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11/12/2012

Free space associated to metric space.

Let $(M, d, 0)$ be a pointed metric space.

$\mathrm{Lip}_0(M) = \left(\{ f : M \to \mathbb{R} : f(0) = 0 \}, \|\cdot\|_{Lip} \right)$ is a Banach space.

It does not depend on 0

Def.: • For $x \in M$, let $\delta(x) : \mathrm{Lip}_0(M) \to \mathbb{R}$
$$f \longmapsto f(x).$$

• The free space of M is $\mathcal{F}(M) = \overline{\mathrm{span}}\, \{ \delta(x) : x \in M \}$ in $\mathrm{Lip}_0(M)^*$.

and $B_{\mathrm{Lip}_0(M)}$ is compact for the topology of pointwise convergence, so that $\mathcal{F}(M)^* = \mathrm{Lip}_0(M)$.

Consider $L : M \to N$ Lipschitz with $L(0_M) = 0_N$. There is a unique $\hat{L}$ such that and $\|\hat{L}\| = \|L\|_{Lip}$. Furthermore δ_M and δ_N are nonlinear isometries.

$$\begin{tikzcd} \mathcal{F}(M) \arrow[r, "\hat{L}"] & \mathcal{F}(N) \\ M \arrow[u, "\delta_M"] \arrow[r, "L"] & N \arrow[u, "\delta_N"'] \end{tikzcd}$$

N.B.: $\mathcal{F}(\mathbb{R}) \cong L^1$ and $\mathcal{F}(M) \underset{2010}{\cong} L_1 \Rightarrow M$ metric subspace of a metric tree (Godard): M looks very much like $\mathbb{R}$.

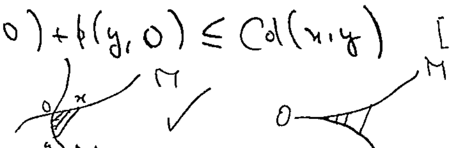
but $\mathcal{F}(\mathbb{R}^2) \not\hookrightarrow L^1$ (Naor – Schechtman 2003)

Open question: Is $\mathcal{F}(\mathbb{R}^2)$ isomorphic to $\mathcal{F}(\mathbb{R}^3)$ — certainly not, but ...

<u>Free space of unions of metric spaces</u>: [Godard 2010] If $M = \bigcup_{i \in I} M_i$ with the M_i s.t. $\exists\, \alpha, \beta > 0$ $\alpha \leq d(x, y) \leq \beta$ for $x \in M_i$, $y \in M_j$, $i \neq j$, then $\mathcal{F}(M) \simeq \left(\sum_{i \in I} \mathcal{F}(M_i) \right)_{\ell_1} \oplus_1 \ell_1(\overset{*}{I})$

you remove one point.

Proposition 1: If $W = M \cup N$ with $M \cap N = \{0_W\}$ and, if there is a $C \geq 1$ such that if $(x, y) \in M \times N$, then $d(x, 0) + d(y, 0) \leq C d(x, y)$ [conversion]

[this means that M and N make an angle: ],

then $\mathcal{F}(M \cup N) \overset{C}{\simeq} \mathcal{F}(M) \oplus_1 \mathcal{F}(N)$.

Proof: Let $\phi : \mathrm{Lip}_0(M) \oplus \mathrm{Lip}_0(N) \to \mathrm{Lip}_0(M \cup N)$
$$(f, g) \longmapsto h = \text{"} f \cup g \text{"} = \text{the } h \text{ s.t. } h_{|M} = f \text{ and } h_{|N} = g.$$

Let us compute $|\phi(f, g)(x) - \phi(f, g)(y)| = |f(x) - g(y)| \leq \|f\| d(x, 0) + \|g\| d(y, 0)$ $\leq C \max(\|f\|, \|g\|) d(x, y)$

So ϕ is well defined, $\|\phi\| \leq C$, $\|\phi^{-1}\| = 1$. It is w^*-w^*-continuous: thus, there is a unique φ s.t.

$\psi : \mathcal{F}(M \cup N) \longrightarrow \mathcal{F}(M) \oplus_1 \mathcal{F}(N)$ s.t. $\psi^* = \phi$ and $\|\psi\| \leq C$.

this can be extended to infinite unions.

$\boxed{2Q}$ $\boxed{1}$. "non convex unions"

$\boxed{2}$ More complex intersections

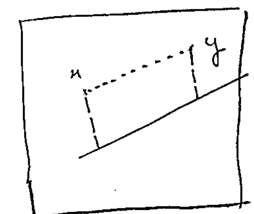
ad 1: ¿ $\mathcal{F}(\ )$? Does it embed linearly into L^1?

$\mathcal{F}(\ \ ) \hookrightarrow L^1$ $\quad \mathcal{F}(\ \ ) \hookrightarrow L^1$ but distorsion increases

Now consider $(\ \ )$ and you're back with Naor-Schechtman.

ad 2: Free spaces of quotient spaces:

$F \subset M$ closed, $\tilde{M}$ the space you get when all points of F are identified, with $\tilde{d}(\tilde{x}, \tilde{y}) = \min \{ d(x,y), d(x,F) + d(y,F) \}$



Let $\mathrm{Lip}_0(M/F) = \{ f \in \mathrm{Lip}_0(M) : f \text{ is constant on } F \}$
$\subset \mathrm{Lip}_0(M)$.

