A SIMPLE PROOF OF CLASSIFICATION THEOREM FOR POSITIVE NATURAL PRODUCTS

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Abstract. A simplification of the proof of classification theorem for natural notions of stochastic independence is given. This simplification is made possible after adding the positivity condition to the algebraic axioms for a (non-symmetric) universal product (= a natural product). Indeed this simplification is nothing but a simplification, under the positivity, of the proof of the claim that, for any natural product, the 'wrong-ordered' coefficients all vanish in the expansion form. The known proof of this claim involves a cumbersome process of solving a system of quadratic equations in 102 unknowns, but in our new proof under the positivity we can avoid such a process.

1. Introduction

In non-commutative probability theory, there exist several different notions of stochastic independence, for example, tensor independence (= classical independence), free independence, Boolean independence and monotone independence (see the references in [1, 4] for the detailed explanations).

This phenomena is very specific to the non-commutative situation because in the commutative case there exisits the only one notion of independence, i.e. tensor independence. We expect that based on the various notions of independence one can develope various probability theories in some way parallel to classical probability theory.

A classification program for universal notions of stochastic independence was carried out in a series of papers [5, 6, 1, 3, 4]. In [5] Schürmann proposed that universal notions of stochastic independence should be formulated as universal products among non-commutative probability spaces. In [6, 1] Speicher (in the expansion form), and Ghorbal and Schürmann (in form of the canonical axioms) proved that the only possible 'symmetric'(or 'commutative') universal products are 3 products: tensor, free and boolean product. Extending this result to the non-symmetric case, we proved in [3] (in the expansion form) and in [4] (in the form of canonical axioms) that the only possible 'non-symmetric' universal products (= natural products) are 5 products: tensor, free, boolean, monotone, and anti-monotone product.

However the proof of the classification theorem for natural products (Theorem 2.2 in [4]) contains a cumbersome step with complicated calculations, unfortunately. The claim of the step is that the coefficient $t(\pi, \lambda; \sigma)$, which is associated to a given natural product, vanishes whenever the partition σ is 'wrong-ordered'. Here the explanations of coefficient $t(\pi, \lambda; \sigma)$ and 'wrong-orderedness' will be

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given in Section 3 of the present paper. In [4], to complete this step, a system of quadratic equations in 102 unknowns was solved by hands. This is heavy calculations that consists of 22 pages in [4] in its compact form.

The aim of this paper is to improve this cumbersome proof for the claim (V): the vanishment of wrong coefficients $\{t(\pi, \lambda; \sigma)\}$, into the more clear proof, without using such a big system of equations. However this simplification for the proof of the claim (V) is made possible after adding the condition of positivity (P) to the algebraic axioms for a natural product. Up to now we dont know if such a simplification for the proof of the claim (V) is possible or not, without using the positivity (P).

This paper consists of the following sections. In Section 2 we prepare some notations concerning partitions of a finite linearly ordered set. In Section 3, after introducing some conditions on a product among algebraic probability spaces (universality, associativity, positivity, etc), we explain the relation beteween the classification theorem for natural products and the vanishment result (V). In Section 4 we give a simple proof of the vanishment result (V) under the positivity assumption (P), which avoids the use of a big system of equations.

Through out the paper, $\mathbb{C}$ is the field of complex numbers, $\mathbb{N}^*$ is the set of all natural numbers ($\neq 0$), and $\#A$ or $|A|$ denotes the cardinality of a finite set A.

2. Notations on Partitions

Here we describe some notations on partitions used in this paper (see [4]).

Let S be a finite linearly ordered set. A collection $\pi = \{U_1, U_2, \cdots, U_p\}$ of subsets of S is called a *partition* of S if $S = \cup_{i=1}^{p} U_i$, $U_i \neq \emptyset$, and $U_i \cap U_j = \emptyset$ for all $i, j \in \{1, 2, \cdots, p\}$. A pair $(\pi, \lambda) = \{U_1 \prec U_2 \prec \cdots \prec U_p\}$ of a partition π and a linear ordering λ among blocks in π is called a *linearly ordered partition* of S (see [3]). A collection $\pi = \{U_1, U_2, \cdots, U_p\}$ of finite sequences U of elements from S is called a *BGS-partition* of S if $\#\{i_1, i_2, \cdots, i_k\} = k$ for each $U = (i_1 i_2 \cdots i_k) \in \pi$ and if $\overline{\pi} := \{\overline{U_1}, \overline{U_2}, \cdots \overline{U_p}\}$ is a (usual) partition of S (see [1]). Here we put $\overline{U} := \{i_1, i_2, \cdots, i_k\}$ for $U = (i_1 i_2 \cdots i_k) \in \pi$. For each block U in a BGS-partition π, we put $lg(U) := \#(\overline{U})$ the length of U.

Denote by $\mathcal{P}(S)$, $\mathcal{LP}(S)$ and $\vec{\mathcal{P}}(S)$, the set of all partitions, linearly ordered partitions and BGS-partitions of S, respectively. For each BGS-partition $\sigma = \{U_1, U_2, \cdots, U_p\}$ in $\vec{\mathcal{P}}(S)$, there exists naturally the associated usual partition in $\mathcal{P}(S)$ given by $\overline{\sigma} := \{\overline{U_1}, \overline{U_2}, \cdots, \overline{U_p}\}$. Conversely we identify $\mathcal{P}(S)$ as the subset of $\vec{\mathcal{P}}(S)$ through the natural correspondence $\pi \mapsto \pi'$ given by $\pi \ni U_q = \{i_1 < i_2 < \cdots < i_k\} \mapsto U' = (i_1 i_2 \cdots i_k) \in \pi'$.

