n,m\geq 1$, i.e., $X_1^*YZ_1$ and $\langle\Omega,Y\Omega\rangle X_1^*Z_1$ coincide on the subspace generated by $\mathcal{A}_1$ and $\mathcal{A}_2$ from Ω.

Definition 5.9. Let X and Y be two normal operators on a Hilbert space H, not necessarily bounded. We say that X and Y are *monotonically independent* w.r.t. Ω, if the $*$-algebras $\mathrm{alg}_0(X) = \{h(X); h \in C_b(\mathbb{C}), h(0) = 0\}$ and $\mathrm{alg}_0(Y) = \{h(Y); h \in C_b(\mathbb{C}), h(0) = 0\}$ are monotonically independent w.r.t. Ω.

Let us now introduce the model we shall use for calculations with monotonically independent operators.

Proposition 5.10. *Let μ, ν be two probability measures on $\mathbb{C}$ and define normal operators X and Y on $L^2(\mathbb{C} \times \mathbb{C}, \mu \otimes \nu)$ by*

$$\begin{aligned} \mathrm{Dom}\, X &= \left\{\psi \in L^2(\mathbb{C} \times \mathbb{C}, \mu \otimes \nu); \int_{\mathbb{C}} \left| x \int_{\mathbb{C}} \psi(x,y)\mathrm{d}\nu(y)\right|^2 \mathrm{d}\mu(x) < \infty\right\}, \\ \mathrm{Dom}\, Y &= \left\{\psi \in L^2(\mathbb{C} \times \mathbb{C}, \mu \otimes \nu); \int_{\mathbb{C}\times\mathbb{C}} |y\psi(x,y)|^2 \mathrm{d}\mu \otimes \nu(x,y) < \infty\right\}, \end{aligned}$$

$$\begin{aligned} (X\psi)(x,y) &= x \int_{\mathbb{C}} \psi(x,y')\mathrm{d}\nu(y'), \\ (X\psi)(x,y) &= y\psi(x,y). \end{aligned}$$

Then $\mathcal{L}(X, \mathbf{1}) = \mu$, $\mathcal{L}(Y, \mathbf{1}) = \nu$, and X and Y are monotonically independent w.r.t. the constant function $\mathbf{1}$.

Proof. Denote by P_2 the orthogonal projection onto the space of functions in $L^2(\mathbb{C} \times \mathbb{C}, \mu \otimes \nu)$ which do not depend on the second variable, and by M_x multiplication by the first variable, then $X = M_x P_2$. This operator is normal, we have

$$h(X)\psi(x,y) = \big(h(x) - h(0)\big) \int_{\mathbb{C}} \psi(x,y)\mathrm{d}\nu(y) + h(0)\psi(x,y)$$

and $\langle \mathbf{1}, h(X)\mathbf{1}\rangle = \int_{\mathbb{C}} h(x)\mathrm{d}\mu(x)$ for all $h \in C_b(\mathbb{C})$, i.e. $\mathcal{L}(X, \mathbf{1}) = \mu$. The operator Y is multiplication by the second variable, it is clearly normal. We have

$$h(Y)\psi(x,y) = h(y)\psi(x,y)$$

and $\langle \mathbf{1}, h(Y)\mathbf{1}\rangle = \int_{\mathbb{C}} h(y)\mathrm{d}\nu(y)$ for all $h \in C_b(\mathbb{C})$, i.e. $\mathcal{L}(Y, \mathbf{1}) = \nu$.

Let $f_1, \ldots, f_n, g_1, \ldots, g_n \in C_b(\mathbb{C})$, $f_1(0) = \cdots = f_n(0) = 0$. Then

$$f_n(X)g_{n-1}(Y)\cdots g_1(Y)f_1(X)\mathbf{1} = \prod_{k=1}^{n-1} \int_{\mathbb{C}} g_k(y)\mathrm{d}\nu(y)\, f_1 \cdots f_n$$

and

$$\begin{aligned} \langle \mathbf{1}, f_n(X)g_{n-1}(Y)\cdots g_1(Y)f_1(X)\mathbf{1}\rangle &= \prod_{k=1} \int_{\mathbb{C}} g_k(y)\mathrm{d}\nu(y) \int_{\mathbb{C}} f_1(x)\cdots f_n(x)\mathrm{d}\mu(x) \\ &= \prod_{k=1}^{n-1} \langle \mathbf{1}, g_k(Y)\mathbf{1}\rangle\langle \mathbf{1} f_1(X)\cdots f_n(X)\mathbf{1}\rangle, \end{aligned}$$

i.e. the condition for monotone independence is satisfied in this case. Similarly one checks the expectation of $g_n(Y)f_n(X)\cdots g_1(Y)f_1(X)$, $f_n(X)g_n(Y)\cdots f_1(X)g_1(Y)$, and $g_n(Y)f_{n-1}(X)\cdots f_1(X)g_1(Y)$. □

The following theorem shows that any pair of monotonically independent normal operators can be reduced to this model.

Theorem 5.11. *Let X and Y be two normal operators on a Hilbert space H that are monotonically independent with respect to $\Omega\in H$ and let $\mu=\mathcal{L}(X,\Omega)$, $\nu=\mathcal{L}(Y,\Omega)$.*

Then there exists an isometry $W:L^2(\mathbb{C}\times\mathbb{C},\mu\otimes\nu)\to H$ such that

$$\begin{aligned} W^*h(X)W\psi(x,y) &= \big(h(x)-h(0)\big)\int\psi(x,y)\mathrm{d}\nu(y)+h(0)\psi(x,y), \\ W^*h(Y)W\psi(x,y) &= h(y)\psi(x,y) \end{aligned} \tag{5.1}$$

for $x,y\in\mathbb{C}$, $\psi\in L^2(\mathbb{C}\times\mathbb{C},\mu\otimes\nu)\cong L^2(\sigma_X,\mu)\otimes L^2(\sigma_Y,\nu)$ and $h\in C_b(\mathbb{C})$.

We have $WL^2(\mathbb{C}\times\mathbb{C},\mu\otimes\nu)=\overline{\mathrm{alg}\{h(X),h(Y);h\in C_b(\mathbb{C})\}\Omega}$.

If the vector $\Omega\in H$ is cyclic for the algebra $\mathrm{alg}(X,Y)=\mathrm{alg}\{h(X),h(Y);h\in C_b(\mathbb{C})\}$ generated by X and Y, then W is unitary.

Proof. Define W on simple tensors of bounded continuous functions by

$$Wf\otimes g=g(Y)f(X)\Omega$$

for $f,g\in C_b(\mathbb{C})$. It follows from the monotone independence of X and Y that this defines an isomorphism, since

$$\begin{aligned} \langle Wf_1\otimes g_1,Wf_2\otimes g_2\rangle &= \langle\Omega,f_1(X)^*g_1(Y)^*g_2(Y)f_2(X)\Omega\rangle \\ &= \langle\Omega,f_1(X)^*f_2(X)\Omega\rangle\langle\Omega,g_1(Y)^*g_2(Y)\Omega\rangle \\ &= \int\overline{f_1(t)}f_2(t)\mathrm{d}\mu(t)\int\overline{g_1(t)}g_2(t)\mathrm{d}\nu(t). \end{aligned}$$

Since $C_b(\mathbb{C})\otimes C_b(\mathbb{C})$ is dense in $L^2(\mathbb{C}\times\mathbb{C},\mu\otimes\nu)$, W extends to a unique isomorphism on $L^2(\mathbb{C}\times\mathbb{C},\mu\otimes\nu)$.

The relations

$$\begin{aligned} &\langle Wf_1\otimes g_1,h(X)Wf_2\otimes g_2\rangle=\langle\Omega,f_1(X)^*g_1(Y)^*h(X)g_2(Y)f_2(X)\Omega\rangle \\ &\qquad=\langle\Omega,f_1(X)^*\big(h(X)-h(0)\big)f_2(X)\Omega\rangle\langle\Omega,g_1(Y)^*\Omega\rangle\langle\Omega,g_2(Y)\Omega\rangle \\ &\qquad\qquad+h(0)\langle\Omega,f_1(X)^*g_1(Y)^*g_2(Y)f_2(X)\Omega\rangle \\ &=\langle\Omega,g_2(Y)\Omega\rangle\big\langle Wf_1\otimes g_1,W\big((h-h(0)1\big)f_1\otimes 1\big\rangle+h(0)\langle Wf_1\otimes g_1,Wf_1\otimes g_2\rangle \\ &\qquad=\left\langle Wf_1\otimes g_1,W\left(\int g_2(y)\mathrm{d}\nu(y)(h-h(0)1)f_1\otimes 1+h(0)f_2\otimes g_2\right)\right\rangle \end{aligned}$$

and

$$\begin{aligned} \langle Wf_1\otimes g_1,h(Y)Wf_2\otimes g_2\rangle &= \langle\Omega,f_1(X)^*g_1(Y)^*h(Y)g_2(Y)f_2(X)\Omega\rangle \\ &= \langle Wf_1\otimes g_1,Wf_2\otimes(hg_2)\rangle \end{aligned}$$

shows that we have the desired formulas for simple tensors of functions $f_1, f_2, g_1, g_2 \in C_b(\mathbb{C})$. The general case follows by linearity and continuity. Remark 5.8(c) implies

$$\begin{aligned} WL^2(\mathbb{C}\times\mathbb{C}, \mu\otimes\nu) &= \overline{\text{span}\,\{g(Y)f(X)\Omega; f,g\in C_b(\mathbb{C})\}} \\ &= \overline{\text{alg}\{h(X),h(Y); h\in C_b(\mathbb{C})\}\Omega}. \end{aligned}$$

If Ω is cyclic, then W is surjective and therefore unitary. □

Remark 5.12. It follows that the joint law of two monotonically independent, normal operators is uniquely determined by their marginal distributions, in the sense that the restriction of $\Phi(\cdot) = \langle\Omega, \cdot\,\Omega\rangle$ to $\text{alg}(X,Y) = \text{alg}\{h(X),h(Y); h \in C_b(\mathbb{C})\}$ is uniquely determined by $\mathcal{L}(X,\Omega)$ and $\mathcal{L}(Y,\Omega)$. But by Lemma **??**, also computations for unbounded functions of X and Y, e.g., concerning the operators $X+Y$ for self-adjoint X and Y, or $\sqrt{X}Y\sqrt{Y}$ for positive X and Y, reduce to the model introduced in Proposition 5.10.

5.3.1. *Additive monotone convolution on $\mathcal{M}_1(\mathbb{R})$.*

Definition 5.13. [Mur00] Let μ and ν be two probability measures on $\mathbb{R}$ with reciprocal Cauchy transforms F_μ and F_ν. Then we define the additive monotone convolution $\lambda = \mu \triangleright \nu$ of μ and ν as the unique probability measure on $\mathbb{R}$ with reciprocal Cauchy transform $F_\lambda = F_\mu \circ F_\nu$.

It follows from Subsection 5.1 that the additive monotone convolution is well-defined. Let us first recall some basic properties of the additive monotone convolution.

Proposition 5.14. [Mur00] *The additive monotone convolution is associative and $*$-weakly continuous in both arguments. It is affine in the first argument and convolution from the right by a Dirac measure corresponds to translation, i.e. $\mu\triangleright\delta_x = T_x^{-1}\mu$ for $x\in\mathbb{R}$, where $T_x:\mathbb{R}\to\mathbb{R}$ is defined by $T_x(t) = t+x$.*

This convolution is not commutative, i.e. in general we have $\mu\triangleright\nu \neq \nu\triangleright\mu$.

Let $x\in\mathbb{R}$ and $0\le p\le 1$. Then one can compute, e.g.,

$$\delta_x \triangleright \big(p\delta_1 + (1-p)\delta_{-1}\big) = q\delta_{z_1} + (1-q)\delta_{z_2}$$

where

$$\begin{aligned} z_1 &= \frac{1}{2}\left(x + \sqrt{x^2+4(2p-1)x+4}\right), \\ z_2 &= \frac{1}{2}\left(x - \sqrt{x^2+4(2p-1)x+4}\right), \\ q &= \frac{x+4p-2+\sqrt{x^2+4(2p-1)x+4}}{2\sqrt{x^2+4(2p-1)x+4}}. \end{aligned}$$

This example shows that convolution from the left by a Dirac mass is in general not equal to a translation and that the additive monotone convolution is not affine in the second argument.

Note that the continuity and the fact that the monotone convolution is affine in the first argument imply the following formula

$$\mu \rhd \nu = \int_{\mathbb{R}} \delta_x \rhd \nu \, \mathrm{d}\mu(x) \tag{5.2}$$

for all $\mu, \nu \in \mathcal{M}_1(\mathbb{R})$.

The following proposition is the key to treating the additive monotone convolution for general probability measures on $\mathbb{R}$.

Proposition 5.15. *Let μ and ν be two probability measures on $\mathbb{R}$ and denote by M_x and M_y the self-adjoint operators on $L^2(\mathbb{R}\times\mathbb{R}, \mu\otimes\nu)$ defined by multiplication with the coordinate functions. Denote by P_2 the orthogonal projection onto the subspace of functions which do not depend on the second coordinate, $L^2(\mathbb{R}\times\mathbb{R}, \mu\otimes \nu) \ni \psi \mapsto \int_{\mathbb{R}} \psi(\cdot, y)\mathrm{d}\nu(y) \in L^2(\mathbb{R}\times\mathbb{R}, \mu\otimes\nu)$. Then $M_xP_2 = P_2M_x$ and M_y are self-adjoint and monotonically independent w.r.t. the constant function and the operator $z - M_xP_2 - M_y$ has a bounded inverse for all $z \in \mathbb{C}\backslash\mathbb{R}$, given by*

$$\big((z - M_xP_2 - M_y)^{-1}\psi\big)(x, y) = \frac{\psi(x, y)}{z - y} + \frac{x\int_{\mathbb{R}} \frac{\psi(x,y')}{z-y'}\mathrm{d}\nu(y')}{(z - y)(1 - xG_\nu(z))}. \tag{5.3}$$

Proof. M_xP_2 and M_y are monotonically independent by 5.10.

