Spatial Quantum Dynamics

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Markov semigroups, E_0–semigroups, product systems

Irreversible dynamics, reversible dynamics, and dilations

$\mathcal{A}, \mathcal{B}, \ldots$ unital C^*–algebras. If von Neumann, then all maps normal.

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$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad T_t \quad} & \mathcal{B} \\ {\scriptstyle \text{embedding } i}\big\downarrow & & \big\uparrow {\scriptstyle \mathbb{E} \text{ (expectation: } i\circ\mathbb{E} \text{ is cond. exp.)}} \\ \mathcal{A} & \xrightarrow[\quad \vartheta_t \quad]{} & \mathcal{A} \end{array}$$

Markov semigroups, E_0–semigroups, product systems

Dilations, noises, and cocycle perturbations

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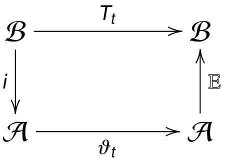
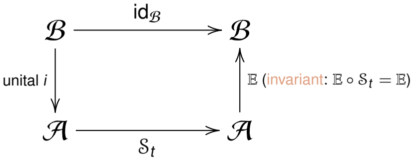


Noise:
(Interaction picture)



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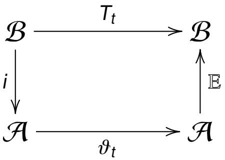

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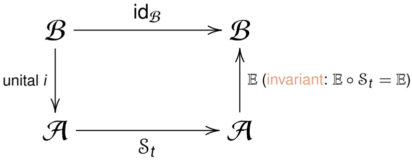

Unitary left cocycle: Unitaries $u_t \in \mathcal{A}$ such that $u_{s+t} = u_t \mathcal{S}_t(u_s)$

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$\Longrightarrow \quad \mathcal{S}^u_t \; := \; u_t \mathcal{S}_t(\bullet) u^*_t$ is E_0–semigroup (cocycle perturbation)

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A basic problem of quantum probability:

Markov semigroups, E_0–semigroups, product systems

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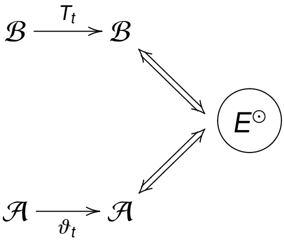
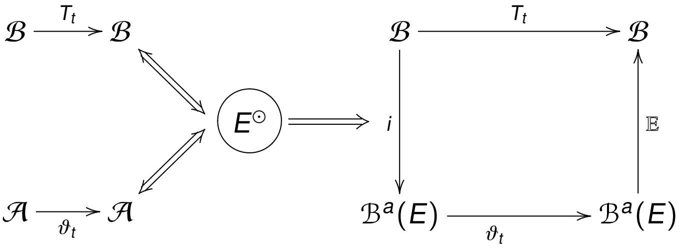
A basic problem of quantum probability:

> Given T, find $\mathcal{S}$ and u_t such that $\vartheta := \mathcal{S}^u$ is a dilation.

Markov semigroups, E_0–semigroups, product systems

Semigroups and product systems





$\mathcal{B}$ $\xrightarrow{T_t}$ $\mathcal{B}$
$\mathcal{A}$ $\xrightarrow{\vartheta_t}$ $\mathcal{A}$
$E^{\odot}$
$\mathcal{B}$ $\xrightarrow{T_t}$ $\mathcal{B}$
i
$\mathbb{E}$
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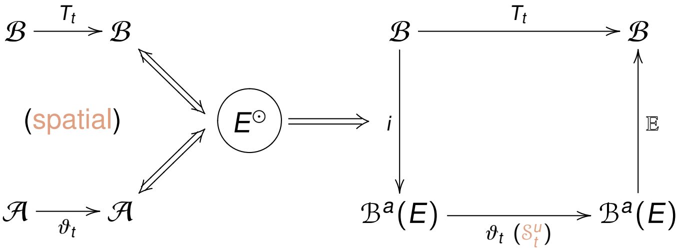

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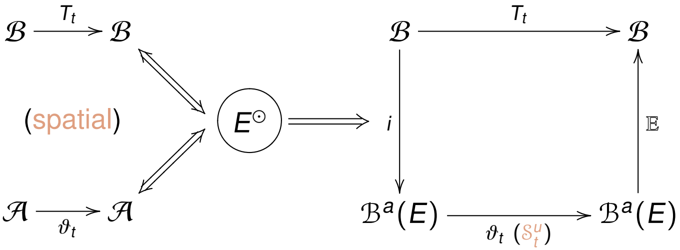