For usual partitions $\pi, \sigma \in \mathcal{P}(S)$, we write $\sigma \leq \pi$ when σ is a refinement of π, i.e. when $\langle\ \forall U \in \sigma, \exists V \in \pi$ s.t. $U \subset V\ \rangle$. When the ordered set S is given by $\{1, 2, \cdots, n\}$ with the natural order, we write $\mathcal{P}(n)$, $\mathcal{LP}(n)$, $\vec{\mathcal{P}}(n)$ in stead of $\mathcal{P}(S)$, $\mathcal{LP}(S)$, $\vec{\mathcal{P}}(S)$, respectively.

3. Classification Theorem and Vanishment Result

In this section we describe the relation between the classfication theorem and the vanishment result (V). For the detailes see [4].

An *algebraic probability space* $(\varphi, \mathcal{A})$ is a pair of an associative $\mathbb{C}$-algebra $\mathcal{A}$ and a $\mathbb{C}$-linear functional φ over $\mathcal{A}$. Denote by $\mathcal{K}$ the class of all algebraic probability spaces $(\varphi, \mathcal{A})$. We do not assume the existence of unit elements for these algebras $\mathcal{A}$. Denote by $\mathcal{A}'$ the set of all $\mathbb{C}$-linear finctionals φ over $\mathcal{A}$. Also denote by $\mathcal{A}_1 \sqcup \mathcal{A}_2$ the free product of algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. Then for any algebra homomorphisms $j_l : \mathcal{B}_l \to \mathcal{A}_l$ $(l = 1, 2)$, there exists a unique algebra homomorphism $j_1 \amalg j_2 : \mathcal{B}_1 \sqcup \mathcal{B}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ such that $i_l \circ j_l = (j_1 \amalg j_2) \circ \iota_l$ for all $l = 1, 2$. Here $i_l : \mathcal{A}_l \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ $(l = 1, 2)$ and $\iota_l : \mathcal{B}_l \to \mathcal{B}_1 \sqcup \mathcal{B}_2$ $(l = 1, 2)$ are the natural embedings. We write the 'expectation' of $a \in \mathcal{A}$ w.r.t φ by $\varphi[a]$ in stead of $\varphi(a)$.

Any map $\Box : \mathcal{K} \times \mathcal{K} \to \mathcal{K} : \big((\varphi_1, \mathcal{A}_1), (\varphi_2, \mathcal{A}_2)\big) \mapsto (\varphi_1 \Box \varphi_2, \mathcal{A}_1 \sqcup \mathcal{A}_2)$ is called a *product* over $\mathcal{K}$. For simplicity we used here the same symbol $\Box$ to denote two different levels of operations.

A *natural product* $\Box$ is a product over $\mathcal{K}$ satisfying the following four conditions.

(N1) *universality*: For any algebra homomorphisms $j_l : \mathcal{B}_l \to \mathcal{A}_l$ and any $\varphi_l \in \mathcal{A}_l{}'$ $(l = 1, 2)$, we have

$$(\varphi_1 \Box \varphi_2) \circ (j_2 \amalg j_2) = (\varphi_1 \circ j_1) \Box (\varphi_2 \circ j_2).$$

(N2) *associativity*: For all $(\varphi_l, \mathcal{A}_l) \in \mathcal{K}$ $(l = 1, 2, 3)$, we have

$$(\varphi_1 \Box \varphi_2) \Box \varphi_3 = \varphi_1 \Box (\varphi_2 \Box \varphi_3)$$

under the identification $(\mathcal{A}_1 \sqcup \mathcal{A}_2) \sqcup \mathcal{A}_3 = \mathcal{A}_1 \sqcup (\mathcal{A}_2 \sqcup \mathcal{A}_3)$.

(N3) *extension*: For all $(\varphi_l, \mathcal{A}_l) \in \mathcal{K}$ $(l = 1, 2)$, we have

$$(\varphi_1 \Box \varphi_2) \circ i_l = \varphi_l \quad (l = 1, 2),$$

where i_l are the natural embedings of $\mathcal{A}_l$ to the free product $\mathcal{A}_1 \sqcup \mathcal{A}_2$.

(N4) *factorization*: For all $(\varphi_l, \mathcal{A}_l) \in \mathcal{K}$ $(l = 1, 2)$, we have

$$(\varphi_1 \Box \varphi_2)[i_1(a) i_2(b)] = (\varphi_1 \Box \varphi_2)[i_2(b) i_1(a)] = \varphi_1[a] \varphi_2[b]$$

for all $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$.

The notion of natural product is nothing but a modification, to the non-symmetric case, of the notion of universal product of Schürmann in [5]. In [4] we proved the following classification theorem for natural products.

Theorem 3.1. *There exist exactly 5 natural products: tensor, free, Boolean, monotone and anti-monotone product.*

Here we omit the definitions of these 5 products (tensor, free, Boolean, monotone and anti-monotone), because in our discussions in the present paper, we have no need to know them. For the detailed explanation on these products and independences, see the references in [1, 4].

