What is Quantum Probability?
Why do we need Quantum Probability?

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Quantum probability spaces

The most fundamental definition in quantum (or noncommutative) probability:

Definition

A quantum probability space is a pair (A, φ) consisting of a (Von Neumann) algebra A and a (normal) state $\varphi : A \to \mathbb{C}$.

Question

How can this definition be motivated? What does it "mean"?

Recall

Definition

A "classical" probability space is a triple $(\Omega, \mathcal{F}, P)$ where

- Ω is a set, the sample space, the set of all possible outcomes.
- $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ is the set of events.
- $P : \mathcal{F} \to [0, 1]$ assign to each event its probability.

This description of randomness is based on the idea that randomness is due to a lack of information.
If we knew which $\omega \in \Omega$ is realized, then the randomness disappears.

"Classical" probability spaces as special cases of quantum probability spaces

Example (Classical $\subseteq$ Quantum)

To a classical probability space $(\Omega, \mathcal{F}, P)$ we can associate a quantum probability space (A, φ), take

- $A = L^\infty(\Omega, \mathcal{F}, P)$, the algebra of bounded measurable fonctions $f : \Omega \to \mathbb{C}$, called the algebra of random variables or observables.
- $\varphi : A \ni f \mapsto E(f) = \int_\Omega f \mathrm{d}P$, which assigns to each random variable/observable its expected value.

$(\Omega, \mathcal{F}, P)$ and (A, P) are essentially equivalent (by the spectral theorem).

Quantum probability spaces

Exemple (Quantum mechanics)

Let H be a Hilbert space, with a unit vector ψ (or a density matrix ρ). Then the quantum probability space associated to (H, ψ) (or (H, ρ) is given by

- $A = B(H)$, the algebra bounded linear operators $X : H \to H$. Self-adjoint (or normal) operators can be considered as quantum random variables or observables.
- $\varphi : B(H) \ni X \mapsto \varphi(X) = \langle \psi, X\psi \rangle$ (or $\varphi(X) = \mathrm{tr}(\rho X)$).

Question

Is "quantum randomness" different from "classical randomness"?

Suppose from now on that H is a finite dimensional complex Hilbert space.

Spectral theorem

If $X \in B(H)$ is an observable (i.e. a self-adjoint operator = hermitian matrix), then it can be written as

$$X = \sum_{\lambda \in \sigma(X)} \lambda E_\lambda$$

where $\sigma(X)$ denotes the spectrum of X (= set of eigenvalues) and E_λ the orthogonal projection onto the eigenspace of X associated to the eigenvalue λ.

Von Neumann's Collapse Postulate

A measurement of the observable X on a quantum system in the state ρ yields the value $\lambda \in \sigma(X)$ with probability

$$p_\lambda = \mathrm{tr}(\rho E_\lambda)$$

where tr denotes the normalised trace.
If the observed value is λ, then the state "collapses" to

$$\tilde{\rho}_\lambda = \frac{E_\lambda \rho E_\lambda}{\mathrm{tr}(\rho E_\lambda)}.$$

Quantum $\to$ Classical

To each observable X in a quantum probability space we can associate a classical probability space, $\Omega = \sigma(X)$, $P(\{\lambda\}) = \mathrm{tr}(\rho E_\lambda)$.

Dictionary "Classical $\leftrightarrow$ Quantum"

	Classical	Quantum
sample space	a set $\Omega = \{\omega_1, \ldots, \omega_n\}$	a Hilbert space $H = \mathbb{C}^n$
events	subsets of Ω that form a σ-algebra (also a Boolean algebra)	the orthogonal projections in H, they form a lattice, which is not Boolean (or distributive), e.g., in general $E \wedge (F_1 \vee F_2) \neq (E \wedge F_1) \vee (E \wedge F_2)$
random variables/ observables	measurable functions $f : \Omega \to \mathbb{R}$ form a commutative (von Neumann) algebra to each event $E \in \mathcal{F}$ we get a r.v. $\mathbf{1}_E$	self-adjoint operators $X : H \to H$, $X^* = X$ span a non-commutative (von Neumann) algebra event are observables with values in $\{0, 1\}$. Note that $E_\lambda = \mathbf{1}_{\{\lambda\}}(X)$.