The first term on the right-hand-side of Equation (5.3) is obtained from ψ by multiplication with a bounded function, the second by composition of multiplications with bounded functions and the projection P_2. Equation (5.3) therefore clearly defines a bounded operator. To check that it is indeed the inverse of $z - M_xP_2 - M_y$ is straightforward,

$$\begin{aligned}
&(z - M_xP_2 - M_y)\left(\frac{\psi(x, y)}{z - y} + \frac{x\int_{\mathbb{R}} \frac{\psi(x,y')}{z-y'}\mathrm{d}\nu(y')}{(z - y)\big(1 - xG_\nu(z)\big)}\right)\\
&= \psi(x,y) + \frac{x\int \frac{\psi(x,y')}{z-y'}\mathrm{d}\nu(y')}{1 - xG_\nu(z)} - x\int_{\mathbb{R}} \frac{\psi(x,y')}{z-y'}\mathrm{d}\nu(y') - x\int_{\mathbb{R}} \frac{x\int_{\mathbb{R}} \frac{\psi(x,y'')}{z-y''}\mathrm{d}\nu(y'')}{(z-y')\big(1 - xG_\nu(z)\big)}\mathrm{d}\nu(y')\\
&= \psi(x,y) + \frac{\Big((z-y) - (z-y)\big(1 - xG_\nu(z)\big) - xG_\nu(z)(z-y)\Big)x\int_{\mathbb{R}} \frac{\psi(x,y')}{z-y'}\mathrm{d}\nu(y')}{(z-y)\big(1 - xG_\nu(z)\big)}\\
&= \psi(x,y)
\end{aligned}$$

□

Theorem 5.16. *Let X and Y be two self-adjoint operators on a Hilbert space H that are monotonically independent w.r.t. to a unit vector $\Omega \in H$. Assume furthermore that Ω is cyclic, i.e. that*

$$\overline{\mathrm{alg}\{h(X), h(Y); h \in C_b(\mathbb{R})\}\Omega} = H.$$

Then $X+Y$ is essentially self-adjoint and the distribution w.r.t. Ω of its closure is equal to the additive monotone convolution of the distributions of X and Y w.r.t. to Ω, i.e.

$$\mathcal{L}(X+Y,\Omega)=\mathcal{L}(X,\Omega)\triangleright\mathcal{L}(Y,\Omega).$$

Proof. Let $\mu=\mathcal{L}(X,\Omega)$, $\nu=\mathcal{L}(Y,\Omega)$.

By Theorem 5.11 and Lemma **??** it is sufficient to consider the case where X and Y are given by Proposition 5.10. Proposition 5.15 shows that $z-X-Y$ admits a bounded inverse and therefore that $\mathrm{Ran}\,(z-X-Y)$ is dense for $z\in\mathbb{C}\backslash\mathbb{R}$. By [RS80, Theorem VIII.3] this is equivalent to $X+Y$ being essentially self-adjoint.

Using Equation (5.3), we can compute the Cauchy transform of the distribution of the closure of $X+Y$. Let $z\in\mathbb{C}^+$, then we have

$$\begin{aligned}
G_{X+Y}(z)&=\langle\Omega,(z-X-Y)^{-1}\Omega\rangle-=\big\langle\mathbf{1},(z-M_xP_2-M_y)^{-1}\mathbf{1}\big\rangle\\
&=\left\langle\mathbf{1},\frac{1}{z-y}+\frac{xG_\nu(z)}{(z-y)(1-xG_\nu(z))}\right\rangle=\int_{\mathbb{R}\times\mathbb{R}}\frac{1}{(z-y)(1-xG_\nu(z))}\mathrm{d}\mu\otimes\nu\\
&=\int_{\mathbb{R}}\frac{G_\nu(z)}{1-xG_\nu(z)}\mathrm{d}\mu(x)=G_\mu\left(\frac{1}{G_\nu(z)}\right)=G_\mu\big(F_\nu(z)\big),
\end{aligned}$$

or

$$F_{X+Y}(z)=\frac{1}{G_{X+Y}(z)}=\frac{1}{G_\mu\big(F_\nu(z)\big)}=F_\mu\big(F_\nu(z)\big)=F_{\mu\triangleright\nu}(z).$$

□

5.3.2. *Multiplicative monotone convolution on $\mathcal{M}_1(\mathbb{R}_+)$.*

Definition 5.17. [Ber05a] Let μ and ν be two probability measures on the positive half-line $\mathbb{R}_+$ with transforms K_μ and K_ν. Then the multiplicative monotone convolution of μ and ν is defined as the unique probability measure $\lambda=\mu\rhd\nu$ on $\mathbb{R}_+$ with transform $K_\lambda=K_\mu\circ K_\nu$.

It follows from Subsection 5.1 that the multiplicative monotone convolution on $\mathcal{M}_1(\mathbb{R}_+)$ is well-defined.

Let us first recall some basic properties of the multiplicative monotone convolution.

Proposition 5.18. *The multiplicative monotone convolution $\mathcal{M}_1(\mathbb{R}_+)$ is associative and $*$-weakly continuous in both arguments. It is affine in the first argument and convolution from the right by a Dirac measure corresponds to dilation, i.e. $\mu\rhd\delta_\alpha=D_\alpha^{-1}\mu$ for $\alpha\in\mathbb{R}_+$, where $D_\alpha:\mathbb{R}_+\to\mathbb{R}_+$ is defined by $D_\alpha(t)=\alpha t$.*

This convolution is not commutative, i.e. in general we have $\mu\rhd\nu\neq\nu\rhd\mu$. As in the additive case is not affine in the second argument, either, and convolution from the left by a Dirac mass is in general not equal to a dilation.

We want to extend [Fra06, Corollary 4.3] to unbounded positive operators, i.e. we want to show that if X and Y are two positive operators such that $X-\mathbf{1}$

and Y are monotonically independent, then the distribution of $\sqrt{X}Y\sqrt{X}$ is equal to the multiplicative monotone convolution of the distributions of X and Y. By Theorem 5.11, it is sufficient to do the calculations for the case where X and Y are constructed from multiplication with the coordinate functions and the projection P_2.

Proposition 5.19. *Let μ and ν be two probability measures on $\mathbb{R}_+$, $\nu\neq\delta_0$, and let M_y be the self-adjoint operator on $L^2(\mathbb{R}_+\times\mathbb{R}_+,\mu\otimes\nu)$ defined by multiplication with the coordinate function $(x,y)\mapsto y$. Define S_x on $L^2(\mathbb{R}_+\times\mathbb{R}_+,\mu\otimes\nu)$ by*

(5.4)

$$\begin{aligned}\mathrm{Dom}\,S_x &= \left\{\psi\in L^2(\mathbb{R}_+\times\mathbb{R}_+,\mu\otimes\nu);\int_{\mathbb{R}_+}x\psi(x,y)\mathrm{d}\nu(y)\in L^2(\mathbb{R}_+,\mu)\right\},\\ (S_x\psi)(x,y) &= (x-1)\int_{\mathbb{R}_+}\psi(x,y)\mathrm{d}\nu(y)+\psi(x,y)\end{aligned}$$

Then $S_x-\mathbf{1}$ and M_y are monotonically independent w.r.t. to the constant function and the operator $z-\sqrt{S_x}M_y\sqrt{S_x}$ has a bounded inverse for all $z\in\mathbb{C}\backslash\mathbb{R}$, given by

$$\left((z-\sqrt{S_x}M_y\sqrt{S_x})^{-1}\psi\right)(x,y)=\frac{\psi(x,y)+g(x)}{z-y}+h(x). \tag{5.5}$$

where

$$\begin{aligned} g(x) &= \frac{\sqrt{x}-x}{(1-x)zG_\nu(z)+x}\int_{\mathbb{R}_+}\psi(x,y)\mathrm{d}\nu(y)\\ &\quad+\frac{z(x-1)}{(1-x)zG_\nu(z)+x}\int_{\mathbb{R}_+}\frac{\psi(x,y)}{z-y}\mathrm{d}\nu(y),\\ h(x) &= \frac{(\sqrt{x}-1)^2G_\nu(z)}{(1-x)zG_\nu(z)+x}\int_{\mathbb{R}_+}\psi(x,y)\mathrm{d}\nu(y)\\ &\quad+\frac{\sqrt{x}-x}{(1-x)zG_\nu(z)+x}\int_{\mathbb{R}_+}\frac{\psi(x,y)}{z-y}\mathrm{d}\nu(y). \end{aligned}$$

Proof. Fix $z\in\mathbb{C}^+$. Let $x>0$, then

$$\mathrm{Im}\frac{z}{z-x}=-\frac{x\mathrm{Im}\,z}{(\mathrm{Re}\,z-x)^2+(\mathrm{Im}\,z)^2}<0,$$

and therefore

$$\mathrm{Im}\,zG_\nu(z)=\mathrm{Im}\int_{\mathbb{R}_+}\frac{z}{z-x}\mathrm{d}\nu(x)<0.$$

Similarly, we get $\mathrm{Im}\,zG_\nu(z)>0$ for $z\in\mathbb{C}^-$. It follows that the functions in front of the integrals in the definitions of g and h are bounded as functions of x, and therefore g and h are square-integrable. Since $\frac{1}{z-y}$ is bounded, too, we see that Equation (5.5) defines a bounded operator.

Let us now check that it is the inverse of $z-\sqrt{S_x}M_y\sqrt{S_x}$.

Using the notation of the previous subsection, we can write S_x also as $S_x = M_{x-1}P_2 + \mathbf{1} = M_xP_2 + P_2^\perp$, where $P_2^\perp$ is the projection onto the orthogonal complement of the subspace of functions which do not depend on y. Its square root can be written as $\sqrt{S_x} = M_{\sqrt{x}}P_2 + P_2^\perp = M_{\sqrt{x}-1}P_2 + \mathbf{1}$, it acts as

$$\left(\sqrt{S_x}\psi\right)(x,y) = \left(\sqrt{x}-1\right)\int_{\mathbb{R}_+}\psi(x,y)\mathrm{d}\nu(y) + \psi(x,y)$$

on a function $\psi\in\mathrm{Dom}\sqrt{S_x}\subseteq L^2(\mathbb{R}_+\times\mathbb{R}_+,\mu\otimes\nu)$.

Since h does not depend on y, we have $\sqrt{S_x}h = \sqrt{x}h$. For g we get

$$\begin{aligned}\left(\sqrt{S_x}\frac{g}{z-y}\right)(x) &= (\sqrt{x}-1)\int_{\mathbb{R}_+}\frac{g(x)}{z-y}\mathrm{d}\nu(y) + \frac{g(x)}{z-y}\\ &= \left((\sqrt{x}-1)G_\nu(z) + \frac{1}{z-y}\right)g(x).\end{aligned}$$

Set $\varphi = \frac{\psi+g}{z-y} + h$. Applying $\sqrt{S_x}$ to φ, we get

$$\begin{aligned}\left(\sqrt{S_x}\varphi\right)(x,y) &= \frac{\psi(x,y)}{z-y} + \frac{\sqrt{x}-x}{(z-y)\big((1-x)zG_\nu(z)+x\big)}\int_{\mathbb{R}_+}\psi(x,y)\mathrm{d}\nu(y)\\ &\quad + \frac{z(x-1)}{(z-y)\big((1-x)zG_\nu(z)+x\big)}\int_{\mathbb{R}_+}\frac{\psi(x,y)}{z-y}\mathrm{d}\nu(y)\\ &= \frac{\psi(x,y)+g(x)}{z-y}.\end{aligned}$$

From this we get

$$\left(\left(z-\sqrt{S_x}M_y\sqrt{S_x}\right)\varphi\right)(x,y) = \psi(x,y)$$

after some tedious, but straightforward computation. □

Remark 5.20. It $\nu=\delta_0$, then $M_y = 0$ on $L^2(\mathbb{R}_+\times\mathbb{R}_+,\mu\otimes\nu)$, and therefore $\sqrt{S_x}M_y\sqrt{S_x}=0$. This is of course a positive operator, and its distribution is δ_0.

Theorem 5.21. *Let X and Y be two positive self-adjoint operators on a Hilbert space H such that $X-\mathbf{1}$ and Y are monotonically independent w.r.t. to a unit vector $\Omega\in H$. Assume furthermore that Ω is cyclic, i.e.*

$$\overline{\mathrm{alg}\{h(X),h(Y);h\in C_b(\mathbb{R}_+)\}\Omega} = H.$$

Then $\sqrt{X}Y\sqrt{X}$ is essentially self-adjoint and the distribution w.r.t. Ω of its closure is equal to the multiplicative monotone convolution of the distributions of X and Y w.r.t. Ω, i.e.

$$\mathcal{L}\left(\sqrt{X}Y\sqrt{X},\Omega\right) = \mathcal{L}(X,\Omega)\rhd\mathcal{L}(Y,\Omega).$$

Proof. Let $\mu = \mathcal{L}(X,\Omega)$, $\nu = \mathcal{L}(Y,\Omega)$.

By Theorem 5.11 it is sufficient to consider the case $X = S_x$ and $Y = M_y$. In this case Proposition 5.19 shows that $z - \sqrt{X}Y\sqrt{X}$ has a bounded inverse for all $z \in \mathbb{C}\backslash\mathbb{R}$. This implies that $\mathrm{Ran}(z - \sqrt{X}Y\sqrt{X})$ is dense for all $z \in \mathbb{C}\backslash\mathbb{R}$ and that $\sqrt{X}Y\sqrt{X}$ is essentially self-adjoint, cf. [RS80, Theorem VIII.3].

Using Equation (5.5), we can compute the Cauchy transform of the distribution of the closure of $\sqrt{X}Y\sqrt{X}$. Let $z \in \mathbb{C}^+$, then we have

$$\begin{aligned} G_{\sqrt{X}Y\sqrt{X}}(z) &= \left\langle \Omega, \left(z - \sqrt{X}Y\sqrt{X}\right)^{-1}\Omega\right\rangle = \left\langle \mathbf{1}, \left(z - \sqrt{S_x}M_y\sqrt{S_x}\right)^{-1}\mathbf{1}\right\rangle \\ &= \left\langle \mathbf{1}, \frac{1+g_1}{z-y} + h_1\right\rangle \end{aligned}$$

where

$$\begin{aligned} g_1(x) &= \frac{\sqrt{x} - x + (x-1)zG_\nu(x)}{(1-x)zG_\nu(z) + x} = \frac{\sqrt{x}}{(1-x)zG_\nu(z)+x} - 1, \\ h_1(x) &= \frac{(1-\sqrt{x})G_\nu(z)}{(1-x)zG_\nu(z)+x}. \end{aligned}$$

Therefore

$$\begin{aligned} G_{\sqrt{X}Y\sqrt{X}}(z) &= \int_{\mathbb{R}_+\times\mathbb{R}_+} \left(\frac{1+g_1(x)}{z-y} + h_1(x)\right) \mathrm{d}\mu\otimes\nu(x,y) \\ &= \int_{\mathbb{R}_+} \frac{G_\nu(z)}{(1-x)zG_\nu(z)+x}\mathrm{d}\mu(x) = \frac{G_\nu(z)}{zG_\nu(z)-1} G_\mu\left(\frac{zG_\nu(z)}{zG_\nu(z)-1}\right). \end{aligned} \tag{5.6}$$

Using the relation

$$G_\mu(z) = \frac{1}{z}\left(\psi_\mu\left(\frac{1}{z}\right) + 1\right)$$

to replace the Cauchy transforms by the ψ-transforms, this becomes

$$\psi_{\sqrt{X}Y\sqrt{X}}\left(\frac{1}{z}\right) = \psi_\mu\left(\frac{\psi_\nu(1/z)}{\psi_\nu(1/z)+1}\right),$$

or finally

$$K_{\sqrt{X}Y\sqrt{X}}(z) = K_\mu\big(K_\nu(z)\big) = K_{\mu\triangleright\nu}(z).$$

□

5.3.3. *Multiplicative monotone convolution on $\mathcal{M}_1(\mathbb{T})$.*

Definition 5.22. [Ber05a] Let μ and ν be two probability measure on the unit circle $\mathbb{T}$ with transforms K_μ and K_ν. Then the multiplicative monotone convolution of μ and ν is defined as the unique probability measure $\lambda = \mu \triangleright \nu$ on $\mathbb{T}$ with transform $K_\lambda = K_\mu \circ K_\nu$.