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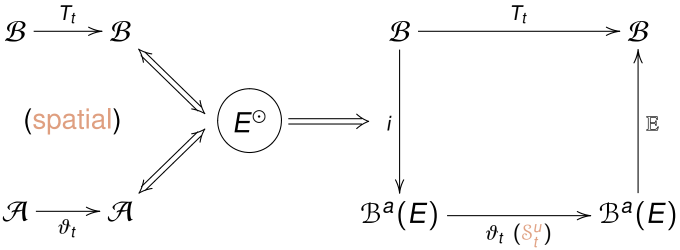

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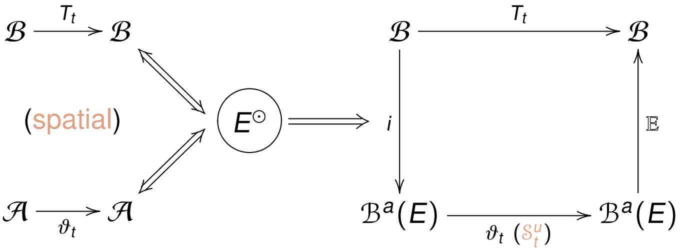

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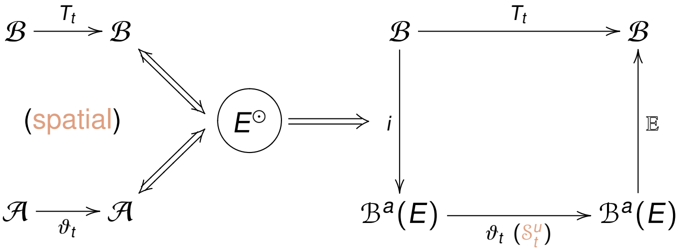

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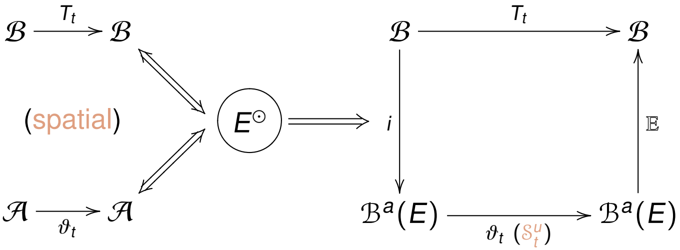

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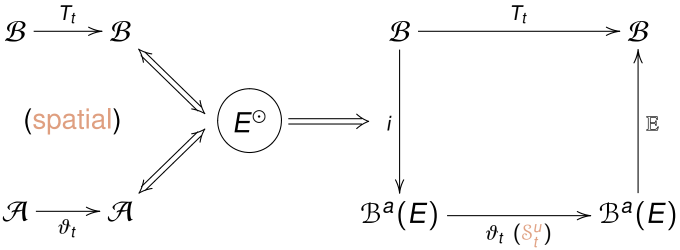

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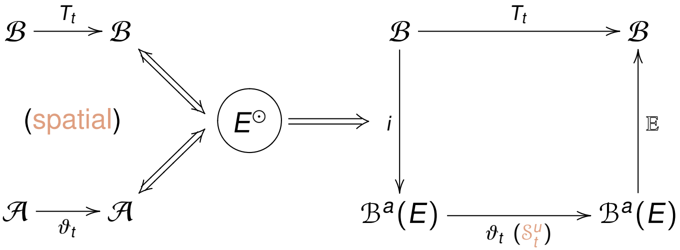

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Markov semigroups, E_0–semigroups, product systems

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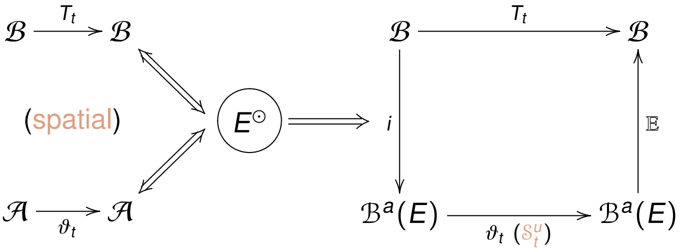

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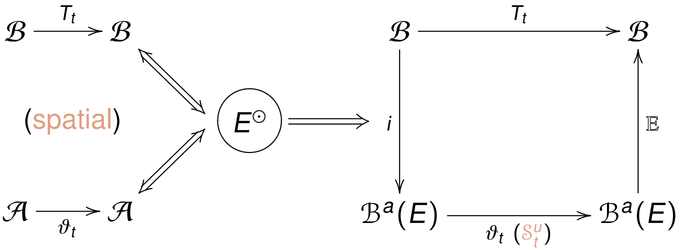

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Also multi-parameter case. (Or more general semigroups.)

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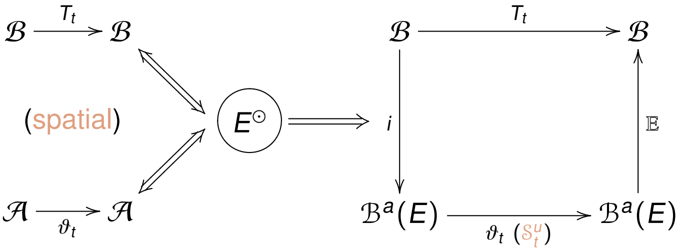

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($\Rightarrow$ relations with multivariate operator theory.)

