Easy quantum groups:

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27/11/2012

Introduction: $G \curvearrowright X$ in order to understand a space X, consider group action.

You want sthg more noncommutative to act on X.

X topological compact space $\longleftrightarrow$ $\mathcal{C}(X)$ [Gel'fand theory!]

Then $\mathcal{C}(G)$ "coacts" on $\mathcal{C}(X)$ in the category sense: the arrows turn around.

Then make $\mathcal{C}(X)$ noncommutative – but you should also make $\mathcal{C}(G)$ noncommutative.

Here: $m: G\times G \longrightarrow G$ becomes $\mathcal{C}(G) \longrightarrow \mathcal{C}(G\times G) = \mathcal{C}(G)\otimes\mathcal{C}(G)$
$(g,h) \longmapsto gh$ $\quad f \longmapsto [(g,h)\longmapsto f(gh)]$

Idea of a qg: G group $\rightsquigarrow$ $\mathcal{C}(G)$ commutative.

G^+ q.g. but it is virtual. $\rightsquigarrow$ $\mathcal{C}(G^+)$ noncommutative

We can say virtually that $G^+ \subset H^+$ and mean $\mathcal{C}(H^+) \twoheadrightarrow \mathcal{C}(G^+)$.

There are also "group actions" Δ, S, ε which makes it a Hopf algebra.

Now, a special way of getting qg: Liberation:

S_n permutation group represented as matrices.

$\mathcal{C}(S_n) = C^*(u_{ij} : 1\leq i,j\leq n : u_{ij} \text{ projections}, \sum_i u_{ik} = \sum_i u_{kj} = 1, [u_{ij}, u_{kl}]=0)$

What is its liberation? You drop commutation!

Wang '98: $\mathcal{C}(S_n^+) = C^*(u_{ij} : 1\leq i,j\leq n : u_{ij} \text{ projection}, \sum u_{ik} = \sum u_{kj} = 1)$ is the q. permutation group

Similarly: O_n group of orthogonal matrices.

$\mathcal{C}(O_n) = C^*(u_{ij} : (u_{ij}) \text{ orthogonal}, u_{ij}^* = u_{ij}, [u_{ij}, u_{kl}]=0)$

$\mathcal{C}(O_n^+) = C^*(\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_})$ is the free orthogonal qg.

You might also deform the commutation relations.

By Woronowicz, every compact matrix qg is determined by its intertwiner spaces.

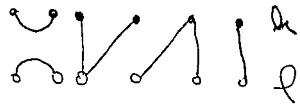
$\mathcal{C}(G^+) = C^*(u_{ij} : (u_{ij}) \text{ orthogonal}, u_{ij} = u_{ij}^*, (R))$ with $1\leq i,j\leq n$. Then take the Hom spaces.

$\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \{T : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l} \text{ linear} : Tu^{\otimes k} = u^{\otimes l}T\}$

these are equivalent objects.

$S_n^+ \longleftrightarrow \mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p : p \in NC(k,l))$

$O_n^+ \longleftrightarrow \qquad\qquad\qquad\qquad = \mathrm{span}(T_p : p \in NC_2(k,l))$.

$NC(k,l)$  non crossing partitions.

$NC_2(k,l)$ if all the strings are pairs.

If $p \in NC(k,l)$, $T_p : (\mathbb{C}^n)^{\otimes k} \longrightarrow (\mathbb{C}^n)^{\otimes l}$

$e_{i_1} \otimes \dots \otimes e_{i_k} \longmapsto \sum_{j_1,\dots,j_l} \delta_p(i,j)\, e_{j_1} \otimes \dots \otimes e_{j_l}$. where $\delta_p(i,j) = 1$ only if equal indices are connected

Example (for a crossing partition, $T_{\times}(e_a \otimes e_b) = e_b \otimes e_a$

Banica and Speicher extended this philosophy:

Def: (2009) $S_n \subseteq G_n^+ \subseteq O_n^+$ compact qg (this means $C(S_n) \twoheadleftarrow C(G_n^+) \twoheadleftarrow C(O_n^+)$)

is <u>easy</u> if its Hom spaces are given by $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p : p \in D(k,l))$,

where $D(k,l) \subseteq \underbrace{P(k,l)}$ set of all partitions.

By Woronowicz, this means G_n^+ is completely determined by the collection

$(D(k,l))_{k,l \geq 0}$, i.e., by its category of partitions.

Intertwiner spaces are tensor categories: $T_{\underbrace{p \otimes q}} \longleftrightarrow T_p \otimes T_q$, $T_{pq} \longleftrightarrow T_p T_q$, $T_{p^*} = T_p^*$

tensor product of partitions.

Thus the category of partitions is closed under operations reflecting the properties of a tensor category: $p \otimes q$, pq, p^*; it contains $|$ and $\sqcap$

$p \otimes q$: writing diagrams next to each other.

pq : writing one above the other, $\begin{matrix} \dots k \\ \dots l \\ \dots m \end{matrix}$ the l in the middle are the same, and you remove the loops.

$\begin{matrix} \dots k \\ p \\ \dots l \end{matrix} \mapsto \begin{matrix} \dots l \\ p^* \\ \dots k \end{matrix}$ upside down!

. The philosophy of easy qg have a nice combinatorics; by its combinatorics, we may control the qg features; you have a combinatorical object.

- It generalises liberation of groups ($G \to G^+$) and hence yields new examples.
- S_n^+ and O_n^+ are important: we want to understand the G_n in between, or more generally $S_n \subseteq G \subseteq O_n^+$.
- Free probability and action of qg: whenever you have (Ω, P), you get $(L^\infty(\Omega), \int \cdot \, dP)$ an algebra; but what is independence for its noncommutative counterpart? It is free independence. These are dual concepts:

$$\begin{array}{ccc} \text{commutative rv} & \xleftrightarrow{\text{dual}} & \text{noncommu rv} \\ \text{independence} & \longleftrightarrow & \text{free independence.} \end{array}$$

For instance: de Finetti: • (x_n) classical rv; if its distribution is invariant under S_n, then the (x_n) are i.i.d.

• (x_n) non commu rv; if ———————— S_n^+, then the (x_n) are freely i.i.d.

<u>Classification of easy qg; categories of partitions:</u>

BS '09, W 12: in between $S_n^+ \subseteq G \subseteq O_n^+$ there are exactly 7 easy qg.

(but they might coincide for $n=3$ eg)

There are further results by Banica, Curran, Speicher, Weber.

Q: and between S_n and O_n^+?

$$\boxed{S_n^+ \not\subseteq O_n^+}$$

~~It~~ in between.

no easy in between; but there might be others. Open Q: Banica: many? 10 ?? 10 $\leftarrow$ O_n^+ in between.

$$\boxed{S_n \not\subseteq O_n}$$

$abc = cba$: collapses to O_n if commutativity enforced.

Recent joint work with Sven Raum (student of Stefan Vaes): there are (a continuum) uncountably many. There is a link to <u>varieties</u> of groups (universal C^* algebras, some relations, class of all groups satisfying them) [Ol'shanskii '70] and classifying them is an open question from the 30's!

Hence there is a very rich class of easy qg.

The Hom spaces are different. But there might be equal categories and they should be nonisomorphic.

The uncountably many collapse to O_n^+

Then one could want to go beyond O_n^+, to U_n^+: colouring partitions.

Or consider a subgroup of U_n^+.

Q: And the non-Kac qg's? $O_n^+ \rightsquigarrow O_n^+(Q)$ or $A_o(F)$: $p \mapsto T_p^F$ $\leftarrow$ deformed by F.