It follows from Subsection 5.1 that the multiplicative monotone convolution on $\mathcal{M}_1(\mathbb{T})$ is well-defined.

Let us first recall some basic properties of the multiplicative monotone convolution.

Proposition 5.23. *The multiplicative monotone convolution on $\mathcal{M}_1(\mathbb{T})$ is associative and $*$-weakly continuous in both arguments. It is affine in the first argument and convolution from the right by a Dirac measure corresponds to rotation, i.e. $\mu \rhd \delta_{e^{i\vartheta}} = R_\vartheta^{-1}\mu$ for $\vartheta \in [0, 2\pi[$, where $R_\vartheta : \mathbb{T} \to \mathbb{T}$ is defined by $R_\vartheta(t) = e^{i\vartheta}t$.*

This convolution is not commutative, i.e. in general we have $\mu \rhd \nu \neq \nu \rhd \mu$. As in the additive case is not affine in the second argument, either, and convolution from the left by a Dirac mass is in general not equal to a rotation.

Probability measures on the unit circle arise as distributions of unitary operators and they are completely characterized by their moments. Therefore the following theorem is a straightforward consequence of [Ber05a] (see also [Fra06, Theorem 4.1 and Corollary 4.2]).

Theorem 5.24. *Let U and V be two unitary operators on a Hilbert space H, $\Omega \in H$ a unit vector and assume furthermore that $U - \mathbf{1}$ and V are monotonically independent w.r.t. Ω. Then the products UV and VU are also unitary and their distribution w.r.t. Ω is equal to the multiplicative monotone convolution of the distributions of U and V, i.e.*

$$\mathcal{L}(UV, \Omega) = \mathcal{L}(VU, \Omega) = \mathcal{L}(U, \Omega) \rhd \mathcal{L}(V, \Omega). \tag{5.7}$$

Remark 5.25. Note that the order of the convolution product on the right-hand-side of Equation (5.7) depends only on the order in which the operators $U - \mathbf{1}$ and $V - \mathbf{1}$ are monotonically independent, but not on the order in which U and V are multiplied.

5.4. **Boolean Convolutions.**

Definition 5.26. Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{B}(H)$ be two $*$-algebras of bounded operators on a Hilbert space H, and let $\Omega \in H$ be a unit vector. We say that $\mathcal{A}_1$ and $\mathcal{A}_2$ are boolean independent w.r.t. Ω, if we have

$$\langle \Omega, X_1 X_2 \cdots X_k \Omega \rangle = \prod_{\kappa=1}^{k} \langle \Omega, X_\kappa \Omega \rangle$$

for all $k \in \mathbb{N}$, $\varepsilon \in \mathbb{A}_k$, $X_1 \in \mathcal{A}_{\varepsilon_1}, \ldots, X_k \in \mathcal{A}_{\varepsilon_k}$.

Remark 5.27. The algebras are not required to be unital. If one of them is unital, say $\mathcal{A}_1$, then the restriction of $\Phi(\cdot) = \langle \Omega, \cdot\, \Omega \rangle$ to the other algebra, say $\mathcal{A}_2$, has to be a homomorphism, since the boolean independence implies

$$\langle \Omega, XY\Omega \rangle = \langle \Omega, X\mathbf{1}Y\Omega \rangle = \langle \Omega, X\Omega \rangle \langle \Omega, Y\Omega \rangle$$

for $X, Y \in \mathcal{A}_2$.

Definition 5.28. Let X and Y be two normal operators on a Hilbert space H, not necessarily bounded. We say that X and Y are boolean independent, if the $*$-algebras $\mathrm{alg}_0(X) = \{h(X) : h \in C_b(\mathbb{C}), h(0) = 0\}$ and $\mathrm{alg}_0(Y) = \{h(Y) : h \in C_b(\mathbb{C}), h(0) = 0\}$ are boolean independent.

We will start by characterizing up to unitary transformations the general form of two boolean independent normal operators. Given a measure space $(M, \mathcal{M}, \mu)$, we shall denote by $L^2(M, \mu)_0$ the orthogonal complement of the constant function, i.e.

$$L^2(M, \mu)_0 = \left\{ \psi \in L^2(M, \mu); \int_M \psi \mathrm{d}\mu = 0 \right\}.$$

Proposition 5.29. *Let μ, ν be two probability measures on $\mathbb{C}$ and define normal operators N_x and N_y on $\mathbb{C} \oplus L^2(\mathbb{C}, \mu)_0 \oplus L^2(\mathbb{C}, \nu)_0$ by*

$$\mathrm{Dom}\, N_x = \left\{ \left(\begin{array}{c} \alpha \\ \psi_1 \\ \psi_2 \end{array} \right) \in \mathbb{C} \oplus L^2(\mathbb{C}, \mu)_0 \oplus L^2(\mathbb{C}, \nu)_0; \int_{\mathbb{C}} \left| x\big(\psi_1(x) + \alpha\big) \right|^2 \mathrm{d}\mu(x) < \infty \right\},$$

$$\mathrm{Dom}\, N_y = \left\{ \left(\begin{array}{c} \alpha \\ \psi_1 \\ \psi_2 \end{array} \right) \in \mathbb{C} \oplus L^2(\mathbb{C}, \mu)_0 \oplus L^2(\mathbb{C}, \nu)_0; \int_{\mathbb{C}} \left| y\big(\psi_2(y) + \alpha\big) \right|^2 \mathrm{d}\nu(y) < \infty \right\},$$

$$N_x \left(\begin{array}{c} \alpha \\ \psi_1 \\ \psi_2 \end{array} \right) = \left(\begin{array}{c} \int_{\mathbb{C}} x\big(\psi_1(x) + \alpha\big) \mathrm{d}\mu(x) \\ x(\psi_1 + \alpha) - \int_{\mathbb{C}} x\big(\psi_1(x) + \alpha\big) \mathrm{d}\mu(x) \\ 0 \end{array} \right),$$

$$N_y \left(\begin{array}{c} \alpha \\ \psi_1 \\ \psi_2 \end{array} \right) = \left(\begin{array}{c} \int_{\mathbb{C}} y\big(\psi_2(y) + \alpha\big) \mathrm{d}\nu(y) \\ 0 \\ x(\psi_2 + \alpha) - \int_{\mathbb{C}} y\big(\psi_2(y) + \alpha\big) \mathrm{d}\nu(y) \end{array} \right).$$

Then N_x and N_y are boolean independent w.r.t. the vector $\omega = \left(\begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right)$ and we have $\mathcal{L}(N_x, \omega) = \mu$, $\mathcal{L}(N_y, \omega) = \nu$.

Proof. Under the identification $\mathbb{C} \oplus L^2(\mathbb{C}, \mu)_0 \oplus L^2(\mathbb{C}, \nu)_0 \cong L^2(\mathbb{C}, \mu) \oplus L^2(\mathbb{C}, \nu)_0$, where

$$\left(\begin{array}{c} \alpha \\ \psi_1 \\ \psi_2 \end{array} \right) \cong \left(\begin{array}{c} \psi_1 + \alpha \\ \psi_2 \end{array} \right),$$

the operator N_x becomes multiplication by the variable x on $L^2(\mathbb{C}, \mu)$. It is clearly normal and we have

$$h(N_x) \left(\begin{array}{c} \alpha \\ \psi_1 \\ \psi_2 \end{array} \right) = \left(\begin{array}{c} \int_{\mathbb{C}} h(x)\big(\alpha + \psi_1(x)\big) \mathrm{d}\mu(x) \\ h(\alpha + \psi_1) - \int_{\mathbb{C}} h(x)\big(\alpha + \psi_1(x)\big) \mathrm{d}\mu(x) \\ h(0)\psi_2 \end{array} \right)$$

and $\langle\omega, h(N_x)\omega\rangle = \int_{\mathbb{C}} h\mathrm{d}\mu$ for all $h \in C_b(\mathbb{C})$, i.e. $\mathcal{L}(N_x, \omega) = \mu$. Similarly

$$h(N_y)\begin{pmatrix}\alpha\\ \psi_1\\ \psi_2\end{pmatrix} = \begin{pmatrix}\int_{\mathbb{C}} h(y)\big(\alpha+\psi_2(y)\big)\mathrm{d}\nu(y)\\ h(0)\psi_1\\ h(\alpha+\psi_2) - \int_{\mathbb{C}} h(y)\big(\alpha+\psi_2(y)\big)\mathrm{d}\nu(y)\end{pmatrix}$$

for all $h \in C_b(\mathbb{C})$, and $\mathcal{L}(N_y, \omega) = \nu$.

Let $f_1, \ldots, f_n, g_1, \ldots, g_n \in C_b(\mathbb{C})$, with $f_1(0) = \cdots = f_n(0) = g_1(0) = \cdots = g_n(0) = 0$. Then

$$f_n(N_x)g_{n-1}(N_y)\cdots g_1(N_y)f_1(N_x)\omega = \begin{pmatrix}\prod_{k=1}^n \int_{\mathbb{C}} f_k\mathrm{d}\mu \prod_{\ell=1}^{n-1}\int_{\mathbb{C}} g_\ell\mathrm{d}\nu\\ \prod_{k=1}^{n-1}\int_{\mathbb{C}} f_k\mathrm{d}\mu \prod_{\ell=1}^{n-1}\int_{\mathbb{C}} g_\ell\mathrm{d}\nu\left(f_n - \int_{\mathbb{C}} f_n\mathrm{d}\mu\right)\\ 0\end{pmatrix}$$

and therefore

$$\begin{aligned}\langle\omega, f_n(N_x)g_{n-1}(N_y)\cdots g_1(N_y)f_1(N_x)\omega\rangle &= \prod_{k=1}^n \int_{\mathbb{C}} f_k\,\mathrm{d}\mu \prod_{\ell=1}^{n-1}\int_{\mathbb{C}} g_\ell\,\mathrm{d}\nu\\ &= \prod_{k=1}^n\langle\omega, f_k(N_x)\omega\rangle\prod_{\ell=1}^{n-1}\langle\omega, g_\ell(N_y)\omega\rangle\end{aligned}$$

i.e. the condition for boolean independence is satisfied in this case. Similarly one checks the expectation of $g_n(N_y)f_n(N_x)\cdots g_1(N_y)f_1(N_x)$, $f_n(N_x)g_n(N_y)\cdots f_1(N_x)g_1(N_y)$, and $g_n(N_y)f_{n-1}(N_x)\cdots f_1(N_x)g_1(N_y)$. □

We shall now show that any pair of boolean independent normal operators can be reduced to this model.

Theorem 5.30. *Let X and Y be two normal operators on a Hilbert space H that are boolean independent w.r.t. to $\Omega \in H$ and let $\mu = \mathcal{L}(X,\Omega)$, $\nu = \mathcal{L}(Y,\Omega)$.*

Then there exists an isometry $W : \mathbb{C}\oplus L^2(\mathbb{C},\mu)_0 \oplus L^2(\mathbb{C},\nu)_0 \to H$ such that

(5.8)
$$\begin{aligned}W^*h(X)W\begin{pmatrix}\alpha\\ \psi_1\\ \psi_2\end{pmatrix} &= \begin{pmatrix}\int_{\mathbb{C}} h(x)\big(\alpha+\psi_1(x)\big)\mathrm{d}\mu(x)\\ h(\alpha+\psi_1) - \int_{\mathbb{C}} h(x)\big(\alpha+\psi_1(x)\big)\mathrm{d}\mu(x)\\ h(0)\psi_2\end{pmatrix},\\ W^*h(Y)W\begin{pmatrix}\alpha\\ \psi_1\\ \psi_2\end{pmatrix} &= \begin{pmatrix}\int_{\mathbb{C}} h(y)\big(\alpha+\psi_2(y)\big)\mathrm{d}\nu(y)\\ h(0)\psi_1\\ h(\alpha+\psi_2) - \int_{\mathbb{C}} h(y)\big(\alpha+\psi_2(y)\big)\mathrm{d}\nu(y)\end{pmatrix}\end{aligned}$$

for all $h \in C_b(\mathbb{C})$, $\alpha \in \mathbb{C}$, $\psi_1 \in L^2(\mathbb{C},\mu)_0$, $\psi_2 \in L^2(\mathbb{C},\nu)_0$.

We have $W\big(\mathbb{C}\oplus L^2(\mathbb{C},\mu)_0\oplus L^2(\mathbb{C},\nu)_0\big) = \overline{\mathrm{alg}\{h(X), h(Y) : h \in C_b(\mathbb{C})\}\Omega}$.

If the vector $\Omega \in H$ is cyclic for the algebra $\mathrm{alg}(X,Y) = \mathrm{alg}\{h(X), h(Y) : h \in C_b(\mathbb{C})\}$ generated by X and Y, then W is unitary.

Proof. For a probability measure μ on $\mathbb{C}$, let

$$C_b(\mathbb{C})_{\mu,0} = \left\{ f \in C_b(\mathbb{C}); \int_{\mathbb{C}} f(z)\mathrm{d}\mu(x) = 0 \right\},$$

then $C_b(\mathbb{C})_{\mu,0}$ is dense in $L^2(\mathbb{C},\mu)_0$.

Define $W : \mathbb{C} \oplus C_b(\mathbb{C})_{\mu,0} \oplus C_b(\mathbb{C})_{\nu,0} \to H$ by

$$W \begin{pmatrix} \alpha \\ f \\ g \end{pmatrix} = \big(\alpha + f(X) + g(Y)\big)\Omega.$$

This is an isometry, since

$$\begin{aligned} \left\langle W \begin{pmatrix} \alpha_1 \\ f_1 \\ g_1 \end{pmatrix}, W \begin{pmatrix} \alpha_2 \\ f_2 \\ g_2 \end{pmatrix} \right\rangle &= \big\langle \big(\alpha_1 + f_1(X) + g_1(Y)\big)\Omega, \big(\alpha_2 + f_2(X) + g_2(Y)\big)\Omega \big\rangle \\ &= \overline{\alpha_1}\alpha_2 + \int_{\mathbb{C}} \overline{f_1(x)} f_2(x)\mathrm{d}\mu(x) + \int_{\mathbb{C}} \overline{g_1(y)} g_2(y)\mathrm{d}\mu(y), \end{aligned}$$

where the mixed terms all vanish because $\langle \Omega, f_i(X)\Omega\rangle = \langle \Omega, g_i(Y)\Omega\rangle = 0$ for $i = 1, 2$. Therefore W extends in a unique way to an isometry on $\mathbb{C} \oplus L^2(\mathbb{C},\mu)_0 \oplus L^2(\mathbb{C},\nu)_0$

Let now $h \in C_b(\mathbb{C})$, then we get

$$\begin{aligned} &\left\langle W \begin{pmatrix} \alpha_1 \\ f_1 \\ g_1 \end{pmatrix}, h(X) W \begin{pmatrix} \alpha_2 \\ f_2 \\ g_2 \end{pmatrix} \right\rangle \\ &\quad = \big\langle \big(\alpha_1 + f_1(X) + g_1(Y)\big)\Omega, \big(h(X) - h(0)\mathbf{1}\big)\big(\alpha_2 + f_2(X) + g_2(Y)\big)\Omega \big\rangle \\ &\qquad + h(0) \left\langle W \begin{pmatrix} \alpha_1 \\ f_1 \\ g_1 \end{pmatrix}, W \begin{pmatrix} \alpha_2 \\ f_2 \\ g_2 \end{pmatrix} \right\rangle \\ &= \big\langle \big(\alpha_1 + f_1(X)\big)\Omega, \big(h(X) - h(0)\mathbf{1}\big)\big(\alpha_2 + f_2(X)\big)\Omega \big\rangle + h(0) \left\langle \begin{pmatrix} \alpha_1 \\ f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ f_2 \\ g_2 \end{pmatrix} \right\rangle, \end{aligned}$$

because the boolean independence and $\langle\Omega, g_i(Y)\Omega\rangle = 0$ imply that all other terms vanish. But since $\langle\Omega, f_i(Y)\Omega\rangle = 0$, this is equal to

$$\begin{aligned}
&\Big\langle\big(\alpha_1+f_1(X)\big)\Omega, h(X)\big(\alpha_2+f_2(X)\big)\Omega\Big\rangle \\
&\qquad + h(0)\left(\left\langle\begin{pmatrix}\alpha_1\\ f_1\\ g_1\end{pmatrix},\begin{pmatrix}\alpha_2\\ f_2\\ g_2\end{pmatrix}\right\rangle - \overline{\alpha_1}\alpha_2 - \langle f_1, f_2\rangle\right) \\
&\qquad = \left\langle\begin{pmatrix}\alpha_1\\ f_1\\ g_1\end{pmatrix},\begin{pmatrix}\int h(x)\big(f_2(x)+\alpha_2\big)\mathrm{d}\mu(x)\\ h(f_2+\alpha_2) - \int h(x)\big(f_2(x)+\alpha_2\big)\mathrm{d}\mu(x)\\ h(0)g_2\end{pmatrix}\right\rangle.
\end{aligned}$$

This proves the first formula. The second formula follows by symmetry.

