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[illegible] de Mar Jiménez

On smooth extensions of mappings on Banach spaces.

(X,Z) a pair of spaces

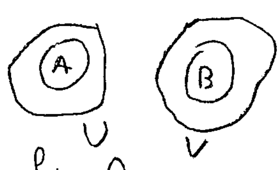
$f: A \subseteq X \longrightarrow Z$ function with some nice property (P). The question is:

When is there $F: X \longrightarrow Z$ with (P) extending f? F is called a (P)-extension of f.

1) P = continuity, X, Z are assumed topological.

There is Tietze's theorem: (1915): X is normal iff every continuous function $f: A \text{ closed} \longrightarrow \mathbb{R}$ has a continuous extension.

The key is Urysohn's lemma: X is normal iff for all A, B closed in X, disjoint, there is $f: X \xrightarrow{\text{continuous}} [0,1]$ s.t. $f_{|A} = 1$, $f_{|B} = 0$.



Proposition: If X is paracompact and Z is any B-space, every continuous $f: A \overset{\text{closed}}{\subseteq} X \longrightarrow Z$ has a continuous extension.

2) P = Lipschitz, X, Z are assumed metric.

There is McShane's theorem: (1934) If $f: A \subseteq X \longrightarrow \mathbb{R}$ is Lipschitz, the inf convolute $F(x) = \inf_{a \in A}(f(a) + \mathrm{Lip} f\, d(a,x))$ is a Lipschitz extension with same Lipschitz constant.

There is a counterexample: Naor (2001): $X = \ell_p$, $Z = L_2$ with $1<p<2$
$X = L_2$, $Z = L_q$ with $2<q<\infty$.

These pairs do not enjoy (E) "$\exists C\ \forall f: A \subseteq X \longrightarrow Z\ \exists F: X \longrightarrow Z$ $F|_A = f$, $\mathrm{Lip} F \leq C\, \mathrm{Lip} f$"

3) Linear extension: (X,Z) Banach spaces.

There is the Hahn-Banach theorem.

Also: Maurey's theorem (1974): If X has type 2 and Z has cotype 2, then there is

the linear extension property. [$\lambda = \lambda(X,Z) \geq 1$ such that every bounded linear $T: Y \overset{\text{closed subspace}}{\subseteq} X \longrightarrow Z$ has a bounded linear extension $\tilde{T}: X \longrightarrow Z$ and $\|\tilde{T}\| \leq \lambda \|T\|$.

4) Smooth extensions.

(X,Z) Banach spaces, A closed $\subseteq X$, $f: A \to Z$ (1) When is there a C^1-smooth $F: X \to Z$ such that $F|_A = f$.

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(2) When is there a C^1-smooth and Lipschitz $F: X \longrightarrow Z$ s.t. $F|_A = f$ and $\mathrm{Lip}\, F \leq C \,\mathrm{Lip}\, f$, where $C = C(X,Z)$.

If $\dim X < \infty$, there is a huge number of papers: Whitney 1936: C^k-smooth, $X=\mathbb{R}$. Glaeser (1958): C^1-smooth, $X=\mathbb{R}^n$. Fefferman (2007): C^k-smooth, $X=\mathbb{R}^n$.

If $\dim X = \infty$, $\overbrace{X \text{ separable}}^{\Leftarrow X^* \text{ separable}}$, with C^1-smooth norm, then [Azagra, Fry, Keener, 2008] every C^1-smooth $f: Y$ $\underbrace{\text{closed subspace}}_{\text{set?!}} \subseteq X \longrightarrow \underbrace{\mathbb{R}}_{Z?!}$ has a C^1-smooth extension.

4-1: Smooth extension from closed subsets.

Def: (*) (X,Z) has (*) if $\exists\, C=C(X,Z) \geq 1$ $\forall \varepsilon>0$ $\forall A \subset X$ $\forall f: A \longrightarrow Z$ Lipschitz $\exists\, g: X \longrightarrow Z$ C^1-smooth Lipschitz with $\mathrm{Lip}\, g \leq C \,\mathrm{Lip}\, f$ and $\|g(x)-f(x)\| \leq \varepsilon$ on A.

(A) (X,Z) has (A) if the same holds at least for $A = X$.

(E) (X,Z) has (E) if the same holds without requiring g C^1-smooth, but $\varepsilon = 0$.

We have $\left.\begin{matrix} (A) \\ (E) \end{matrix}\right\} \Rightarrow (*) \Rightarrow (A)$

But do we have $(*) \Rightarrow (E)$. Yes if Z is a dual space!

Do we have $(A) \Rightarrow (*)$? No in general?