The strategy to prove Theorem 3.1 is to reduce this Theorem 3.1 to the next Theorem 3.2 through Theorem 3.3 and Theorem 3.4. It is the same strategy as that in the case of 'symmetric' products in [1] but in our non-symmetric case we must take care of some ordere structures on partitions.

Let us prepare some notations for the description of these Theorems. Let $(\pi, \lambda) = \{V_1 \prec V_2 \prec \cdots \prec V_p\} \in \mathcal{LP}(n)$ be a linearly ordered partition, and $(\varphi_l, \mathcal{A}_l)_{l=1}^p$ be a family of algebraic probability spaces. Then we simply write

$a_1a_2\cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$ to denote the situation that $a_1 \in \mathcal{A}_{l_1}$, $a_2 \in \mathcal{A}_{l_2}$, $\cdots$, $a_n \in \mathcal{A}_{l_n}$ and that $l_k = q$ if and only if $k \in V_q$. We always identify $a \in \mathcal{A}_l$ with its natural image $i_l(a) \in \amalg_{l=1}^p \mathcal{A}_l$. Let $a_1a_2\cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$ and $\sigma \in \vec{\mathcal{P}}(n)$ with $\overline{\sigma} \leq \pi$ be fixed, then we put for each $U = (i_1i_2\cdots i_k) \in \sigma$

$$\varphi_U[a_1, a_2, \cdots, a_n] := \varphi_{l(U)}[a_U] := \varphi_{l(U)}[a_{i_1}a_{i_2}\cdots a_{i_k}].$$

Here $l(U)$ is the label $l \in \{1, 2, \cdots, p\}$ such that $l_{i_s} = l$ for all $i_s \in \overline{U}$.

A *quasi-universal product* $\Box$ is a product over $\mathcal{K}$ satisfying the following three conditions.

(Q1) *associativity*: same as (N2)

(Q2) *quasi-universal calculation rule for mixed moments*: There exists a family of constants

$$\left\{ t(\pi, \lambda; \rho) \,\middle|\, \begin{array}{l} \rho \in \mathcal{P}(n), \ \ \rho \leq \pi, \\ (\pi, \lambda) \in \mathcal{LP}(n), \ \ n \in \mathbb{N}^* \end{array} \right\}$$

such that, for any p-tuple $(\varphi_l, \mathcal{A}_l)_{l=1}^p$ of algebraic probability spaces, and $\varphi = \Box_{l=1}^p \varphi_l$, we have

$$\varphi[a_1a_2\cdots a_n] = \sum_{\rho \leq \pi} t(\pi, \lambda; \rho) \prod_{U \in \rho} \varphi_U[a_1, a_2, \cdots, a_n],$$

whenever $a_1a_2\cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$ with $(\pi, \lambda) = \{V_1 \prec V_2 \prec \cdots \prec V_p\}$.

(Q3) *normalization*:

$$t(1) = t(12) = t(21) = 1.$$

Here we put $t(s) := t(\pi, \lambda) := t(\pi, \lambda; \pi)$, where $s = (s_1s_2\cdots s_n)$ is the associated sequence to (π, λ) defind by the condition that $s_i = l$ if and only if $i \in V_l$.

Theorem 3.2. *There exist exactly 5 quasi-universal products: tensor, free, Boolean, monotone and anti-monotone product.*

Theorem 3.2 was proved in [3] based on the method of Speicher in [6]. The following expansion theorem (Theorem 3.3) was shown in [4] as a result by a direct application of theory of universal families of Ben Ghorbal and Schürmann [1].

Theorem 3.3. *Let a natural product $\Box$ be given. Then there exists uniquely a family of constants*

$$\left\{ t(\pi, \lambda; \sigma) \,\middle|\, \begin{array}{l} \sigma \in \vec{\mathcal{P}}(n), \ \ \overline{\sigma} \leq \pi, \\ (\pi, \lambda) \in \mathcal{LP}(n), \ \ n \in \mathbb{N}^* \end{array} \right\}$$

such that, for any p-tuple $(\varphi_l, \mathcal{A}_l)_{l=1}^p$ of algebraic probability spaces, and $\varphi = \Box_{l=1}^p \varphi_l$, we have

$$\varphi[a_1a_2\cdots a_n] = \sum_{\substack{\sigma \in \vec{\mathcal{P}}(n) \\ \overline{\sigma} \leq \pi}} t(\pi, \lambda; \sigma) \prod_{U \in \sigma} \varphi_U[a_1, a_2, \cdots, a_n],$$

whenever $a_1a_2\cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$ with $(\pi, \lambda) = \{V_1 \prec V_2 \prec \cdots \prec V_p\}$.

Let $\sigma \in \vec{\mathcal{P}}(n)$ be a BGS-partition. A block $U = (i_1 i_2 \cdots i_k)$ in σ is said to be *wrong-ordered* if there exist $a, b \in \{1, 2, \cdots, k\}$ such that $a < b$ but $i_a > i_b$. A BGS-partition σ is said to be *wrong-ordered* if there exists in σ a wrong-ordered block U. Let $\{t(\pi, \lambda; \sigma)\}$ be the coefficients associated to a natural product $\Box$ in Theorem 3.3. A coefficient $t(\pi, \lambda, \sigma)$ is said to be *wrong-ordered* if σ is wrong-ordered. In [4] we proved the following vanishment result. We denote it by (V).

Theorem 3.4. *For any natural product, its wrong-ordered coefficients all vanish.*

Theorem 3.4 implies that any natural product is a quasi-universal product, and hence we reach to the classification Theorem 3.1 through Theorem 3.2.