Let $f, g \in C_b(\mathbb{C})$, $f(0) = 0$, and note that

$$\begin{aligned}
&\left|\left|f(X)g(Y)\Omega - \int_{\mathbb{C}} g\mathrm{d}\nu\, f(X)\Omega\right|\right|^2 \\
= \;& \langle\Omega, g(Y)^*|f(X)|^2 g(Y)\Omega - \int_{\mathbb{C}} g\mathrm{d}\nu\, \langle\Omega, g(Y)^*|f(X)|^2\Omega \\
& - \int_{\mathbb{C}} \overline{g}\mathrm{d}\nu\, \langle\Omega, |f(X)|^2 g(Y)\Omega\rangle + \left(\int_{\mathbb{C}} g\mathrm{d}\nu\right)^2 \langle\Omega, |f(X)|^2\Omega\rangle \\
= \;& 0,
\end{aligned}$$

i.e. $f(X)g(Y)\Omega = \int_{\mathbb{C}} g\mathrm{d}\nu\, f(X)\Omega$. Similarly $f(Y)g(X)\Omega = \int_{\mathbb{C}} g\mathrm{d}\mu\, f(Y)\Omega$ and thus

$$\begin{aligned}
\overline{\mathrm{alg}\{h(X), h(Y) : h \in C_b(\mathbb{C})\}\Omega} &= \overline{\mathrm{span}\,\{\Omega, f(X)\Omega, f(Y)\Omega; f \in C_b(\mathbb{C})\}} \\
&= W\big(\mathbb{C}\oplus L^2(\mathbb{C},\mu)_0 \oplus L^2(\mathbb{C},\nu)_0\big).
\end{aligned}$$

If Ω is cyclic, then W is surjective and therefore unitary. □

Remark 5.31. As in the monotone case, cf. Remark 5.12, this theorem shows that joint law of bounded functions on X and Y is uniquely determined by $\mathcal{L}(X,\Omega)$ and $\mathcal{L}(Y,\Omega)$. Furthermore, the characterisation and computation of the law of unbounded functions of X and Y like, e.g., $X+Y$ or $\sqrt{X}Y\sqrt{Y}$, is also reduced to the model introduced in Proposition 5.29.

5.4.1. *Additive boolean convolution on $\mathcal{M}_1(\mathbb{R})$.*

Definition 5.32. [SW97] Let μ and ν be two probability measures on $\mathbb{R}$ with reciprocal Cauchy transforms F_μ and F_ν. Then we define the additive monotone convolution $\lambda = \mu \uplus \nu$ of μ and ν as the unique probability measure λ on $\mathbb{R}$ with reciprocal Cauchy transform given by

$$F_\lambda(z) = F_\mu(z) + F_\nu(z) - z$$

for $z \in \mathbb{C}^+$.

That the additive boolean convolution is well-defined follows from Subsection 5.1. It is commutative and associative, $*$-weakly continuous, but not affine, cf. [SW97].

Proposition 5.33. *Let μ and ν be two probabilities on $\mathbb{R}$ and define operators N_x and N_y as in Proposition 5.29. Then N_x and N_y are self-adjoint and boolean independent w.r.t. $\omega = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Furthermore, the operator $z - N_x - N_y$ has a bounded inverse for all $z \in \mathbb{C}\backslash\mathbb{R}$, given by*

$$(z - N_x - N_y)^{-1} \begin{pmatrix} \alpha \\ \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \beta \\ \frac{\psi_1 + \beta x - c_x}{z-x} \\ \frac{\psi_2 + \beta y - c_y}{z-y} \end{pmatrix}, \tag{5.9}$$

where

$$\beta = \frac{\alpha G_\mu(z) G_\nu(z) + G_\nu(z) \int_\mathbb{R} \frac{\psi_1(x)}{z-x} \mathrm{d}\mu(x) + G_\mu(z) \int_\mathbb{R} \frac{\psi_2(y)}{z-y} \mathrm{d}\nu(y)}{G_\mu(z) + G_\nu(z) - z G_\mu(z) G_\nu(z)}, \tag{5.10}$$

and $c_x, c_y \in \mathbb{C}$ have to be chosen such that

$$\int_\mathbb{R} \frac{\psi_1(x) + \beta x - c_x}{z - x} \mathrm{d}\mu(x) = 0 = \int_\mathbb{R} \frac{\psi_2(y) + \beta y - c_y}{z - y} \mathrm{d}\nu(y). \tag{5.11}$$

Note that Equation (5.11) yields the following formulas for the constants c_x, c_y,

$$\begin{aligned} c_x &= \frac{\int \frac{\psi_1(x)}{z-x} \mathrm{d}\mu(x) + \beta\big(z G_\mu(z) - 1\big)}{G_\mu(z)}, \\ c_y &= \frac{\int \frac{\psi_2(y)}{z-y} \mathrm{d}\nu(y) + \beta\big(z G_\nu(z) - 1\big)}{G_\nu(z)}. \end{aligned}$$

Proof. N_x and N_y are boolean independent by Proposition 5.29.

For $z \in \mathbb{C}^+$, we have $\mathrm{Im}\, F_\mu(z) \geq \mathrm{Im}\, z > 0$, $\mathrm{Im}\, F_\nu(z) \geq \mathrm{Im}\, z > 0$, and therefore

$$\mathrm{Im} \frac{G_\mu(z) + G_\nu(z) - z G_\mu(z) G_\nu(z)}{G_\mu(z) G_\nu(z)} = \mathrm{Im}\,\big(F_\mu(z) + F_\nu(z) - z\big) > 0.$$

This shows that the denominator of the right-hand-side of Equation (5.10) can not vanish for $z \in \mathbb{C}^+$. Since $G_\mu(\overline{z}) = \overline{G_\mu(z)}$, $G_\nu(\overline{z}) = \overline{G_\nu(z)}$, it can not vanish for z with $\mathrm{Im}\, z < 0$, either. The functions $\frac{1}{z-x}$ and $\frac{x}{z-x}$ are bounded on $\mathbb{R}$ for $z \in \mathbb{C}\backslash\mathbb{R}$, therefore Equation (5.9) defines a bounded operator.

Let

$$\varphi_1 = \frac{\psi_1 + \beta x - c_x}{z - x} \qquad \text{and} \qquad \varphi_2 = \frac{\psi_2 + \beta y - c_y}{z - y},$$

then

$$(z - N_x - N_y)\begin{pmatrix} \beta \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} z\beta + d_x + d_y \\ (z-x)\varphi_1 - \beta x - d_x \\ (z-y)\varphi_2 - \beta y - d_y \end{pmatrix} = \begin{pmatrix} z\beta + d_x + d_y \\ \psi_1 - c_x - d_x \\ \psi_2 - c_y - d_y \end{pmatrix}$$

where

$$d_x = \int x\big(\varphi_1(x) + \beta\big)\mathrm{d}\mu(x), \qquad d_y = \int y\big(\varphi_2(y) + \beta\big)\mathrm{d}\nu(y).$$

Since $\psi_1 \in L^2(\mathbb{R}, \mu)_0$, $\psi_2 \in L^2(\mathbb{R}, \nu)_0$, integrating over the second and third component gives $c_x = -d_x$ and $c_y = -d_y$. Therefore

$$(z - N_x - N_y)\begin{pmatrix} \beta \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} z\beta - c_x - c_y \\ \psi_1 \\ \psi_2 \end{pmatrix}$$

We have to show that the first component is equal to α. We get

$$z\beta - c_x - c_y = z\beta - \frac{\int \frac{\psi_1(x)}{z-x}\mathrm{d}\mu(x) + \beta\big(zG_\mu(z) - 1\big)}{G_\mu(z)} - \frac{\int \frac{\psi_1(x)}{z-x}\mathrm{d}\mu(x) + \beta\big(zG_\nu(z) - 1\big)}{G_\nu(z)}$$
$$= \beta\frac{G_\mu(z) + G_\nu(z) - zG_\mu(z)G_\nu(z)}{G_\mu(z)G_\nu(z)} - \frac{1}{G_\mu(z)}\int \frac{\psi_1(x)}{z-x}\mathrm{d}\mu(x) - \frac{1}{G_\nu(z)}\int \frac{\psi_2(y)}{z-y}\mathrm{d}\nu(y)$$

Substituting Equation (5.10) into this expression, we get the desired result $z\beta - c_x - c_y = \alpha$. □

Theorem 5.34. *Let X and Y be two self-adjoint operators on a Hilbert space H that are boolean independent w.r.t. a unit vector $\Omega \in H$ and assume that Ω is cyclic, i.e. that*

$$\overline{\mathrm{alg}\{h(X), h(Y); h \in C_b(\mathbb{R})\}\Omega} = H.$$

Then $X + Y$ is essentially self-adjoint and the distribution w.r.t. Ω of the closure of $X + Y$ is equal to the boolean convolution of the distributions of X and Y w.r.t. Ω, i.e.

$$\mathcal{L}(X + Y, \Omega) = \mathcal{L}(X, \Omega) \uplus \mathcal{L}(Y, \Omega).$$

Proof. Let $\mu = \mathcal{L}(X, \Omega)$, $\nu = \mathcal{L}(Y, \Omega)$.

By Theorem 5.30 and Lemma **??** it is sufficient to consider the case where X and Y are defined as in Proposition 5.29. Then Proposition 5.33 shows that $z - X - Y$ admits a bounded inverse for all $z \in \mathbb{C}\backslash\mathbb{R}$ and therefore that $\mathrm{Ran}\,(z - X - Y)$ is dense. By [RS80, Theorem VIII.3] this is equivalent to $X + Y$ being essentially self-adjoint.

Using Equation (5.9), we can compute the Cauchy transform of the distribution of the closure of $X+Y$. Let $z \in \mathbb{C}^+$, then

$$\begin{aligned}
G_{X+Y}(z) &= \langle \Omega, (z-X-Y)^{-1}\Omega\rangle = \big\langle \omega, (z-N_x-N_y)^{-1}\omega\big\rangle \\
&= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \frac{G_\mu(z)G_\nu(z)}{G_\mu(z)+G_\nu(z)-zG_\mu(z)G_\nu(z)} \begin{pmatrix} 1 \\ \frac{x-\frac{zG_\mu(z)-1}{G_\mu(z)}}{z-x} \\ \frac{y-\frac{zG_\nu(z)-1}{G_\nu(z)}}{z-y} \end{pmatrix} \right\rangle \\
&= \frac{G_\mu(z)G_\nu(z)}{G_\mu(z)+G_\nu(z)-zG_\mu(z)G_\nu(z)}.
\end{aligned}$$

Replacing all Cauchy transforms by their reciprocals, this becomes

$$F_{X+Y}(z) = F_\mu(z) + F_\nu(z) - z = F_{\mu \uplus \nu}(z).$$

□

5.4.2. *Multiplicative boolean convolution on $\mathcal{M}_1(\mathbb{R}_+)$.* Bercovici defined a boolean convolution for probability measures in the positive half-line, cf. [Ber06].

Definition 5.35. [Ber06] Let μ and ν be two probability measures on $\mathbb{R}_+$ with transforms K_μ and K_ν. If the holomorphic function defined by

$$K(z) = \frac{K_\mu(z)K_\nu(z)}{z} \tag{5.12}$$

for $z \in \mathbb{C}\backslash\mathbb{R}_+$ belongs to the class $\mathcal{P}$ introduced in Subsection 5.1, then the boolean convolution $\lambda = \mu \boxtimes \nu$ is defined as the unique probability measure λ on $\mathbb{R}_+$ with transform $K_\lambda = K$.

But in general the function K defined in Equation (5.12) does not belong to $\mathcal{P}$ and in that case the convolution of μ and ν is not defined. Bercovici has shown that for any probability measure μ on $\mathbb{R}_+$ not concentrated in one point there exists an $n \in \mathbb{N}$ such that the n-fold convolution product $\mu^{\boxtimes n}$ of μ with itself is not defined, cf. [Ber06, Proposition 3.1].

This is of course related to the problem that in general the product of two positive operators is not positive. One might hope that taking e.g. $\sqrt{X}Y\sqrt{X}$ could lead to a better definition of the multiplicative boolean convolution, since this operator will automatically be positive. This leads to a convolution that is always defined, but that is not associative, cf. []

5.4.3. *Multiplicative boolean convolution on $\mathcal{M}_1(\mathbb{T})$.* For completeness we recall the results of [Fra04] for the multiplicative boolean convolution on $\mathcal{M}_1(\mathbb{T})$.

Definition 5.36. [Fra04] Let μ and ν be two probability measures on the unit circle $\mathbb{T}$ with transforms K_μ and K_ν. Then the multiplicative monotone convolution $\lambda = \mu \boxtimes \nu$ is defined as the unique probability on $\mathbb{T}$ with transform K_λ given by

$$K_\lambda(z) = \frac{K_\mu(z)K_\nu(z)}{z}$$

for $z \in \mathbb{D}$.

It is easy to deduce from Subsection 5.1 that the multiplicative boolean convolution on $\mathcal{M}_1(\mathbb{T})$ is well-defined. It is associative, commutative, $*$-weakly continuous in both arguments, but not affine.