Dictionary “Classical ↔ Quantum”

	Classical	Quantum
probability distribution/ state	a countably additive function $P:\mathcal{F}\to[0,1]$ determined by n pos. real numbers $p_k=P(\{\omega_k\})$ s.t. $\sum_{k=1}^n p_k=1$ $P(E)=\sum_{\omega\in E}P(\{\omega\})$	a density matrix, i.e. a pos. operator with $\mathrm{tr}(\rho)=1$ $\Pr(X=\lambda)=\mathrm{tr}(\rho E_\lambda)$, $\Pr(X\in E)=\mathrm{tr}(\rho\mathbf{1}_E(X))$, $\mathbf{1}_E(X)=\sum_{\lambda\in E\cap\sigma(X)}E_\lambda$.
expectation	$E(f)=\int_\Omega f\mathrm{d}P$ $=\sum_{k=1}^n f(\omega)P(\{\omega\})$	$E(X)=\mathrm{tr}(\rho X)$
variance	$\mathrm{Var}(f)=E(f^2)-E(f)^2$	$\mathrm{Var}_\rho(X)$ $=\mathrm{tr}(\rho X^2)-\big(\mathrm{tr}(\rho X)\big)^2$.

Dictionary "Classical $\leftrightarrow$ Quantum"

	Classical	Quantum
Extreme points	the set of all probability distribution on Ω is a compact convex set exactly n extreme points δ_{ω_k}, $k = 1, \ldots, n$. if $P = \delta_{\omega_k}$, then the the distribution of any r.v. f is contentrated in one point (namely $f(\omega_k)$).	the extreme points of the set $\mathcal{S}(H)$ of states on H are exactly the one-dim. projections onto the rays $\mathbb{C}u$, $u \in H$ a unit vector. if $\rho = P_u$ then $\mathrm{Var}(X) =$ $= \|(X - \langle u, Xu \rangle)u\|^2$ thus $\mathrm{Var}(X) = 0$ if and only if if u is an eigenvector of X. Degeneracy of the state does not kill the uncertainty of the obervables!

Dictionary "Classical ↔ Quantum"

	Classical	Quantum
Product spaces systems	given two systems described by $(\Omega_i, \mathcal{F}_i, P_i)$, $i = 1, 2$ then $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, P_1 \otimes P_2)$ describes both independent systems as a single system	given two systems described by (H_i, ρ_i), $i = 1, 2$ then $(H_1 \otimes H_2, \rho_1 \otimes \rho_2)$ describes both independent systems as a single system $\to$ independence $\to$ entanglement
reversible dynamics	bijective (measurable) maps $T : \Omega \to \Omega$ $f \mapsto f \circ T$ (for r.v.) $P \mapsto P \circ T^{-1}$ (for prob.)	unitary operators $U : H \to H$ Heisenberg: $X \mapsto U^* X U$ Schrödinger: $\rho \mapsto U \rho U^*$ or $\psi \mapsto U\psi$.

Three distinguishing features (1)

Theorem

E, F projections on H s.t. $EF \neq FE$. Then

$$E \vee F \not\leq E + F.$$

Corollary

E, F projections on H s.t. $EF \neq FE$. Then, for some state ρ,

$$\mathrm{tr}\rho(E \vee F) \not\leq \mathrm{tr}\rho E + \mathrm{tr}\rho F.$$

Theorem (Heisenberg Uncertainty)

X, Y observables, ρ a state. Then

$$\begin{aligned}\mathrm{Var}_\rho(X)\mathrm{Var}_\rho(Y) &\geq \left(\mathrm{tr}\rho\frac{1}{2}(XY+YX)\right)^2+\left(\mathrm{tr}\rho\frac{i}{2}(XY-YX)\right)^2\\ &\geq \frac{1}{2}\big(\mathrm{tr}\rho i[X,Y]\big)^2.\end{aligned}$$

Extreme points

Extremal states (= one-dimensional projections) are called pure states.
The set of all pure states on an n-dimensional complex Hilbert space is a real manifold of dimesnsion $2n-2$.
The set of all extremal probability measures on a sample space of n points has cardinality n.