However the proof of the vanishment result (V) (Theorem 3.4) given in [4] is a cumbersome one consisting of elementary but heavy calculations, unfortunately. Therefore for the clear understanding of the classification theorem, it is desirable to improve this heavy proof into the more light one. We give in Section 4 such a simplified proof for the vanishment result under some additional condition of positivity (P) for natural products.

Now let us define the positivity for a product $\Box$. Let $\mathcal{A}$ be a $*$-algebra and φ be a linear functional over $\mathcal{A}$. The unitization $(\widetilde{\varphi}, \widetilde{\mathcal{A}})$ of $(\varphi, \mathcal{A})$ is the pair of a unital $*$-algebra $\widetilde{\mathcal{A}}$ and a unital linear functional $\widetilde{\varphi}$ over $\widetilde{\mathcal{A}}$, defind by $\widetilde{\mathcal{A}} := \mathbb{C}1_{\widetilde{\mathcal{A}}} \oplus \mathcal{A}$ with $1_{\widetilde{\mathcal{A}}}$ an artificial unit, and $\widetilde{\varphi}[1_{\widetilde{\mathcal{A}}}] = 1$, $\widetilde{\varphi}[a] = \varphi[a]$, $a \in \mathcal{A}$. If $\mathcal{A}$ is a unital $*$-algebra and φ is a unital linear functional over $\mathcal{A}$ ($\varphi[1_{\mathcal{A}}] = 1$), then φ is a state on $\mathcal{A}$ if and only if $\widetilde{\varphi}$ is a state on $\widetilde{\mathcal{A}}$. A $*$-probability space $(\mathcal{A}, \varphi)$ is a pair of a $*$-algebra $\mathcal{A}$ and a linear functional φ over $\mathcal{A}$ such that $\widetilde{\varphi}$ is a state on $\widetilde{\mathcal{A}}$.

A product $\Box$ over $\mathcal{K}$ is said to be *positive* if it satisfies the following condition.

(P) *positivity*: For any $*$-algebras $\mathcal{A}_l$ and any functionals $\varphi_l \in \mathcal{A}_l'$ ($l = 1, 2$), $\widetilde{\varphi_1 \Box \varphi_2}$ is a state over $\widetilde{\mathcal{A}_1 \sqcup \mathcal{A}_2}$ whenever $\widetilde{\varphi}_l$ is a state over $\widetilde{A}_l$ for each $l = 1, 2$.

The five products (tensor, free, Boolean, monotone and anti-monotone) are positive.

Using the positivity (P), we can prove without heavy calculations the following Theorem 3.5 which we denote by (V^+).

Theorem 3.5. *For any positive natural product, its wrong-ordered coefficients all vanish.*

The proof (without heavy calculation) of (V^+) (Theorem 3.5) will be presented in Section 4. (V^+) implies that any positive natural product is a quasi-universal product, and hence we immediately reach to the following classification Theorem 3.6 through Theorem 3.2.

Theorem 3.6. *There exist exactly 5 positive natural products: tensor, free, Boolean, monotone and anti-monotone product.*

Now by the same argument in [2] that the four conditions (N1),(N2),(N3) & (P) implies the condition (N4), we get the following Theorem 3.7.

A *positive universal product* is a product over $\mathcal{K}$ satisfying the four conditions (N1), (N2), (N3) and (P). A product over $\mathcal{K}$ is said to be *degenerate* if $(\varphi_1 \Box \varphi_2)[a_1 a_2 \cdots a_n] = 0$ whenever $a_1 a_2 \cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$ with $|\pi| \geq 2$.

Theorem 3.7. *There exist exactly 5 non-degenerate positive universal products: tensor, free, Boolean, monotone and anti-monotone product.*

This is the same theorem as Theorem 2.5 in [2], but this time the proof is improved so that it is dependent on Theorem 3.6 and hence on (V^+) (Theorem 3.5), and not dependent on (V) (Theorem 3.4) with heavy calculations.

4. A Simple Proof of Vanishment Result

In this section we prove the vanishment result (V^+)(= Theorem 3.5) not using heavy algebraic calculations, but using the positivity.

For our purpose it is sufficient to show the following Proposition 4.1 from which we conclude that $t(\pi, \lambda; \sigma_0) = 0$ for all $\sigma_0 \in \vec{\mathcal{P}}(n) \setminus \mathcal{P}(n)$.

Proposition 4.1. *Let $\Box$ be a natural product. Then for each $n \in \mathbb{N}^*$, each $(\pi, \lambda) \in \mathcal{LP}(n)$ and each $\sigma_0 \in \vec{\mathcal{P}}(n) \setminus \mathcal{P}(n)$ with $\overline{\sigma_0} \leq \pi$, there exist $*$-probability spaces $(\mathcal{A}_l, \varphi_l)_{l=1}^{|\pi|}$ and a sequence of elements $a_1, a_2, \cdots, a_n$ with $a_1 a_2 \cdots a_n \in \mathcal{A}_{(\pi,\lambda)}$ such that for $\varphi = \Box_{l=1}^{|\pi|} \varphi_l$ we have*

$$\begin{cases} \text{(1)} \prod_{W \in \sigma} \varphi_{l(W)}[a_W] = \delta_{\sigma,\sigma_0} \text{ for all } \sigma \in \vec{\mathcal{P}}(n) \text{ with } \overline{\sigma} \leq \pi, \\ \text{(2)} \ \varphi\big[a_1 a_2 \cdots a_n\big] = 0. \end{cases}$$