Theorem 5.37. [Fra04, Theorem 2.2] *Let U and V be two unitary operators on a Hilbert space H, $\Omega \in H$ a unit vector and assume furthermore that $U - \mathbf{1}$ and $V - \mathbf{1}$ are boolean independent w.r.t. Ω. Then the products UV and VU are also unitary and their distribution w.r.t. Ω is equal to the multiplicative boolean convolution of the distributions of U and V, i.e.*

$$\mathcal{L}(UV,\Omega) = \mathcal{L}(VU,\Omega) = \mathcal{L}(U,\Omega) \mathbin{\underline{\circledast}}\, \mathcal{L}(V,\Omega).$$

6. The Five Universal Independences

In classical probability theory there exists only one canonical notion of independence. But in quantum probability many different notions of independence have been used, e.g., to obtain central limit theorems or to develop a quantum stochastic calculus. If one requires that the joint law of two independent random variables should be determined by their marginals, then an independence gives rise to a product. Imposing certain natural condition, e.g., that functions of independent random variables should again be independent or an associativity property, it becomes possible to classify all possible notions of independence. This program has been carried out in recent years by Schürmann [Sch95a], Speicher [Spe97], Ben Ghorbal and Schürmann [BGS99][BGS02], and Muraki [Mur03, Mur02]. In this section we will present the results of these classifications. Furthermore we will formulate a category theoretical approach to the notion of independence and show that boolean, monotone, and anti-monotone independence can be reduced to tensor independence in a similar way as the bosonization of Fermi independence [HP86] or the symmetrization of [Sch93, Section 3].

The free product of unital associative algebras will play an important role in our discussion.

Example 6.1. The coproduct in the category of unital algebras $\mathfrak{Alg}$ is the *free product* of $*$-algebras *with* identification of the units. Let us recall its defining universal property. Let $\{\mathcal{A}_k\}_{k\in I}$ be a family of unital $*$-algebras and $\coprod_{k\in I}\mathcal{A}_k$ their free product, with canonical inclusions $\{i_k : \mathcal{A}_k \to \coprod_{k\in I}\mathcal{A}_k\}_{k\in I}$. If $\mathcal{B}$ is any unital $*$-algebra, equipped with unital $*$-algebra homomorphisms $\{i'_k : \mathcal{A}_k \to \mathcal{B}\}_{k\in I}$, then there exists a unique unital $*$-algebra homomorphism $h : \coprod_{k\in I}\mathcal{A}_k \to \mathcal{B}$ such that

$$h \circ i_k = i'_k, \qquad \text{for all} \quad k \in I.$$

It follows from the universal property that for any pair of unital $*$-algebra homomorphisms $j_1 : \mathcal{A}_1 \to \mathcal{B}_1$, $j_2 : \mathcal{A}_2 \to \mathcal{B}_2$ there exists a unique unital $*$-algebra homomorphism $j_1 \coprod j_2 : \mathcal{A}_1 \coprod \mathcal{A}_2 \to \mathcal{B}_1 \coprod \mathcal{B}_2$ such that the diagram

$$\begin{tikzcd}
 & \mathcal{A}_1 \arrow[r, "j_1"] \arrow[dl, "i_{\mathcal{A}_1}"'] & \mathcal{B}_1 \arrow[dr, "i_{\mathcal{B}_1}"] & \\
\mathcal{A}_1 \coprod \mathcal{A}_2 \arrow[rrr, dotted, "j_1 \coprod j_2"] & & & \mathcal{B}_1 \coprod \mathcal{B}_2 \\
 & \mathcal{A}_2 \arrow[r, "j_2"'] \arrow[ul, "i_{\mathcal{A}_2}"] & \mathcal{B}_2 \arrow[ur, "i_{\mathcal{B}_2}"'] &
\end{tikzcd}$$

commutes.

The free product $\coprod_{k \in I} \mathcal{A}_k$ can be constructed as a sum of tensor products of the $\mathcal{A}_k$, where neighboring elements in the product belong to different algebras. For simplicity, we illustrate this only for the case of the free product of two algebras. Let

$$\mathbb{A} = \bigcup_{n \in \mathbb{N}} \{\epsilon \in \{1,2\}^n | \epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_n\}$$

and decompose $\mathcal{A}_i = \mathbb{C}\mathbf{1} \oplus \mathcal{A}_i^0$, $i = 1, 2$, into a direct sum of vector spaces. As a coproduct $\mathcal{A}_1 \coprod \mathcal{A}_2$ is unique up to isomorphism, so the construction does not depend on the choice of the decompositions.

Then $\mathcal{A}_1 \coprod \mathcal{A}_2$ can be constructed as

$$\mathcal{A}_1 \coprod \mathcal{A}_2 = \bigoplus_{\epsilon \in \mathbb{A}} \mathcal{A}^\epsilon,$$

where $\mathcal{A}^\emptyset = \mathbb{C}$, $\mathcal{A}^\epsilon = \mathcal{A}^0_{\epsilon_1} \otimes \cdots \otimes \mathcal{A}^0_{\epsilon_n}$ for $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$. The multiplication in $\mathcal{A}_1 \coprod \mathcal{A}_2$ is inductively defined by

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_m) = \begin{cases} a_1 \otimes \cdots \otimes (a_n \cdot b_1) \otimes \cdots \otimes b_m & \text{if } \epsilon_n = \delta_1, \\ a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m & \text{if } \epsilon_n \neq \delta_1, \end{cases}$$

for $a_1 \otimes \cdots \otimes a_n \in \mathcal{A}^\epsilon$, $b_1 \otimes \cdots \otimes b_m \in \mathcal{A}^\delta$. Note that in the case $\epsilon_n = \delta_1$ the product $a_n \cdot b_1$ is not necessarily in $\mathcal{A}^0_{\epsilon_n}$, but is in general a sum of a multiple of the unit of $\mathcal{A}_{\epsilon_n}$ and an element of $\mathcal{A}^0_{\epsilon_n}$. We have to identify $a_1 \otimes \cdots a_{n-1} \otimes 1 \otimes b_2 \otimes \cdots b_m$ with $a_1 \otimes \cdots \otimes a_{n-1} \cdot b_2 \otimes \cdots b_m$.

Since $\coprod$ is the coproduct of a category, it is commutative and associative in the sense that there exist natural isomorphisms

$$\begin{aligned} \gamma_{\mathcal{A}_1, \mathcal{A}_2} &: \mathcal{A}_1 \coprod \mathcal{A}_2 \stackrel{\cong}{\to} \mathcal{A}_2 \coprod \mathcal{A}_1, \\ \alpha_{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3} &: \mathcal{A}_1 \coprod \left(\mathcal{A}_2 \coprod \mathcal{A}_3\right) \stackrel{\cong}{\to} \left(\mathcal{A}_1 \coprod \mathcal{A}_2\right) \coprod \mathcal{A}_3 \end{aligned} \tag{6.1}$$

for all unital $*$-algebras $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$. Let $i_\ell : \mathcal{A}_\ell \to \mathcal{A}_1 \coprod \mathcal{A}_2$ and $i'_\ell : \mathcal{A}_\ell \to \mathcal{A}_2 \coprod \mathcal{A}_1$, $\ell = 1, 2$ be the canonical inclusions. The commutativity constraint

$\gamma_{\mathcal{A}_1,\mathcal{A}_2} : \mathcal{A}_1 \coprod \mathcal{A}_2 \to \mathcal{A}_2 \coprod \mathcal{A}_1$ maps an element of $\mathcal{A}_1 \coprod \mathcal{A}_2$ of the form $i_1(a_1)i_2(b_1)\cdots i_2(b_n)$ with $a_1, \ldots, a_n \in \mathcal{A}_1$, $b_1, \ldots, b_n \in \mathcal{A}_2$ to

$$\gamma_{\mathcal{A}_1,\mathcal{A}_2}\big(i_1(a_1)i_2(b_1)\cdots i_2(b_n)\big) = i'_1(a_1)i'_2(b_1)\cdots i'_2(b_n) \in \mathcal{A}_2 \coprod \mathcal{A}_1.$$

Exercise 6.2. We also consider non-unital algebras. Show that the *free product* of $*$-algebras *without* identification of units is a coproduct in the category $\mathfrak{nuAlg}$ of non-unital (or rather not necessarily unital) algebras. Give an explicit construction for the free product of two non-unital algebras.

Exercise 6.3. Show that the following defines a a functor from the category of non-unital algebras $\mathfrak{nuAlg}$ to the category of unital algebras $\mathfrak{Alg}$. For an algebra $\mathcal{A} \in \mathrm{Ob}\,\mathfrak{nuAlg}$, $\tilde{\mathcal{A}}$ is equal to $\tilde{\mathcal{A}} = \mathbb{C}\mathbf{1} \oplus \mathcal{A}$ as a vector space and the multiplication is defined by

$$(\lambda\mathbf{1} + a)(\lambda'\mathbf{1} + a') = \lambda\lambda'\mathbf{1} + \lambda' a + \lambda a' + aa'$$

for $\lambda, \lambda' \in \mathbb{C}$, $a, a' \in \mathcal{A}$. We will call $\tilde{\mathcal{A}}$ the *unitization* of $\mathcal{A}$. Note that $\mathcal{A} \cong 0\mathbf{1} + \mathcal{A} \subseteq \tilde{\mathcal{A}}$ is not only a subalgebra, but even an ideal in $\tilde{\mathcal{A}}$.

How is the functor defined on the morphisms?

Show that the following relation holds between the free product *with* identification of units $\coprod_{\mathfrak{Alg}}$ and the free product *without* identification of units $\coprod_{\mathfrak{nuAlg}}$,

$$\widetilde{\mathcal{A}_1 \coprod_{\mathfrak{nuAlg}} \mathcal{A}_2} \cong \tilde{\mathcal{A}}_1 \coprod_{\mathfrak{Alg}} \tilde{\mathcal{A}}_2$$

for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{Ob}\,\mathfrak{nuAlg}$.

Note furthermore that the range of this functor consists of all algebras that admit a decomposition of the form $\mathcal{A} = \mathbb{C}\mathbf{1} \oplus \mathcal{A}_0$, where $\mathcal{A}_0$ is a subalgebra. This is equivalent to having a one-dimensional representation. The functor is not surjective, e.g., the algebra $\mathcal{M}_2$ of 2×2-matrices can not be obtained as a unitization of some other algebra.

6.1. **Classical stochastic independence and the product of probability spaces.** Two random variables $X_1 : (\Omega, \mathcal{F}, P) \to (E_1, \mathcal{E}_1)$ and $X_2 : (\Omega, \mathcal{F}, P) \to (E_2, \mathcal{E}_2)$, defined on the same probability space $(\Omega, \mathcal{F}, P)$ and with values in two possibly distinct measurable spaces $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$, are called *stochastically independent* (or simply *independent*) w.r.t. P, if the σ-algebras $X_1^{-1}(\mathcal{E}_1)$ and $X_2^{-1}(\mathcal{E}_2)$ are independent w.r.t. P, i.e. if

$$P\big((X_1^{-1}(M_1) \cap X_2^{-1}(M_2)\big) = P\big((X_1^{-1}(M_1)\big)P\big(X_2^{-1}(M_2)\big)$$

holds for all $M_1 \in \mathcal{E}_1$, $M_2 \in \mathcal{E}_2$. If there is no danger of confusion, then the reference to the measure P is often omitted.

This definition can easily be extended to arbitrary families of random variables. A family $\big(X_j : (\Omega, \mathcal{F}, P) \to (E_j, \mathcal{E}_j)\big)_{j\in J}$, indexed by some set J, is called independent, if

$$P\left(\bigcap_{k=1}^{n} X_{j_k}^{-1}(M_{j_k})\right) = \prod_{k=1}^{n} P\big(X_{j_k}^{-1}(M_{j_k})\big)$$

holds for all $n \in \mathbb{N}$ and all choices of indices $k_1, \ldots, k_n \in J$ with $j_k \neq j_\ell$ for $j \neq \ell$, and all choices of measurable sets $M_{j_k} \in \mathcal{E}_{j_k}$.

There are many equivalent formulations for independence, consider, e.g., the following proposition.

Proposition 6.4. *Let X_1 and X_2 be two real-valued random variables. The following are equivalent.*

(i) *X_1 and X_2 are independent.*

(ii) *For all bounded measurable functions f_1, f_2 on $\mathbb{R}$ we have*

$$\mathbb{E}\big(f_1(X_1)f_2(X_2)\big) = \mathbb{E}\big(f_1(X_1)\big)\mathbb{E}\big(f_2(X_2)\big).$$

(iii) *The probability space $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), P_{(X_1,X_2)})$ is the product of the probability spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_{X_1})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), P_{X_2})$, i.e.*

$$P_{(X_1,X_2)} = P_{X_1} \otimes P_{X_2}.$$

We see that stochastic independence can be reinterpreted as a rule to compute the joint distribution of two random variables from their marginal distribution. More precisely, their joint distribution can be computed as a product of their marginal distributions. This product is associative and can also be iterated to compute the joint distribution of more than two independent random variables.

The classifications of independence for non-commutative probability spaces [Spe97, BGS99, BG01, Mur03, Mur02] that we are interested in are based on redefining independence as a product satisfying certain natural axioms.

6.1.1. *Example: Tensor Independence in the Category of Algebraic Probability Spaces.* By the *category of algebraic probability spaces* $\mathfrak{AlgProb}$ we denote the category of associative unital algebras over $\mathbb{C}$ equipped with a unital linear functional. A morphism $j : (\mathcal{A}_1, \varphi_1) \to (\mathcal{A}_2, \varphi_2)$ is a quantum random variable, i.e. an algebra homomorphism $j : \mathcal{A}_1 \to \mathcal{A}_2$ that preserves the unit and the functional, i.e. $j(\mathbf{1}_{\mathcal{A}_1}) = \mathbf{1}_{\mathcal{A}_2}$ and $\varphi_2 \circ j = \varphi_1$.

The tensor product we will consider on this category is just the usual tensor product $(\mathcal{A}_1 \otimes \mathcal{A}_2, \varphi_1 \otimes \varphi_2)$, i.e. the algebra structure of $\mathcal{A}_1 \otimes \mathcal{A}_2$ is defined by

$$\begin{aligned} \mathbf{1}_{\mathcal{A}_1 \otimes \mathcal{A}_2} &= \mathbf{1}_{\mathcal{A}_1} \otimes \mathbf{1}_{\mathcal{A}_2}, \\ (a_1 \otimes a_2)(b_1 \otimes b_2) &= a_1 b_1 \otimes a_2 b_2, \end{aligned}$$

and the new functional is defined by

$$(\varphi_1 \otimes \varphi_2)(a_1 \otimes a_2) = \varphi_1(a_1)\varphi_2(a_2),$$

for all $a_1, b_1 \in \mathcal{A}_1$, $a_2, b_2 \in \mathcal{A}_2$.

This becomes a tensor category with inclusions with the inclusions defined by

$$\begin{aligned} i_{\mathcal{A}_1}(a_1) &= a_1 \otimes \mathbf{1}_{\mathcal{A}_2}, \\ i_{\mathcal{A}_2}(a_2) &= \mathbf{1}_{\mathcal{A}_1} \otimes a_2, \end{aligned}$$

for $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$.