And this is exactly $\mathrm{Lip}_0(\tilde{M})$.

two ways of going: directly / to F, teleport to, from F

Define $\mathcal{F}(M/F) = \mathcal{F}(\tilde{M}, \tilde{d}, \tilde{0})$. Consider $\delta_{M/F}(x) : \mathrm{Lip}_0(M/F) \to \mathbb{R}$, $f \mapsto f(x)$

We have $\mathcal{F}(M/F) \cong \overline{\mathrm{span}} \{ \delta_{M/F}(x) : \tilde{x} \in \tilde{M} \}$ in $\mathrm{Lip}_0(M/F)^*$,

where $\delta_{M/F} : \tilde{M} \longrightarrow \mathrm{Lip}_0(M/F)^*$ is a nonlinear isometry.

<u>Lemma</u>: Suppose $0 \in F \subset_{\text{closed}} M$ and suppose that $\exists E : \mathrm{Lip}_0(F) \longrightarrow \mathrm{Lip}_0(M)$, $\underbrace{\text{“}w^*\text{-}w^*\text{-continuous linear extension operator”}}_{\text{special extension}}$. Then $\mathcal{F}(M) \underset{\|E\|+1}{\simeq} \mathcal{F}(F) \oplus_1 \mathcal{F}(M/F)$.

<u>Proof</u>: $\phi : \mathrm{Lip}_0(F) \oplus_\infty \mathrm{Lip}_0(M/F) \longrightarrow \mathrm{Lip}_0(M)$ is an onto isomorphism,
$(f, g) \longmapsto Ef + g$

$\|\phi\| \leq \|E\| + 1$ and ϕ w^*-w^*-continuous: if $\mu \in B_{\mathcal{F}(M)}$, let $V = \{ Ef + g \in \mathrm{Lip}_0(M) : \left| \int (Ef+g)\, d\mu \right| < \varepsilon \}$ and $\phi(\{ (f,g) \in \mathrm{Lip}_0(F) \oplus_\infty \mathrm{Lip}_0(M/F) : \left| \int f\, d(\mu_{|F}) \right| < \frac{\varepsilon}{2\|E\|}, \left| \int g\, d\mu \right| < \frac{\varepsilon}{2} \}) \subset V$

then there is $\psi : \mathcal{F}(M) \longrightarrow \mathcal{F}(F) \oplus_1 \mathcal{F}(M/F)$ s.t. $\|\psi\| \leq \|E\| + 1$, $\psi^* = \phi$

<u>Question</u>: When are there special extensions from $\mathrm{Lip}_0(F)$ to $\mathrm{Lip}_0(M)$?

Equivalent question: When is $\mathcal{F}(F)$ complemented in $\mathcal{F}(M)$?

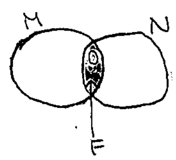
Negative example: [Godefroy-Kalton 2003] If Y is a Banach space, then Y has λ-BAP $\Leftrightarrow$ $\mathcal{F}(Y)$ has λ-BAP. But c_0 has BAP and contains a subspace X without. If $\mathcal{F}(X)$ was complemented in $\mathcal{F}(c_0)$, X would be ↯

Positive example: [Lee-Naor 2005 : $\overset{K>0}{K}$- gentle partition of unity wrt F]

One application: If M is a doubling metric space, then $\mathcal{F}(M) \underset{\uparrow}{\simeq} \mathcal{F}(F) \oplus_1 \mathcal{F}(M/F)$

$C(1+\log D(M))$
universal constant

Another: If F is closed in $\mathbb{R}^N$, then $\mathcal{F}(\mathbb{R}^N) \underset{1+C\sqrt{N}}{\simeq} \mathcal{F}(F) \oplus_1 \mathcal{F}(\mathbb{R}^N/F)$.

Proposition 2 : 

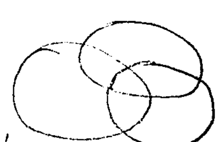
If there is a special extension from $Lip_0(F)$ to $Lip_0(M \cup N)$ and a constant C such that $(x,y) \in M \times N \Rightarrow d(x,y) \geq \frac{1}{C}(d(x,F) + d(y,F))$, then $\mathcal{F}(M \cup N) \simeq \mathcal{F}(F) \oplus_1 \mathcal{F}(M/F) \oplus_1 \mathcal{F}(N/F)$.

Proof: Let $\tilde{M} = M/F$, $\tilde{N} = N/F$, $\widetilde{M \cup N} = (M \cup N/F, \tilde{d})$. Let us apply Prop 1:

Let $\tilde{x} \in M$, $\tilde{y} \in N$. $\tilde{d}(\tilde{x},\tilde{0}) + d(\tilde{y},\tilde{0}) = d(x,F) + d(y,F) \leq C \min\{d(x,y), d(x,F) + d(y,F)\}$
$= C\,\tilde{d}(\tilde{x},\tilde{y})$.

and $\mathcal{F}(M \cup N/F) = \mathcal{F}(\widetilde{M \cup N}) \underset{C}{\simeq} \mathcal{F}(\tilde{M}) \oplus_1 \mathcal{F}(\tilde{N}) = \mathcal{F}(M/F) \oplus_1 \mathcal{F}(N/F)$

and the lemma yields $\mathcal{F}(M \cup N) \underset{1+\|E\|}{\simeq} \mathcal{F}(M \cup N/F) \oplus_1 \mathcal{F}(F)$

Pb: ∞ extension is open. ex of a tree :  : start with the common intersection of all, then continue with the intersection of all but one, etc.

Application:
→ 3-space properties.

What is a distorsion of an L^1-embedding?