At first let us give a construction of $(\mathcal{A}_l, \varphi_l)_{l=1}^{|\pi|}$ and $a_1, a_2, \cdots, a_n$. Let (π, λ) and σ_0 be fixed. Suppose that $(\pi, \lambda) = \{V_1 \prec V_2 \prec \cdots \prec V_{|\pi|}\}$ and $\sigma_0 = \{U_1, U_2, \cdots, U_{|\sigma_0|}\}$. For each block $V \in \pi$, we put $\sigma_0(V) := \{U \in \sigma_0 | \overline{U} \subset V\}$. Then since $\overline{\sigma_0} \leq \pi$, we have $V = \bigcup_{U \in \sigma_0(V)} \overline{U}$.

For each $V \in \pi$, let us construct a $*$-probability space $(\mathcal{A}_V, \varphi_V)$ by

$$\mathcal{A}_V = \bigoplus_{U \in \sigma_0(V)} \mathcal{B}_U, \qquad \varphi_V = \frac{1}{\#(\sigma_0(V))} \Big(\bigoplus_{U \in \sigma_0(V)} \varphi_U \Big),$$

where we put, for each $U \in \sigma_0(V)$, $\mathcal{B}_U = M_{d(U)}(\mathbb{C})$ the matirix algebra, $\varphi_U(\cdot) = \langle e_1^{(U)} | \cdot e_1^{(U)} \rangle$ the state over $\mathcal{B}_U$, $(e_i^{(U)})_{i=1}^{d(U)}$ the natural CONS of $\mathcal{H}_U := \mathbb{C}^{d(U)}$, and $d(U) := lg(U)$ the length of U.

On these $*$-probability spaces $(\mathcal{A}_l, \varphi_l) := (\mathcal{A}_{V_l}, \varphi_{V_l})$, $l = 1, 2, \cdots, |\pi|$, we construct the operators $a_1, a_2, \cdots, a_n$ $(a_1 \in \mathcal{A}_{l_1}, a_2 \in \mathcal{A}_{l_2}, \cdots, a_n \in \mathcal{A}_{l_n})$ as follows. For each block $U = (i_1 i_2 \cdots i_k) \in \sigma_0$ with $\overline{U} \subset V$ and $k \geq 2$, we define the operators $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ in $\mathcal{A}_V$ as the natural extensions

$$a_{i_q} := \widetilde{b_{i_q}} := b_{i_q} \oplus \Big(\oplus_{\substack{U' \in \sigma_0(V) \\ U' \neq U}} 0 \Big)$$

of the operators $b_{i_1}, b_{i_2}, \cdots, b_{i_k}$ in $\mathcal{B}_U$ given by

$$(b_{i_1}, b_{i_2}, b_{i_3}, \cdots, b_{i_{k-1}}, b_{i_k}) = (E_{1,k}, E_{k,k-1}, E_{k-1,k-2}, \cdots, E_{3,2}, E_{2,1}),$$

where $E_{i,j}$ are the matrix units in $\mathcal{B}_U$, i.e. $\langle e_k^{(U)} | E_{i,j} e_l^{(U)} \rangle = \delta_{ik}\delta_{jl}$. When $U \in \sigma_0(V)$ is a singleton block $U = (i)$, we put $a_i := \widetilde{b_i}$ with $b_i = I_U$ the identity matrix of $\mathcal{B}_U$. These operators $a_{i_1}, a_{i_2}, \cdots, a_{i_k}$ over all $U = (i_1 i_2 \cdots i_k) \in \sigma_0$ well-define the operators $a_1, a_2, \cdots, a_n$ since $\{1, 2, \cdots, n\}$ is the disjoint union of all $\overline{U}$ ($U \in \sigma_0$).

Well let us prove the properties (1) and (2) in Proposition 4.1 separately.

Proof of Property (1). We examine, for general $\sigma \in \vec{\mathcal{P}}(n)$ with $\overline{\sigma} \leq \pi$, the value of $\prod_{W \in \sigma} \varphi_{l(W)}[a_W]$, where we put $l(W) = q$ if $\overline{W} \subset V_q \in \pi$. Concerning a BGS-partition σ with $\overline{\sigma} \leq \pi$, we consider the following three cases a), b) and c).

Case a): $\exists W \in \sigma, \exists U, U' \in \sigma_0$ $s.t.$ $\overline{W} \cap \overline{U} \neq \emptyset$, $\overline{W} \cap \overline{U'} \neq \emptyset$, $U \neq U'$. In this case there exists a common V ($\in \pi$) such that $\overline{W} \cup \overline{U} \cup \overline{U'} \subset V$. The blocks W, U, U' can be expressed as $W = (\iota_1 \iota_2 \cdots \iota_s)$, $U = (i_1 i_2 \cdots i_t)$, $U' = (j_1 j_2 \cdots j_u)$, respectively. From the assumption there exist some $s_1, s_2 \in \{1, 2, \cdots, s\}$, $t_0 \in \{1, 2, \cdots, t\}$ and $u_0 \in \{1, 2, \cdots, u\}$ such that

$$\begin{aligned} a_{\iota_{s_1}} &= a_{i_{t_0}} = \widetilde{b_{i_{t_0}}} \quad (b_{i_{t_0}} \in \mathcal{B}_U), \\ a_{\iota_{s_2}} &= a_{j_{u_0}} = \widetilde{b_{j_{u_0}}} \quad (b_{j_{u_0}} \in \mathcal{B}_{U'}). \end{aligned}$$

Since $U \cap U' = \emptyset$ we have $A a_{\iota_{s_1}} B a_{\iota_{s_2}} C = 0$ and $A a_{\iota_{s_2}} B a_{\iota_{s_1}} C = 0$ for all $A, B, C \in \mathcal{A}_V$, and hence $a_{\iota_1} a_{\iota_2} \cdots a_{\iota_s} = 0$. So we have $\prod_{W' \in \sigma} \varphi_{l(W')}[a_{W'}] = 0$.