One gets the category of $*$-algebraic probability spaces, if one assumes that the underlying algebras have an involution and the functional are states, i.e. also positive. Then an involution is defined on $\mathcal{A}_1 \otimes \mathcal{A}_2$ by $(a_1 \otimes a_2)^* = a_1^* \otimes a_2^*$ and $\varphi_1 \otimes \varphi_2$ is again a state.

The notion of independence associated to this tensor product with inclusions by Definition **??** is the usual notion of *Bose* or *tensor independence* used in quantum probability, e.g., by Hudson and Parthasarathy.

Proposition 6.5. *Two quantum random variables $j_1 : (\mathcal{B}_1, \psi_1) \to (\mathcal{A}, \varphi)$ and $j_2 : (\mathcal{B}_2, \psi_2) \to (\mathcal{A}, \varphi)$, defined on algebraic probability spaces $(\mathcal{B}_1, \psi_1), (\mathcal{B}_2, \psi_2)$ and with values in the same algebraic probability space $(\mathcal{A}, \varphi)$ are independent if and only if the following two conditions are satisfied.*

(i): *The images of j_1 and j_2 commute, i.e.*

$$\big[j_1(a_1), j_2(a_2)\big] = 0,$$

for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$.

(ii): *φ satisfies the factorization property*

$$\varphi\big(j_1(a_1) j_2(a_2)\big) = \varphi\big(j_1(a_1)\big)\varphi\big(j_2(a_2)\big),$$

for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$.

We will not prove this Proposition since it can be obtained as a special case of Proposition 6.6, if we equip the algebras with the trivial $\mathbb{Z}_2$-grading $\mathcal{A}^{(0)} = \mathcal{A}$, $\mathcal{A}^{(1)} = \{0\}$.

6.1.2. *Example: Fermi Independence.* Let us now consider the category of $\mathbb{Z}_2$-graded algebraic probability spaces $\mathbb{Z}_2$-$\mathfrak{AlgProb}$. The objects are pairs $(\mathcal{A}, \varphi)$ consisting of a $\mathbb{Z}_2$-graded unital algebra $\mathcal{A} = \mathcal{A}^{(0)} \oplus \mathcal{A}^{(1)}$ and an even unital functional φ, i.e. $\varphi|_{\mathcal{A}^{(1)}} = 0$. The morphisms are random variables that don't change the degree, i.e., for $j : (\mathcal{A}_1, \varphi_1) \to (\mathcal{A}_2, \varphi_2)$, we have

$$j(\mathcal{A}_1^{(0)}) \subseteq \mathcal{A}_2^{(0)} \quad \text{and} \quad j(\mathcal{A}_1^{(1)}) \subseteq \mathcal{A}_2^{(1)}.$$

The tensor product $(\mathcal{A}_1 \otimes_{\mathbb{Z}_2} \mathcal{A}_2, \varphi_1 \otimes \varphi_2) = (\mathcal{A}_1, \varphi_1) \otimes_{\mathbb{Z}_2} (\mathcal{A}_2, \varphi_2)$ is defined as follows. The algebra $\mathcal{A}_1 \otimes_{\mathbb{Z}_2} \mathcal{A}_2$ is the graded tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$, i.e. $(\mathcal{A}_1 \otimes_{\mathbb{Z}_2} \mathcal{A}_2)^{(0)} = \mathcal{A}_1^{(0)} \otimes \mathcal{A}_2^{(0)} \oplus \mathcal{A}_1^{(1)} \otimes \mathcal{A}_2^{(1)}$, $(\mathcal{A}_1 \otimes_{\mathbb{Z}_2} \mathcal{A}_2)^{(1)} = \mathcal{A}_1^{(1)} \otimes \mathcal{A}_2^{(0)} \oplus \mathcal{A}_1^{(0)} \otimes \mathcal{A}_2^{(1)}$, with the algebra structure given by

$$\begin{aligned} \mathbf{1}_{\mathcal{A}_1 \otimes_{\mathbb{Z}_2} \mathcal{A}_2} &= \mathbf{1}_{\mathcal{A}_1} \otimes \mathbf{1}_{\mathcal{A}_2}, \\ (a_1 \otimes a_2) \cdot (b_1 \otimes b_2) &= (-1)^{\deg a_2 \deg b_1} a_1 b_1 \otimes a_2 b_2, \end{aligned}$$

for all homogeneous elements $a_1, b_1 \in \mathcal{A}_1$, $a_2, b_2 \in \mathcal{A}_2$. The functional $\varphi_1 \otimes \varphi_2$ is simply the tensor product, i.e. $(\varphi_1 \otimes \varphi_2)(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2)$ for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$. It is easy to see that $\varphi_1 \otimes \varphi_2$ is again even, if φ_1 and φ_2 are even. The inclusions $i_1 : (\mathcal{A}_1, \varphi_1) \to (\mathcal{A}_1 \otimes_{\mathbb{Z}_2} \mathcal{A}_2, \varphi_1 \otimes \varphi_2)$ and $i_2 : (\mathcal{A}_2, \varphi_2) \to (\mathcal{A}_1 \otimes_{\mathbb{Z}_2} \mathcal{A}_2, \varphi_1 \otimes \varphi_2)$ are defined by

$$i_1(a_1) = a_1 \otimes \mathbf{1}_{\mathcal{A}_2} \quad \text{ and } i_2(a_2) = \mathbf{1}_{\mathcal{A}_1} \otimes a_2,$$

for $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$.

If the underlying algebras are assumed to have an involution and the functionals to be states, then the involution on the $\mathbb{Z}_2$-graded tensor product is defined by $(a_1 \otimes a_2)^* = (-1)^{\deg a_1 \deg a_2} a_1^* \otimes a_2^*$, this gives the category of $\mathbb{Z}_2$-graded $*$-algebraic probability spaces.

The notion of independence associated to this tensor category with inclusions is called *Fermi independence* or *anti-symmetric independence.*

Proposition 6.6. *Two random variables $j_1 : (\mathcal{B}_1, \psi_1) \to (\mathcal{A}, \varphi)$ and $j_2 : (\mathcal{B}_2, \psi_2) \to (\mathcal{A}, \varphi)$, defined on two $\mathbb{Z}_2$-graded algebraic probability spaces $(\mathcal{B}_1, \psi_1)$, $(\mathcal{B}_2, \psi_2)$ and with values in the same $\mathbb{Z}_2$-algebraic probability space $(\mathcal{A}, \varphi)$ are independent if and only if the following two conditions are satisfied.*

(i): *The images of j_1 and j_2 satisfy the commutation relations*

$$j_2(a_2) j_1(a_1) = (-1)^{\deg a_1 \deg a_2} j_1(a_1) j_2(a_2)$$

for all homogeneous elements $a_1 \in \mathcal{B}_1$, $a_2 \in \mathcal{B}_2$.

(ii): *φ satisfies the factorization property*

$$\varphi\big(j_1(a_1) j_2(a_2)\big) = \varphi\big(j_1(a_1)\big)\varphi\big(j_2(a_2)\big),$$

for all $a_1 \in \mathcal{B}_1$, $a_2 \in \mathcal{B}_2$.

Proof. The proof is similar to that of Proposition **??**, we will only outline it. It is clear that the morphism $h : (\mathcal{B}_1, \psi_1) \otimes_{\mathbb{Z}_2} (\mathcal{B}_2, \psi_2) \to (\mathcal{A}, \varphi)$ that makes the diagram in Definition **??** commuting, has to act on elements of $\mathcal{B}_1 \otimes \mathbf{1}_{\mathcal{B}_2}$ and $\mathbf{1}_{\mathcal{B}_1} \otimes \mathcal{B}_2$ as

$$h(b_1 \otimes \mathbf{1}_{\mathcal{B}_2}) = j_1(b_1) \quad \text{ and } \quad h(\mathbf{1}_{\mathcal{B}_1} \otimes b_2) = j_2(b_2).$$

This extends to a homomorphism from $(\mathcal{B}_1, \psi_1) \otimes_{\mathbb{Z}_2} (\mathcal{B}_2, \psi_2)$ to $(\mathcal{A}, \varphi)$, if and only if the commutation relations are satisfied. And the resulting homomorphism is a quantum random variable, i.e. satisfies $\varphi \circ h = \psi_1 \otimes \psi_2$, if and only if the factorization property is satisfied. □ □

6.1.3. *Example: Free Independence.* We will now introduce another tensor product with inclusions for the category of algebraic probability spaces $\mathfrak{AlgProb}$. On the algebras we take simply the free product of algebras with identifications of units introduced in Example 6.1. This is the coproduct in the category of algebras, therefore we also have natural inclusions. It only remains to define a unital linear functional on the free product of the algebras.

Voiculescu's[VDN92] *free product* $\varphi_1 * \varphi_2$ of two unital linear functionals $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}$ and $\varphi_2 : \mathcal{A}_2 \to \mathbb{C}$ can be defined recursively by

$$(\varphi_1 * \varphi_2)(a_1 a_2 \cdots a_m) = \sum_{I \subsetneqq \{1,\ldots,m\}} (-1)^{m-\sharp I+1} (\varphi_1 * \varphi_2) \left(\overrightarrow{\prod_{k\in I}} a_k \right) \prod_{k \notin I} \varphi_{\epsilon_k}(a_k)$$

for a typical element $a_1 a_2 \cdots a_m \in \mathcal{A}_1 \coprod \mathcal{A}_2$, with $a_k \in \mathcal{A}_{\epsilon_k}$, $\epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_m$, i.e. neighboring a's don't belong to the same algebra. $\sharp I$ denotes the number of elements of I and $\overrightarrow{\prod}_{k\in I} a_k$ means that the a's are to be multiplied in the same order in which they appear on the left-hand-side. We use the convention $(\varphi_1 * \varphi_2) \left(\overrightarrow{\prod}_{k\in\emptyset} a_k \right) = 1$.

It turns out that this product has many interesting properties, e.g., if φ_1 and φ_2 are states, then their free product is a again a state. For more details, see [BNT05] and the references given there.

6.1.4. *Examples: Boolean, Monotone, and Anti-monotone Independence.* Ben Ghorbal and Schürmann[BG01, BGS99] and Muraki[Mur03] have also considered the category of non-unital algebraic probability $\mathfrak{nuAlgProb}$ consisting of pairs $(\mathcal{A}, \varphi)$ of a not necessarily unital algebra $\mathcal{A}$ and a linear functional φ. The morphisms in this category are algebra homomorphisms that leave the functional invariant. On this category we can define three more tensor products with inclusions corresponding to the boolean product $\diamond$, the monotone product $\triangleright$ and the anti-monotone product $\triangleleft$ of states. They can be defined by

$$\begin{aligned}
\varphi_1 \diamond \varphi_2 (a_1 a_2 \cdots a_m) &= \prod_{k=1}^{m} \varphi_{\epsilon_k}(a_k), \\
\varphi_1 \triangleright \varphi_2 (a_1 a_2 \cdots a_m) &= \varphi_1 \left(\overrightarrow{\prod_{k:\epsilon_k=1}} a_k \right) \prod_{k:\epsilon_k=2} \varphi_2(a_k), \\
\varphi_1 \triangleleft \varphi_2 (a_1 a_2 \cdots a_m) &= \prod_{k:\epsilon_k=1} \varphi_1(a_k)\ \varphi_2 \left(\overrightarrow{\prod_{k:\epsilon_k=2}} a_k \right),
\end{aligned}$$

for $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}$ and $\varphi_2 : \mathcal{A}_2 \to \mathbb{C}$ and a typical element $a_1 a_2 \cdots a_m \in \mathcal{A}_1 \coprod \mathcal{A}_2$, $a_k \in \mathcal{A}_{\epsilon_k}$, $\epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_m$, i.e. neighboring a's don't belong to the same algebra. Note that for the algebras and the inclusions we use here the free product without units, the coproduct in the category of not necessarily unital algebras.

The monotone and anti-monotone product are not commutative, but related by

$$\varphi_1 \triangleright \varphi_2 = (\varphi_2 \triangleleft \varphi_1) \circ \gamma_{\mathcal{A}_1, \mathcal{A}_2},$$

for all linear functionals $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}$, $\varphi_2 : \mathcal{A}_2 \to \mathbb{C}$, where $\gamma_{\mathcal{A}_1,\mathcal{A}_2} : \mathcal{A}_1 \coprod \mathcal{A}_2 \to \mathcal{A}_2 \coprod \mathcal{A}_1$ is the commutativity constraint (for the commutativity constraint for the free product of unital algebras see Equation (6.1)). The boolean product is commutative, i.e. it satisfies

$$\varphi_1 \diamond \varphi_2 = (\varphi_2 \diamond \varphi_1) \circ \gamma_{\mathcal{A}_1,\mathcal{A}_2},$$

for all linear functionals $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}$, $\varphi_2 : \mathcal{A}_2 \to \mathbb{C}$.

Exercise 6.7. The boolean, the monotone and the anti-monotone product can also be defined for unital algebras, if they are in the range of the unitization functor introduced in Exercise 6.3.

Let $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}$ and $\varphi_2 : \mathcal{A}_2 \to \mathbb{C}$ be two unital functionals on algebras $\mathcal{A}_1$, $\mathcal{A}_2$, which can be decomposed as $\mathcal{A}_1 = \mathbb{C}\mathbf{1} \oplus \mathcal{A}_1^0$, $\mathcal{A}_2 = \mathbb{C}\mathbf{1} \oplus \mathcal{A}_2^0$. Then we define the boolean, monotone, or anti-monotone product of φ_1 and φ_2 as the unital extension of the boolean, monotone, or anti-monotone product of their restrictions $\varphi_1|_{\mathcal{A}_1^0}$ and $\varphi_2|_{\mathcal{A}_2^0}$.

Show that this leads to the following formulas.

$$\begin{aligned}
\varphi_1 \diamond \varphi_2(a_1a_2\cdots a_n) &= \prod_{i=1}^n \varphi_{\epsilon_i}(a_i),\\
\varphi_1 \triangleright \varphi_2(a_1a_2\cdots a_n) &= \varphi_1\left(\prod_{i:\epsilon_i=1} a_i\right)\prod_{i:\epsilon_i=2}\varphi_2(a_i),\\
\varphi_1 \triangleleft \varphi_2(a_1a_2\cdots a_n) &= \prod_{i:\epsilon_i=1}\varphi_1(a_i)\varphi_2\left(\prod_{i:\epsilon_i=2} a_i\right),
\end{aligned}$$

for $a_1a_2\cdots a_n \in \mathcal{A}_1 \coprod \mathcal{A}_2$, $a_i \in \mathcal{A}_{\epsilon_i}^0$, $\epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_n$. We use the convention that the empty product is equal to the unit element.