Recall Naor's counterexample: they satisfy non(E), and thus non(*)

Fry (2008) proved that (L_p, L_q), $1<p,q<\infty$, has property (A)

Recall that if $f: X \longrightarrow Z$ C^1-smooth, then $f': X \longrightarrow \mathcal{L}(X,Z)$ continuous.

If $f: X \longrightarrow Z$ is furthermore Lipschitz, then f' is bounded.

Def.: (MVC) $f: A$ closed $\subseteq X \longrightarrow Z$ satisfies the mean value condition if $\exists\, D: A \longrightarrow \mathcal{L}(X,Z)$ continuous $\forall \varepsilon>0$ $\forall x \in A$ $\exists\, r>0$ $\|f(z)-f(y)-D(x)(z-y)\| \leq \varepsilon \|z-y\|$ for $z,y \in A \cap B(x,r)$

N.B.: (MVC) is necessary

Theorem (with Mar Jiménez Sevilla, 2012): If (X,Z) has (*), $f: A$ closed $\subseteq X \longrightarrow Z$ has (MVC) iff there is a C^1-smooth extension.

Theorem: If (X,Z) has (*), then $f: A$ closed $\subseteq X \longrightarrow Z$ Lipschitz satisfies (MVC) with D bounded iff there is a C^1-smooth-Lipschitz extension $F: X \longrightarrow Z$ with $\mathrm{Lip}\, F \leq C(\mathrm{Lip}\, f + \sup\|D\|)$

NB.: $(*) \Leftarrow$ C^1-smooth-Lipschitz extension

Corollary: X separable, $Z=\mathbb{R}$. Then (i) $(X,\mathbb{R})$ has $\circledast$ $\Longleftrightarrow$ (ii) $(X,\mathbb{R})$ has C^1-smooth Lipschitz extension.

(iii) $\Longleftrightarrow$ X has a C^1-smooth and Lipschitz bump.

(iv) $\Longleftrightarrow$ X^* is separable

(v) $\Longleftrightarrow$ $(X,\mathbb{R})$ has C^1-smooth-extension.

(iii) $\Rightarrow$ (i) Hájek + Johannis

Let us prove (ii) $\Rightarrow$ (iii): Let $f(0)=0$, $f(x)=0$ for $\|x\|\geq 1$. An extension is a bump function!

These properties are also equivalent to: $\forall A,B$ closed $\subseteq X$, $A\cap B=\emptyset$, $\exists f: X\to[0,1]$ C^1 smooth $f|_A=1$, $f|_B=0$.

Examples: • $(\mathbb{R}^4, Z)$

• (X,Z) Hilbert spaces but X is separable.

• $(X,\mathbb{R})$ with X Hilbert.

• $\left(X = \begin{cases} c_0(\Gamma) \\ \text{or} \\ X \text{ sep with } C^1\text{-smooth norm}\end{cases}, \; Z = \begin{matrix}\mathbb{R} \\ \text{or} \\ \ell_\infty(\mathbb{N}) \\ \text{or} \\ c_0(\mathbb{N}) \\ \text{or} \\ C(K), K \text{ compact metric}\end{matrix}\right)$

Counterexample: • (ℓ^1, Z): there is no smooth extension, no smooth Lipschitz extension.

• (L_2, L_q), $2<q<\infty$, (L_p, L_2), $1<p<2$: there is no smooth Lipschitz extension.

4-2: Smooth extension from closed subspaces.

Def: (X,Z) has linear extension if $\exists \lambda(X,Z)\geq 1$ $\forall T: Y$ closed subspace $\subseteq X \xrightarrow[\text{linear}]{\text{bounded}} Z$.

T extends to $\tilde{T}: X \xrightarrow[\text{linear}]{\text{bounded}} Z$ $\|\tilde{T}\|\leq \lambda\|T\|$.

N.B.: The linear extension property is necessary for our problem.

If $T: Y\to Z$ C^1-smooth, $F: X\to Z$ C^1-extension: $F'(0)\in\mathcal{L}(X,Z)$ is the extension we are looking for!

Th: (X,Z) with $\circledast$ and linear extension: for every C^1-smooth $f: Y$ closed subspace $\subseteq X\to Z$ has a C^1-smooth extension

Examples: ① $\left(X = \begin{cases} c_0(\mathbb{N}) \\ \ell_p, 1<p<\infty \\ c_0(\Gamma)\end{cases}, \; Z = C(K)\right)$

② (L_p, L_2) $2<p<\infty$

③ $\left(X = \begin{cases} c_0(\Gamma) \\ \text{separable with } C^1\text{-smooth norm}\end{cases}, \; Z=\mathbb{R}\right)$

Question: $f: c_0\to\mathbb{R}$ C^1-smooth. Is there $F: \ell_\infty\to\mathbb{R}$?

Question: (X WCG with C^1-smooth norm, $\mathbb{R}$). Does it have (A)?