Case b): $\overline{\sigma} \leq \overline{\sigma_0}$ & $\sigma \neq \sigma_0$. In this case there exist $W \in \sigma$ and $U \in \sigma_0$ such that $\overline{W} \subset \overline{U}$ & $W \neq U$. By the way, W and U can be expressed as $W = (j_1 j_2 \cdots j_s)$ and $U = (i_1 i_2 \cdots i_t)$. Note that $\{j_1, j_2, \cdots, j_s\} \subset \{i_1, i_2 \cdots, i_t\}$ and $t \geq 2$.

Let us examine the value of $b_{j_1} b_{j_2} \cdots b_{j_s} \xi$ where $\xi := e_1^{(U)}$. For the vector $b_{j_s}\xi$ to be non-zero it is necessary that $j_s = i_t$, because b_{i_t} is the only element in $\{b_{i_1}, b_{i_2}, \cdots, b_{i_t}\}$ that corresponds to $E_{2,1}$ ($\in \mathcal{B}_U$). Next (when $t \geq 3$), for the vector $b_{j_{s-1}} b_{j_s} \xi$ to be non-zero it is necessary that $j_s = i_t$ and $j_{s-1} = i_{t-1}$, because $b_{i_{t-1}}$ is the only element in $\{b_{i_1}, b_{i_2}, \cdots, b_{i_t}\}$ that corresponds to $E_{3,2}$. Repeating this argument we see that for the vector $b_{j_1} b_{j_2} \cdots b_{j_s} \xi$ to be non-zero it is necessary that $(j_1 j_2 \cdots j_{s-1} j_s) = (i_{t-s+1} i_{t-s+2} \cdots i_{t-1} i_t)$.

Furthermore, for the expectation

$$\varphi_U[b_{j_1} b_{j_2} \cdots b_{j_s}] = \langle \xi | b_{j_1} b_{j_2} \cdots b_{j_s} \xi \rangle$$

to be non-zero it is necessary that $(j_1 j_2 \cdots j_s) = (i_1 i_2 \cdots i_t)$, because b_{i_1} is the only element in $\{b_{i_1}, b_{i_2}, \cdots, b_{i_t}\}$ that corresponds to $E_{1,t}$. But by the assumption we have $W = (j_1 \cdots j_s) \neq (i_1 \cdots i_t) = U$, and hence $\varphi_U[b_W] = \varphi_U[b_{j_1} \cdots b_{j_s}] = 0$. Since $\varphi_{l(W)}[a_W] = \varphi_V[a_W] = \frac{1}{\#(\sigma_0(V))}\varphi_U[b_W] = 0$, we have $\prod_{W' \in \sigma} \varphi_{l(W')}[a_{W'}] = 0$

Case c): $\sigma = \sigma_0$. In this case it is clear that $\prod_{W \in \sigma} \varphi_{l(W)}[a_W] = 1$.

From the three cases a),b) and c), we know that the property (1) holds. □

Proof of Property (2). For each block $U = (i_1 i_2 \cdots i_k) \in \sigma_0$, denote by $WP(U)$ the set of all wrong-ordered pairs $p \subset \overline{U}$ in a block U, that is,

$$WP(U) := \left\{ p \subset \overline{U} \;\middle|\; \begin{array}{l} p = \{l, m\},\ m = i_a,\ l = i_b,\ a < b, \\ l < m,\ \text{for some } a, b \in \{1, 2, \cdots, lg(U)\} \end{array} \right\}.$$

Put $W(U) := \bigcup_{p \in WP(U)} p$ and $W(\sigma_0) := \bigcup_{U \in \sigma_0} W(U)$. Then U is wrong-ordered if and only if $W(U) \neq \emptyset$. Since σ_0 is wrong-ordered, we have $W(\sigma_0) \neq \emptyset$. Let us put $v := \max W(\sigma_0)$. Let U_0 be the block in σ_0 that contains v, and V_0 be the block in π that contains v. Then obviously $v \in U_0 \subset V_0$ and $lg(U_0) \geq 2$.

Since v is the largest 'wrong' element in $\{1, 2, \cdots, n\}$, we have $\{v+1, v+2, \cdots, n\} \cap W(\sigma_0) = \emptyset$, and hence any wrong pair p $(\in WP(U_0))$ that contains v must be of the form $p = \{u, v\}$, $u < v$, for some $u \in W(\sigma_0)$. From this we see that the block U_0 must be of the form

$$\begin{aligned} U_0 &= (i_1 \cdots i_a \cdots i_b \cdots i_k) \\ &= (i_1 \cdots v \cdots u \cdots i_k), \end{aligned}$$

where $i_a = v$, $i_b = u$, $a < b$ and $u < v$ for some $a, b \in \{1, 2, \cdots, k\}$ and for some $u \in \overline{U_0}$.