These products can be defined in the same way for $*$-algebraic probability spaces, where the algebras are unital $*$-algebras having such a decomposition $\mathcal{A} = \mathbb{C}\mathbf{1} \oplus \mathcal{A}_0$ and the functionals are states. To check that $\varphi_1\diamond\varphi_2, \varphi_1\triangleright\varphi_2, \varphi_1\triangleleft\varphi_2$ are again states, if φ_1 and φ_2 are states, one can verify that the following constructions give their GNS representations. Let (π_1, H_1, ξ_1) and (π_2, H_2, ξ_2) denote the GNS representations of $(\mathcal{A}_1, \varphi_1)$ and $(\mathcal{A}_2, \varphi_2)$. The GNS representations of $(\mathcal{A}_1 \coprod \mathcal{A}_2, \varphi_1 \diamond \varphi_2)$, $(\mathcal{A}_1 \coprod \mathcal{A}_2, \varphi_1 \triangleright \varphi_2)$, and $(\mathcal{A}_1 \coprod \mathcal{A}_2, \varphi_1 \triangleleft \varphi_2)$ can all be defined on the Hilbert space $H = H_1 \otimes H_2$ with the state vector $\xi = \xi_1 \otimes \xi_2$. The representations are defined by $\pi(\mathbf{1}) = \mathrm{id}$ and

$$\begin{array}{rclrcll}
\pi|_{\mathcal{A}_1^0} &=& \pi_1 \otimes P_2, & \pi|_{\mathcal{A}_2^0} &=& P_1 \otimes \pi_2, & \text{for} \quad \varphi_1 \diamond \varphi_2,\\
\pi|_{\mathcal{A}_1^0} &=& \pi_1 \otimes P_2, & \pi|_{\mathcal{A}_2^0} &=& \mathrm{id}_{H_2} \otimes \pi_2, & \text{for} \quad \varphi_1 \triangleright \varphi_2,\\
\pi|_{\mathcal{A}_1^0} &=& \pi_1 \otimes \mathrm{id}_{H_2}, & \pi|_{\mathcal{A}_2^0} &=& P_1 \otimes \pi_2, & \text{for} \quad \varphi_1 \triangleleft \varphi_2,
\end{array}$$

where P_1, P_2 denote the orthogonal projections $P_1 : H_1 \to \mathbb{C}\xi_1$, $P_2 : H_2 \to \mathbb{C}\xi_2$. For the boolean case, $\xi = \xi_1 \otimes \xi_2 \in H_1 \otimes H_2$ is not cyclic for π, only the subspace $\mathbb{C}\xi \oplus H_1^0 \oplus H_2^0$ can be generated from ξ.

6.2. **Classification of the universal independences.** The associativity gives us the condition

$$\big((\varphi_1\cdot\varphi_2)\cdot\varphi_3\big)\circ\alpha_{\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3}=\varphi_1\cdot(\varphi_2\cdot\varphi_3), \tag{6.2}$$

for all $(\mathcal{A}_1,\varphi_1),(\mathcal{A}_2,\varphi_2),(\mathcal{A}_3,\varphi_3)$ in $\mathfrak{AlgProb}$. Denote the unique unital functional on $\mathbb{C}\mathbf{1}$ by δ, then the unit properties are equivalent to

$$(\varphi\cdot\delta)\circ\rho_{\mathcal{A}}=\varphi\quad\text{ and }\quad(\delta\cdot\varphi)\circ\lambda_{\mathcal{A}}=\varphi,$$

for all $(\mathcal{A},\varphi)$ in $\mathfrak{AlgProb}$. The inclusions are random variables, if and only if

$$(\varphi_1\cdot\varphi_2)\circ i_{\mathcal{A}_1}=\varphi_1\quad\text{ and }\quad(\varphi_1\cdot\varphi_2)\circ i_{\mathcal{A}_2}=\varphi_2 \tag{6.3}$$

for all $(\mathcal{A}_1,\varphi_1),(\mathcal{A}_2,\varphi_2)$ in $\mathfrak{AlgProb}$. Finally, from the functoriality of $\square$ we get the condition

$$(\varphi_1\cdot\varphi_2)\circ(j_1\coprod j_2)=(\varphi_1\circ j_1)\cdot(\varphi_2\circ j_2) \tag{6.4}$$

for all pairs of morphisms $j_1:(\mathcal{B}_1,\psi_1)\to(\mathcal{A}_1,\varphi_1)$, $j_2:(\mathcal{B}_2,\psi_2)\to(\mathcal{A}_2,\varphi_2)$ in $\mathfrak{AlgProb}$.

Our Conditions (6.2), (6.3), and (6.4) are exactly the axioms (P2), (P3), and (P4) in Ben Ghorbal and Schürmann[BGS99], or the axioms (U2), the first part of (U4), and (U3) in Muraki[Mur03].

Theorem 6.8. *(Muraki*[Mur03]*, Ben Ghorbal and Schürmann*[BG01, BGS99]*) There exist exactly two universal tensor products with inclusions on the category of algebraic probability spaces* $\mathfrak{AlgProb}$*, namely the universal version* $\tilde{\otimes}$ *of the tensor product defined in Section 6.1.1 and the one associated to the free product* $*$ *of states.*

For the classification in the non-unital case, Muraki imposes the additional condition

$$(\varphi_1\cdot\varphi_2)(a_1a_2)=\varphi_{\epsilon_1}(a_1)\varphi_{\epsilon_2}(a_2) \tag{6.5}$$

for all $(\epsilon_1,\epsilon_2)\in\big\{(1,2),(2,1)\big\}$, $a_1\in\mathcal{A}_{\epsilon_1}$, $a_2\in\mathcal{A}_{\epsilon_2}$.

Theorem 6.9. *(Muraki*[Mur03]*) There exist exactly five universal tensor products with inclusions satisfying (6.5) on the category of non-unital algebraic probability spaces* $\mathfrak{nuAlgProb}$*, namely the universal version* $\tilde{\otimes}$ *of the tensor product defined in Section 6.1.1 and the ones associated to the free product* $*$*, the boolean product* $\diamond$*, the monotone product* $\triangleright$ *and the anti-monotone product* $\triangleleft$*.*

The monotone and the anti-monotone are not symmetric, i.e. $(\mathcal{A}_1\coprod\mathcal{A}_2,\varphi_1\triangleright\varphi_2)$ and $(\mathcal{A}_2\coprod\mathcal{A}_2,\varphi_2\triangleright\varphi_1)$ are not isomorphic in general. Actually, the anti-monotone product is simply the mirror image of the monotone product,

$$(\mathcal{A}_1\coprod\mathcal{A}_2,\varphi_1\triangleright\varphi_2)\cong(\mathcal{A}_2\coprod\mathcal{A}_1,\varphi_2\triangleleft\varphi_1)$$

for all $(\mathcal{A}_1,\varphi_1),(\mathcal{A}_2,\varphi_2)$ in the category of non-unital algebraic probability spaces. The other three products are symmetric.

In the symmetric setting of Ben Ghorbal and Schürmann, Condition (6.5) is not essential. If one drops it and adds symmetry, one finds in addition the degenerate product

$$(\varphi_1 \bullet_0 \varphi_2)(a_1 a_2 \cdots a_m) = \begin{cases} \varphi_{\epsilon_1}(a_1) & \text{if} \quad m = 1, \\ 0 & \text{if} \quad m > 1. \end{cases}$$

and families

$$\varphi_1 \bullet_q \varphi_2 = q\big((q^{-1}\varphi_1) \cdot (q^{-1}\varphi_2)\big),$$

parametrized by a complex number $q \in \mathbb{C}\backslash\{0\}$, for each of the three symmetric products, $\bullet \in \{\tilde{\otimes}, *, \diamond\}$.

If one adds the condition that products of states are again states, then one can also show that the constant has to be equal to one.

Exercise 6.10. Consider the category of non-unital $*$-algebraic probability spaces, whose objects are pairs $(\mathcal{A}, \varphi)$ consisting of a not necessarily unital $*$-algebra $\mathcal{A}$ and a state $\varphi : \mathcal{A} \to \mathbb{C}$. Here a state is a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ whose unital extension $\tilde{\varphi} : \tilde{\mathcal{A}} \cong \mathbb{C}\mathbf{1} \oplus \mathcal{A} \to \mathbb{C}$, $\lambda\mathbf{1} + a \mapsto \tilde{\varphi}(\lambda\mathbf{1} + a) = \lambda + \varphi(a)$, to the unitization of $\mathcal{A}$ is a state.

Assume we have products $\cdot : S(\mathcal{A}_1) \times S(\mathcal{A}_2) \to S(\mathcal{A}_1 \coprod \mathcal{A}_2)$ of linear functionals on non-unital algebras $\mathcal{A}_1, \mathcal{A}_2$ that satisfy

$$\begin{aligned} (\varphi_1 \cdot \varphi_2)(a_1 a_2) &= c_1 \varphi_1(a_1)\varphi_2(a_2), \\ (\varphi_1 \cdot \varphi_2)(a_2 a_1) &= c_2 \varphi_1(a_1)\varphi_2(a_2), \end{aligned}$$

for all linear functionals $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}$, $\varphi_2 : \mathcal{A}_2 \to \mathbb{C}$, and elements $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$ with "universal" constants $c_1, c_2 \in \mathbb{C}$, i.e. constants that do not depend on the algebras, the functionals, or the algebra elements. That for every universal independence such constants have to exist is part of the proof of the classifications in [BG01, BGS99, Mur03].

Show that if the products of states are again states, then we have $c_1 = c_2 = 1$. Hint: Take for $\mathcal{A}_1$ and $\mathcal{A}_2$ the algebra of polynomials on $\mathbb{R}$ and for φ_1 and φ_2 evaluation in a point.

The proof of the classification of universal independences can be split into three steps.

Using the "universality" or functoriality of the product, one can show that there exist some "universal constants" - not depending on the algebras - and a formula for evaluating

$$(\varphi_1 \cdot \varphi_2)(a_1 a_2 \cdots a_m)$$

for $a_1 a_2 \cdots a_m \in \mathcal{A}_1 \coprod \mathcal{A}_2$, with $a_k \in \mathcal{A}_{\epsilon_k}$, $\epsilon_1 \neq \epsilon_2 \neq \cdots \neq \epsilon_m$, as a linear combination of products $\varphi_1(M_1)$, $\varphi_2(M_2)$, where M_1, M_2 are "sub-monomials" of $a_1 a_2 \cdots a_m$. Then in a second step it is shown by associativity that only products with *ordered* monomials M_1, M_2 contribute. This is the content of [BGS02, Theorem 5] in the commutative case and of [Mur03, Theorem 2.1] in the general case.

The third step, which was actually completed first in both cases, see [Spe97] and [Mur02], is to find the conditions that the universal constants have to satisfy, if the resulting product is associative. It turns out that the universal coefficients for $m > 5$ are already uniquely determined by the coefficients for $1 \leq m \leq 5$. Detailed analysis of the non-linear equations obtained for the coefficients of order up to five then leads to the classifications stated above.

7. Lévy Processes on Dual Groups

We now want to study quantum stochastic processes whose increments are free or independent in the sense of boolean, monotone, or anti-monotone independence. The approach based on bialgebras that we followed in the first Section works for the tensor product and fails in the other cases because the corresponding products are not defined on the tensor product, but on the free product of the algebra. The algebraic structure which has to replace bialgebras was first introduced by Voiculescu [Voi87, Voi90], who named them dual groups. In this section we will introduce these algebras and develop the theory of their Lévy processes. It turns out that Lévy processes on dual groups with boolean, monotonically, or anti-monotonically independent increments can be reduced to Lévy processes on involutive bialgebra. We do not know if this is also possible for Lévy processes on dual groups with free increments.

In the literature additive free Lévy processes have been studied most intensively, see, e.g., [GSS92, Bia98, Ans02, Ans03, BNT02b, BNT02a].

7.1. Preliminaries on dual groups.

Denote by $\mathfrak{ComAlg}$ the category of commutative unital algebras and let $\mathcal{B} \in \mathrm{Ob}\,\mathfrak{ComAlg}$ be a commutative bialgebra. Then the mapping

$$\mathrm{Ob}\,\mathfrak{ComAlg} \ni \mathcal{A} \mapsto \mathrm{Mor}_{\mathfrak{ComAlg}}(\mathcal{B}, \mathcal{A})$$

can be understood as a functor from $\mathfrak{ComAlg}$ to the category of unital semigroups. The multiplication in $\mathrm{Mor}_{\mathfrak{Alg}}(\mathcal{B}, \mathcal{A})$ is given by the convolution, i.e.

$$f \star g = m_{\mathcal{A}} \circ (f \otimes g) \circ \Delta_{\mathcal{B}}$$

and the unit element is $\varepsilon_{\mathcal{B}} \mathbf{1}_{\mathcal{A}}$. A unit-preserving algebra homomorphism $h : \mathcal{A}_1 \to \mathcal{A}_2$ gets mapped to the unit-preserving semigroup homomorphism $\mathrm{Mor}_{\mathfrak{ComAlg}}(\mathcal{B}, \mathcal{A}_1) \ni f \to h \circ f \in \mathrm{Mor}_{\mathfrak{ComAlg}}(\mathcal{B}, \mathcal{A}_2)$, since

$$h \circ (f \star g) = (h \circ f) \star (h \circ g)$$

for all $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{Ob}\,\mathfrak{ComAlg}$, $h \in \mathrm{Mor}_{\mathfrak{ComAlg}}(\mathcal{A}_1, \mathcal{A}_2)$, $f, g \in \mathrm{Mor}_{\mathfrak{ComAlg}}(\mathcal{B}, \mathcal{A}_1)$.

If $\mathcal{B}$ is even a commutative Hopf algebra with antipode S, then $\mathrm{Mor}_{\mathfrak{ComAlg}}(\mathcal{B}, \mathcal{A})$ is a group with respect to the convolution product. The inverse of a homomorphism $f : \mathcal{B} \to \mathcal{A}$ with respect to the convolution product is given by $f \circ S$.

The calculation

$$\begin{aligned}(f\star g)(ab) &= m_{\mathcal{A}}\circ(f\otimes g)\circ\Delta_{\mathcal{B}}(ab)\\ &= f(a_{(1)}b_{(1)})g(a_{(2)}b_{(2)}) = f(a_{(1)})f(b_{(1)})g(a_{(2)})g(b_{(2)})\\ &= f(a_{(1)})g(a_{(2)})f(b_{(1)})g(b_{(2)}) = (f\star g)(a)(f\star g)(b)\end{aligned}$$

shows that the convolution product $f\star g$ of two homomorphisms $f,g:\mathcal{B}\to\mathcal{A}$ is again a homomorphism. It also gives an indication why non-commutative bialgebras or Hopf algebras do not give rise to a similar functor on the category of non-commutative algebras, since we had to commute $f(b_{(1)})$ with $g(a_{(2)})$.

Zhang [Zha91], Berman and Hausknecht [BH96] showed that if one replaces the tensor product in the definition of bialgebras and Hopf algebras by the free product, then one arrives at a class of algebras that do give rise to a functor from the category of non-commutative algebras to the category of semigroups or groups.