Let H be a separabe Hilbert space and denote by $\mathcal{P}(H)$ the set of orthogonal projections on H.

Definition

A map $P : \mathcal{P}(H) \to [0,1]$ is called an additive probability measure on $\mathcal{P}(H)$ if $P(\mathrm{id}_H) = 1$ and

$$P\left(\sum_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} P(E_k)$$

for all $n \geq 1$ and all families $E_1, \ldots, E_n$ of pairwise orthogonal projections. If the additivity condition holds also for countable families, then we say that P is a σ-additive probability measure on $\mathcal{P}(H)$

Gleason's theorem

Theorem

Let $\dim H \geq 3$. Then each additive measure on $\mathcal{P}(H)$ can be uniquely extended to a state on $B(H)$. Conversly the restriction of every state to $\mathcal{P}(H)$ is a additive measure on $\mathcal{P}(H)$.
The same holds for σ-additive probability measures and normal states: Every σ-additive probability measure can be extended to a normal state and every normal state restricts to a σ-additive probability measure.

The Kochen-Specker theorem

Definition

A valuation (or "dispersion-free" additive probability measure) is an additive probability measure $P : \mathcal{P}(H) \to \{0, 1\}$.

Remark

$\to$ realistic non-contextual models / hidden variables.

Theorem

Let $\dim(H) \geq 3$. Then there exists no valuation on $\mathcal{P}(H)$.

This theorem proves that they exist no "realistic non-contextual" models of quantum probability.

One q-bit

Example: spin-$\frac{1}{2}$ or polarisation of a photon

$H = \mathbb{C}^2$. The most general state vector is of the form

$$\psi = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}.$$

with $\theta \in [0,\pi)$, $\phi \in [0,2\pi)$, $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$, and can be visualized as the point (θ,ϕ) on the unit sphere (Bloch sphere) in $\mathbb{R}^3$, i.e. the vector

$$\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

Example: spin-$\frac{1}{2}$ or polarisation of a photon, cont'd

Density matrices are of the form

$$\rho(x,y,z) = \frac{I + x\sigma_x + y\sigma_y + z\sigma_z}{2}$$

with $x, y, z \in \mathbb{R}$, $x^2 + y^2 + z^2 \leq 1$, where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that

$$|\psi\rangle\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} = \rho\begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}.$$

Example: spin-$\frac{1}{2}$ or polarisation of a photon, cont'd

Observables (self-adjoint operators) are of the form

$$X = a|\psi\rangle\langle\psi| + b|\psi_\perp\rangle\langle\psi_\perp|,$$

for $a, b \in \mathbb{R}$, ψ a unit vector, $\psi_\perp$ orthogonal to ψ (unique up to a phase).
In an experiment, X takes values a and b, with probabilities

$$P(X = a) = \varphi\big(|\psi\rangle\langle\psi|\big) \text{ and } P(X = b) = \varphi\big(|\psi_\perp\rangle\langle\psi_\perp|\big)$$

E.g., for $\phi = \langle\psi', \cdot\,\psi'\rangle$ the vector state associated to
$\psi' = \cos\frac{\theta'}{2}|0\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|1\rangle$, we get

$$P(X = a) = \big|\langle\psi, \psi'\rangle\big|^2 = \frac{1+\cos\vartheta}{2} \text{ and } P(X = b) = \frac{1-\cos\vartheta}{2}$$

where ϑ is the angle between ψ and ψ' on the Bloch sphere.