By the way let us estimate the norm of the vector (= an equivalence class) $[a_v a_{v+1} \cdots a_n]$ in the GNS-representation space associated to $\varphi = \square_{l=1}^{|\pi|} \varphi_l$. First we have from Theorem 3.3

$$\begin{aligned} &\| \; [a_v a_{v+1} \cdots a_n] \; \|^2 \\ &= \varphi\big[(a_v a_{v+1} \cdots a_n)^* (a_v a_{v+1} \cdots a_n)\big] \\ &= \varphi\big[a_n^* \cdots a_{v+1}^* a_v^* a_v a_{v+1} \cdots a_n\big] \\ &= \varphi\big[a_n^* \cdots a_{v+1}^* (a_v^* a_v) a_{v+1} \cdots a_n\big] \\ &= \sum_{\substack{\tau \in \vec{\mathcal{P}}(S) \\ \overline{\tau} \leq \rho}} t(\rho, \mu; \tau) \prod_{T \in \tau} \varphi_{l(T)}[c_T]. \end{aligned}$$

Here S is the linearly ordered finite set given by

$$S = \{-n, -(n-1), \cdots, -(v+1), v, v+1, \cdots, n-1, n\},$$

(ρ, μ) is the linearly ordered partion of S associated to the sequence

$$(l_n, l_{n-1}, \cdots, l_{v+1}, l_v, l_{v+1}, \cdots, l_{n-1}, l_n),$$

and c's are the operators defined by

$$c_{-n} = a_n^*,\ c_{-(n-1)} = a_{n-1}^*,\ \cdots,\ c_{-(v+1)} = a_{v+1}^*,$$
$$c_v = a_v,\ c_{v+1} = a_{v+1},\ \cdots,\ c_{n-1} = a_{n-1},\ c_n = a_n.$$

Also here we put $l(T) = q$ if $\langle\, \overline{T} \cap V_q \neq \emptyset$ or $-\overline{T} \cap V_q \neq \emptyset \,\rangle$ with $-\overline{T} := \{-m | m \in \overline{T}\}$.

For each block $\tau \in \vec{\mathcal{P}}(S)$ with $\overline{\tau} \leq \rho$, there exists a unique block $T_0 \in \tau$ such that $\overline{T_0} \ni v$. Also let R_0 be the unique block in ρ such that $R_0 \ni v$, then we have $\overline{T_0} \subset R_0$. Since $u \notin \{v, v+1, \cdots, n\}$, we have $u \notin \overline{T_0}$, and hence $\{a_u^*, a_u\} \cap \{c_m | m \in \overline{T_0}\} = \emptyset$. From the definition of the operators $a_{i_1}, \cdots, a_{i_a}, \cdots, a_{i_b}, \cdots, a_{i_k}$ $(\in \mathcal{A}_{V_0})$ based on the block $U_0 = (i_1 \cdots i_a \cdots i_b \cdots i_k)$ with $i_a = v$, $i_b = u$, we see that

$$\begin{cases} a_u = \widetilde{E_{N+1,N}^{(U_0)}} \text{ with } N = (k-b)+1, \\ a_v = \widetilde{E_{M+1,M}^{(U_0)}} \text{ with } M = (k-a)+1 \quad \text{for} \quad a \geq 2, \\ a_v = \widetilde{E_{1,k}^{(U_0)}} \quad \text{for} \quad a = 1. \end{cases}$$

So we have $c_v = a_v^* a_v = \widetilde{E_{M,M}^{(U_0)}}$ for $a \geq 2$, and $c_v = a_v^* a_v = \widetilde{E_{k,k}^{(U_0)}}$ for $a = 1$. Put $c_{\overline{T_0}} := \{c_m | m \in \overline{T_0}\}$, $a_{\overline{U_0}} := \{a_m | m \in \overline{U_0}\}$ and $a_{\overline{U_0}}^* := \{a_m^* | m \in \overline{U_0}\}$. Then we have

$$\{c_v\} \quad \subset \quad c_{\overline{T_0}} \quad \subset \quad \big((a_{\overline{U_0}} \cup a_{\overline{U_0}}^*) \setminus \{a_u, a_u^*, a_v, a_v^*\}\big) \cup \{c_v\}.$$

This means $\left\{\widetilde{E_{M,M}^{(U_0)}}\right\} \subset c_{\overline{T_0}}$ and

$$c_{\overline{T_0}} \;\subset\; \left(\big(a_{\overline{U_0}} \cup a_{\overline{U_0}}^*\big) \setminus \left\{ \widetilde{E_{N+1,N}^{(U_0)}}, \widetilde{E_{N,N+1}^{(U_0)}}, \widetilde{E_{M+1,M}^{(U_0)}}, \widetilde{E_{M,M+1}^{(U_0)}} \right\} \right) \cup \{\widetilde{E_{M,M}^{(U_0)}}\}$$

for $a \geq 2$, and means $\left\{\widetilde{E_{k,k}^{(U_0)}}\right\} \subset c_{\overline{T_0}}$ and

$$c_{\overline{T_0}} \;\subset\; \left(\big(a_{\overline{U_0}} \cup a_{\overline{U_0}}^*\big) \setminus \left\{ \widetilde{E_{N+1,N}^{(U_0)}}, \widetilde{E_{N,N+1}^{(U_0)}}, \widetilde{E_{1,k}^{(U_0)}}, \widetilde{E_{k,1}^{(U_0)}} \right\} \right) \cup \{\widetilde{E_{k,k}^{(U_0)}}\}$$

for $a = 1$.