A *dual group* [Voi87, Voi90] (called *H-algebra* or *cogroup* in the category of unital associative $*$-algebras in [Zha91] and [BH96], resp.) is a unital $*$-algebra $\mathcal{B}$ equipped with three unital $*$-algebra homomorphisms $\Delta:\mathcal{B}\to\mathcal{B}\coprod\mathcal{B}$, $S:\mathcal{B}\to\mathcal{B}$ and $\varepsilon:\mathcal{B}\to\mathbb{C}$ (also called *comultiplication*, *antipode*, and *counit*) such that

$$\left(\Delta\coprod\mathrm{id}\right)\circ\Delta = \left(\mathrm{id}\coprod\Delta\right)\circ\Delta, \tag{7.1}$$

$$\left(\varepsilon\coprod\mathrm{id}\right)\circ\Delta = \mathrm{id} = \left(\mathrm{id}\coprod\varepsilon\right)\circ\Delta, \tag{7.2}$$

$$m_{\mathcal{B}}\circ\left(S\coprod\mathrm{id}\right)\circ\Delta = \mathrm{id} = m_{\mathcal{B}}\circ\left(\mathrm{id}\coprod S\right)\circ\Delta, \tag{7.3}$$

where $m_{\mathcal{B}}:\mathcal{B}\coprod\mathcal{B}\to\mathcal{B}$, $m_{\mathcal{B}}(a_1\otimes a_2\otimes\cdots\otimes a_n)=a_1\cdot a_2\cdot\cdots\cdot a_n$, is the multiplication of $\mathcal{B}$. Besides the formal similarity, there are many relations between dual groups on the one side and Hopf algebras and bialgebras on the other side, cf. [Zha91]. For example, let $\mathcal{B}$ be a dual group with comultiplication Δ, and let $R:\mathcal{B}\coprod\mathcal{B}\to\mathcal{B}\otimes\mathcal{B}$ be the unique unital $*$-algebra homomorphism with

$$R_{\mathcal{B},\mathcal{B}}\circ i_1(b)=b\otimes\mathbf{1},\qquad R_{\mathcal{B},\mathcal{B}}\circ i_2(b)=\mathbf{1}\otimes b,$$

for all $b\in\mathcal{B}$. Here $i_1,i_2:\mathcal{B}\to\mathcal{B}\coprod\mathcal{B}$ denote the canonical inclusions of $\mathcal{B}$ into the first and the second factor of the free product $\mathcal{B}\coprod\mathcal{B}$. Then $\mathcal{B}$ is a bialgebra with the comultiplication $\overline{\Delta}=R_{\mathcal{B},\mathcal{B}}\circ\Delta$, see [Zha91, Theorem 4.2], but in general it is not a Hopf algebra.

We will not really work with dual groups, but the following weaker notion. A *dual semigroup* is a unital $*$-algebra $\mathcal{B}$ equipped with two unital $*$-algebra homomorphisms $\Delta:\mathcal{B}\to\mathcal{B}\coprod\mathcal{B}$ and $\varepsilon:\mathcal{B}\to\mathbb{C}$ such that Equations (7.1) and (7.2) are satisfied. The antipode is not used in the proof of [Zha91, Theorem 4.2], and therefore we also get an involutive bialgebra $(\mathcal{B},\overline{\Delta},\varepsilon)$ for every dual semigroup $(\mathcal{B},\Delta,\varepsilon)$.

Note that we can always write a dual semigroup $\mathcal{B}$ as a direct sum $\mathcal{B}=\mathbb{C}\mathbf{1}\oplus\mathcal{B}^0$, where $\mathcal{B}^0=\ker\varepsilon$ is even a $*$-ideal. Therefore it is in the range of the unitization functor and the boolean, monotone, and anti-monotone product can be defined for unital linear functionals on $\mathcal{B}$, cf. Exercise 6.7.

The comultiplication of a dual semigroup can also be used to define a convolution product. The *convolution* $j_1 \star j_2$ of two unital $*$-algebra homomorphisms $j_1, j_2 : \mathcal{B} \to \mathcal{A}$ is defined as

$$j_1 \star j_2 = m_{\mathcal{A}} \circ \left(j_1 \coprod j_2 \right) \circ \Delta.$$

As the composition of the three unital $*$-algebra homomorphisms $\Delta : \mathcal{B} \to \mathcal{B} \coprod \mathcal{B}$, $j_1 \coprod j_2 : \mathcal{B} \coprod \mathcal{B} \to \mathcal{A} \coprod \mathcal{A}$, and $m_{\mathcal{A}} : \mathcal{A} \coprod \mathcal{A} \to \mathcal{A}$, this is obviously again a unital $*$-algebra homomorphism. Note that this convolution can not be defined for arbitrary linear maps on $\mathcal{B}$ with values in some algebra, as for bialgebras, but only for unital $*$-algebra homomorphisms.

7.2. **Definition of Lévy processes on dual groups.**

Definition 7.1. Let $j_1 : \mathcal{B}_1 \to (\mathcal{A}, \Phi), \ldots, j_n : \mathcal{B}_n \to (\mathcal{A}, \Phi)$ be quantum random variables over the same quantum probability space $(\mathcal{A}, \Phi)$ and denote their marginal distributions by $\varphi_i = \Phi \circ j_i$, $i = 1, \ldots, n$. The quantum random variables $(j_1, \ldots, j_n)$ are called tensor independent (respectively boolean independent, monotonically independent, anti-monotonically independent or free), if the state $\Phi \circ m_{\mathcal{A}} \circ (j_1 \coprod \cdots \coprod j_n)$ on the free product $\coprod_{i=1}^n \mathcal{B}_i$ is equal to the tensor product (boolean, monotone, anti-monotone, or free product, respectively) of $\varphi_1, \ldots, \varphi_n$.

Note that tensor, boolean, and free independence do not depend on the order, but monotone and anti-monotone independence do. An n-tuple $(j_1, \ldots, j_n)$ of quantum random variables is monotonically independent, if and only if $(j_n, \ldots, j_1)$ is anti-monotonically independent.

We are now ready to define tensor, boolean, monotone, anti-monotone, and free Lévy processes on dual semigroups.

Definition 7.2. [Sch95] Let $(\mathcal{B}, \Delta, \varepsilon)$ be a dual semigroup. A quantum stochastic process $\{j_{st}\}_{0 \leq s \leq t \leq T}$ on $\mathcal{B}$ over some quantum probability space $(\mathcal{A}, \Phi)$ is called a *tensor (resp. boolean, monotone, anti-monotone, or free) Lévy process on the dual semigroup* $\mathcal{B}$, if the following four conditions are satisfied.

(1) (Increment property) We have

$$\begin{aligned} j_{rs} \star j_{st} &= j_{rt} \quad \text{for all } 0 \leq r \leq s \leq t \leq T, \\ j_{tt} &= \varepsilon \mathbf{1}_{\mathcal{A}} \quad \text{for all } 0 \leq t \leq T. \end{aligned}$$

(2) (Independence of increments) The family $\{j_{st}\}_{0 \leq s \leq t \leq T}$ is tensor independent (resp. boolean, monotonically, anti-monotonically independent, or free) w.r.t. Φ, i.e. the n-tuple $(j_{s_1 t_2}, \ldots, j_{s_n t_n})$ is tensor independent (resp. boolean, monotonically, anti-monotonically independent, or free) for all $n \in \mathbb{N}$ and all $0 \leq s_1 \leq t_1 \leq s_2 \leq \cdots \leq t_n \leq T$.

(3) (Stationarity of increments) The distribution $\varphi_{st} = \Phi \circ j_{st}$ of j_{st} depends only on the difference $t - s$.

(4) (Weak continuity) The quantum random variables j_{st} converge to j_{ss} in distribution for $t \searrow s$.

Remark 7.3. The independence property depends on the products and therefore for boolean, monotone and anti-monotone Lévy processes on the choice of a decomposition $\mathcal{B} = \mathbb{C}\mathbf{1} \oplus \mathcal{B}^0$. In order to show that the convolutions defined by $(\varphi_1 \diamond \varphi_2) \circ \Delta$, $(\varphi_1 \triangleright \varphi_2) \circ \Delta$, and $(\varphi_1 \triangleleft \varphi_2) \circ \Delta$ are associative and that the counit ε acts as unit element w.r.t. these convolutions, one has to use the universal property [BGS99, Condition (P4)], which in our setting is only satisfied for morphisms that respect the decomposition. Therefore we are forced to choose the decomposition given by $\mathcal{B}^0 = \ker \varepsilon$.

The marginal distributions $\varphi_{t-s} := \varphi_{st} = \Phi \circ j_{st}$ form again a convolution semigroup $\{\varphi_t\}_{t\in\mathbb{R}_+}$, with respect to the tensor (boolean, monotone, anti-monotone, or free respectively) convolution defined by $(\varphi_1 \tilde{\otimes} \varphi_2) \circ \Delta$ ($(\varphi_1 \diamond \varphi_2) \circ \Delta$, $(\varphi_1 \triangleright \varphi_2) \circ \Delta$, $(\varphi_1 \triangleleft \varphi_2) \circ \Delta$, or $(\varphi_1 * \varphi_2) \circ \Delta$, respectively). It has been shown that the generator $\psi : \mathcal{B} \to \mathbb{C}$,

$$\psi(b) = \lim_{t \searrow 0} \frac{1}{t}\big(\varphi_t(b) - \varepsilon(b)\big)$$

is well-defined for all $b \in \mathcal{B}$ and uniquely characterizes the semigroup $\{\varphi_t\}_{t\in\mathbb{R}_+}$, cf. [Sch95, BGS99, Fra01].

Denote by S be the *flip* map $S : \mathcal{B} \coprod \mathcal{B} \to \mathcal{B} \coprod \mathcal{B}$, $S = m_{\mathcal{B}\coprod\mathcal{B}} \circ (i_2 \coprod i_1)$, where $i_1, i_2 : \mathcal{B} \to \mathcal{B} \coprod \mathcal{B}$ are the inclusions of $\mathcal{B}$ into the first and the second factor of the free product $\mathcal{B} \coprod \mathcal{B}$. The flip map S acts on $i_1(a_1)i_2(b_1)\cdots i_2(b_n) \in \mathcal{B} \coprod \mathcal{B}$ with $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{B}$ as

$$S\Big(i_1(a_1)i_2(b_1)\cdots i_2(b_n)\Big) = i_2(a_1)i_1(b_1)\cdots i_1(b_n).$$

If $j_1 : \mathcal{B} \to \mathcal{A}_1$ and $j_2 : \mathcal{B} \to \mathcal{A}_2$ are two unital $*$-algebra homomorphisms, then we have $(j_2 \coprod j_1) \circ S = \gamma_{\mathcal{A}_1,\mathcal{A}_2} \circ (j_1 \coprod j_2)$. Like for bialgebras, the opposite comultiplication $\Delta^{\rm op} = S \circ \Delta$ of a dual semigroup $(\mathcal{B}, \Delta, \varepsilon)$ defines a new dual semigroup $(\mathcal{B}, \Delta^{\rm op}, \varepsilon)$.

Lemma 7.4. *Let $\{j_{st} : \mathcal{B} \to (\mathcal{A}, \Phi)\}_{0\le s\le t\le T}$ be a quantum stochastic process on a dual semigroup $(\mathcal{B}, \Delta, \varepsilon)$ and define its time-reversed process $\{j^{\rm op}_{st}\}_{0\le s\le t\le T}$ by*

$$j^{\rm op}_{st} = j_{T-t,T-s}$$

for $0 \le s \le t \le T$.

(i) *The process $\{j_{st}\}_{0\le s\le t\le T}$ is a tensor (boolean, free, respectively) Lévy process on the dual semigroup $(\mathcal{B}, \Delta, \varepsilon)$ if and only if the time-reversed process $\{j^{\rm op}_{st}\}_{0\le s\le t\le T}$ is a tensor (boolean, free, respectively) Lévy process on the dual semigroup $(\mathcal{B}, \Delta^{\rm op}, \varepsilon)$.*

(ii) *The process $\{j_{st}\}_{0\le s\le t\le T}$ is a monotone Lévy process on the dual semigroup $(\mathcal{B}, \Delta, \varepsilon)$ if and only if the time-reversed process $\{j^{\rm op}_{st}\}_{0\le s\le t\le T}$ is an anti-monotone Lévy process on the dual semigroup $(\mathcal{B}, \Delta^{\rm op}, \varepsilon)$.*

Proof. The equivalence of the stationarity and continuity property for the quantum stochastic processes $\{j_{st}\}_{0\leq s\leq t\leq T}$ and $\{j^{\mathrm{op}}_{st}\}_{0\leq s\leq t\leq T}$ is clear.

The increment property for $\{j_{st}\}_{0\leq s\leq t\leq T}$ with respect to Δ is equivalent to the increment property of $\{j^{\mathrm{op}}_{st}\}_{0\leq s\leq t\leq T}$ with respect to Δ^{op}, since

$$\begin{aligned} m_{\mathcal{A}}\circ\left(j^{\mathrm{op}}_{st}\coprod j^{\mathrm{op}}_{tu}\right)\circ\Delta^{\mathrm{op}} &= m_{\mathcal{A}}\circ\left(j_{T-t,T-s}\coprod j_{T-u,T-t}\right)\circ S\circ\Delta \\ &= m_{\mathcal{A}}\circ\gamma_{\mathcal{A},\mathcal{A}}\circ\left(j_{T-u,T-t}\coprod j_{T-t,T-s}\right)\circ\Delta \\ &= m_{\mathcal{A}}\circ\left(j_{T-u,T-t}\coprod j_{T-t,T-s}\right)\circ\Delta \end{aligned}$$

for all $0\leq s\leq t\leq u\leq T$.

If $\{j_{st}\}_{0\leq s\leq t\leq T}$ has monotonically independent increments, i.e. if the n-tuples $(j_{s_1t_2},\ldots,j_{s_nt_n})$ are monotonically independent for all $n\in\mathbb{N}$ and all $0\leq s_1\leq t_1\leq s_2\leq\cdots\leq t_n$, then the n-tuples $(j_{s_nt_n},\ldots,j_{s_1t_1})=(j^{\mathrm{op}}_{T-t_n,T-s_n},\ldots,j^{\mathrm{op}}_{T-t_1,T-s_1})$ are anti-monotonically independent and therefore $\{j^{\mathrm{op}}_{st}\}_{0\leq s\leq t\leq T}$ has anti-monotonically independent increments, and vice versa.

Since tensor and boolean independence and freeness do not depend on the order, $\{j_{st}\}_{0\leq s\leq t\leq T}$ has tensor (boolean, free, respectively) independent increments, if and only $\{j^{\mathrm{op}}_{st}\}_{0\leq s\leq t\leq T}$ has tensor (boolean, free, respectively) independent increments. □ □

Schürmann and Voss [SV12] have given a new proof of Schoenberg's, using ideas from [SSV10], that includes also the free convolution. This shows that Lévy processes on dual semigroups are in one-to-one correspondence with generating functionals, i.e. hermitian, conditionally positive linear functionals that vanish on the identity.

Theorem 7.5. (Schoenberg correspondence) *Let $\{\varphi_t\}_{t\geq 0}$ be a convolution semigroup of unital functionals with respect to the tensor, boolean, monotone, or anti-monotone convolution on a dual semigroup $(\mathcal{B},\Delta,\varepsilon)$ and let $\psi:\mathcal{B}\to\mathbb{C}$ be defined by*

$$\psi(b)=\lim_{t\searrow 0}\frac{1}{t}\big(\varphi_t(b)-\varepsilon(b)\big)$$

for $b\in\mathcal{B}$. Then the following statements are equivalent.

(i) *φ_t is positive for all $t\geq 0$.*
(ii) *ψ is hermitian and conditionally positive.*

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