Put $T_* := \big(\overline{T_0} \cup (-\overline{T_0})\big) \cap \{v, v+1, \cdots, n\}$, then $\overline{T_0} \subset T_* \cup (-T_*)$. For simplicity we denote $E_{i,j}^{(U_0)}$ by $E_{i,j}$. Concerning T_*, we consider the following three cases a), b) and c).

Case a): $\exists m \in T_*$ s.t. $m \notin \overline{U_0}$. In this case there exists $m' \in S$ and $U' \in \sigma_0$ such that $c_{m'} = a_m = \widetilde{b_m}$ or $c_{m'} = a_m^* = \widetilde{b_m^*}$ with $b_m \in U'$ and $U' \neq U_0$. So we have two operators b_m (or b_m^*) $\in \mathcal{B}_{U'}$ and $b_v \in \mathcal{B}_{U_0}$ with $U' \cap U_0 = \emptyset$ so that $\{\widetilde{b_v}, \widetilde{b_m}\} \subset c_{\overline{T_0}}$ or $\{\widetilde{b_v}, \widetilde{b_m^*}\} \subset c_{\overline{T_0}}$. This implies $\varphi_{l(T_0)}[c_{T_0}] = 0$.

Case b): $T_* \subset \overline{U_0}$ & $i_a = v$ $(a \geq 2)$. In this case let d_m $(m \in \overline{T_0})$ be the operators in $\mathcal{B}_{U_0}$ such that $c_m = \widetilde{d_m}$. Then we have in the algebra $\mathcal{B}_{U_0}$

$$\begin{aligned} \{E_{M,M}\} \subset d_{\overline{T_0}} \subset \ & \{E_{1,k}, E_{k,k-1}, \cdots, \widehat{E_{M+1,M}}, \cdots, \widehat{E_{N+1,N}}, \cdots, E_{3,2}, E_{2,1}\} \\ & \cup \{E_{1,2}, E_{2,3}, \cdots, \widehat{E_{N,N+1}}, \cdots, \widehat{E_{M,M+1}}, \cdots, E_{k-1,k}, E_{k,1}\} \\ & \cup \{E_{M,M}\} \end{aligned}$$

with $1 \leq N < M \leq k-1$. Here the symbol '$\widehat{\ }$' denotes the omission. Since there are two gaps $\{N, N+1\}$ and $\{M, M+1\}$ in the circle graph (minus two edges)

$$\{\{1,2\},\{2,3\},\{3,4\},\cdots,\{k-1,k\},\{k,1\}\} \setminus \{\{N,N+1\},\{M,M+1\}\}$$

with $N < M$, the vertex M cannot be connected to the vertex 1. So there is no path such that starting from the vertex 1, passing through the vertex M and finally arriving at the vertex 1 again. This implies that $\varphi_{l(T_0)}[c_{T_0}] = \langle \xi | d_{\iota_1} \cdots d_{\iota_t} \xi \rangle = 0$ with $T_0 = (\iota_1 \cdots \iota_t)$ and $\xi = e_1^{(U_0)}$.

Case c): $T_* \subset \overline{U_0}$ & $i_a = v$ $(a = 1)$. In this case let d_m $(m \in \overline{T_0})$ be the same as above. Then we have in $\mathcal{B}_{U_0}$

$$\begin{aligned} \{E_{k,k}\} \subset d_{\overline{T_0}} \subset \ & \{E_{k,k-1}, \cdots, \widehat{E_{N+1,N}}, \cdots, E_{3,2}, E_{2,1}\} \\ & \cup \{E_{1,2}, E_{2,3}, \cdots, \widehat{E_{N,N+1}}, \cdots, E_{k-1,k}\} \\ & \cup \{E_{k,k}\}. \end{aligned}$$

Since there is one gap $\{N, N+1\}$ in the linear graph (minus one edge)

$$\{\{1,2\},\{2,3\},\{3,4\},\cdots,\{k-1,k\}\} \setminus \{\{N,N+1\}\}$$

with $N \leq k-1$, the vertex k cannot be connected to the vertex 1. So there is no path such that starting from the vertex 1, passing through the vertex k and finally arriving at the vertex 1 again. This implies that $\varphi_{l(T_0)}[c_{T_0}] = \langle \xi | d_{\iota_1} \cdots d_{\iota_t} \xi \rangle = 0$.

For each cases a), b) and c), we have

$$\| \, [a_v a_{v+1} \cdots a_n] \, \|^2 = \sum_{\substack{\tau \in \vec{\mathcal{P}}(S) \\ \overline{\tau} \leq \rho}} t(\rho, \mu; \tau) \prod_{T \in \tau} \varphi_{l(T)}[c_T] \ = \ 0.$$

Hence we get by the Cauchy-Schwartz inequality

$$|\varphi[a_1 \cdots a_{v-1} a_v \cdots a_n]| \leq \| \, [a_1 \cdots a_{v-1}] \, \| \, \| \, [a_v \cdots a_n] \, \| = 0. \qquad \square$$

Now we complete the simplified proof for (V^+) (= Theorem 3.5).

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