Markov semigroups, E_0–semigroups, product systems

Spatial dynamics and spatial product systems

Definition (Powers 1987 ($\mathcal{B}(H)$), MS 2006 (preprint 2001))

An E_0–semigroup ϑ is spatial if it admits a semigroup of intertwining isometries (u_t)*:* $\quad \vartheta_t(a)u_t \;=\; u_t a.$

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A CP-semigroup T is spatial if admits a semigroup (c_t) *such that* $T_t - c_t^* \bullet c_t$ *is CP for all* t.

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Outline

Product systems, units, and CP-semigroups

Product systems, E_0–semigroups, and noises

Unital units and weak dilations of Markov semigroups

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A unit $\xi^{\odot} = (\xi_t)_{t \in \mathbb{R}_+}$ is a section such that $\xi_s \xi_t = \xi_{s+t}$ and $\xi_0 = \mathbf{1}$.

Units and CP-semigroups

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- **vN-case:** Much more interesting. (Much weaker topology!) (Defi. strongly cont. PS by MS 2009 (preprint).)

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Theorem (Arveson 1990, MS 2006 ($\mathcal{B}(H)$), MS variants)

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- vN-case: Last theorem **and** everything else general. (TP $\rightsquigarrow$ vN-TP; total $\rightsquigarrow$ strongly total; cont. $\rightsquigarrow$ strongly cont.)

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Theorem (MS 2010)

An E_0–semigroup is spatial iff it is spatial as Markov semigroup.

Outline

Product systems + unital units $\Longrightarrow$ weak dilations

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We summarize:

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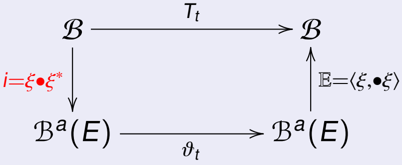

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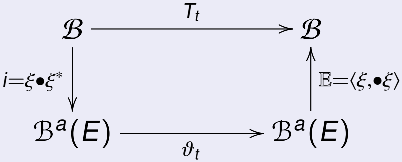

Moreover, if $(E^{\odot}, \xi^{\odot})$ is the GNS-construction of the Markov semigroup T, then (E, ϑ, ξ) is the unique minimal dilation of T.

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$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad T_t \quad} & \mathcal{B} \\ {\scriptstyle i=\xi\bullet\xi^*}\big\downarrow & & \big\uparrow{\scriptstyle \mathbb{E}=\langle\xi,\bullet\xi\rangle} \\ \mathcal{B}^a(E) & \xrightarrow[\quad \vartheta_t \quad]{} & \mathcal{B}^a(E) \end{array}$$

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Remark

Works for product systems over arbitrary right-reversible monoids!
The crucial ingredient is the GNS-system.

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- The PS of a weak automorphism dilation is contained in the Picard group of $\mathcal{B}$.

Spatial dynamics and spatial product systems

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Corollary

A strict (normal) E_0–semigroup on $\mathcal{B}^a(E)$, E (strongly) full, is spatial iff is is stably cocycle equivalent to a noise.

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Spatial dynamics and spatial product systems

Spatial product systems $\Longrightarrow$ noises

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- Since $\omega^\odot$ central, left action of E_t survives $\liminf_t$
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- Every strict primary noise arises that way from its spatial product system $(E^{\odot}, \omega^{\odot})$.

Dilations of Markov semigroups to noises

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Dilations of Markov semigroups to noises

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- Amplifications remain weak dilation and noise, respectively.
- Identification can be done to identify new ξ with new ω. $\square$

Dilations of Markov semigroups to noises

Unclear
- Cocycle adapted?
- Fulfills QSDE?
  - Hudson-Parthasarathy 1984. Uniformly continuous Markov semigroups on $\mathcal{B}(H)$.
    $\mathcal{B}\big(\Gamma(L^2(\mathbb{R}_+, K)) \otimes H\big) = \mathcal{B}^a\big(\Gamma(L^2(\mathbb{R}_+, K) \otimes \mathcal{B}(H))\big)$.
  - Kümmerer-Speicher 1992. Uniformly continuous Markov semigroups on $\mathcal{B}(H)$.
    $\mathcal{B}\big(\mathcal{F}(L^2(\mathbb{R}_+, K)) \otimes H\big) = \mathcal{B}^a\big(\mathcal{F}(L^2(\mathbb{R}_+, K) \otimes \mathcal{B}(H))\big)$.
  - Goswami-Sinha 1999. Uniformly continuous Markov semigroups on $\mathcal{B} \subset \mathcal{B}(H)$.
    Calculus for $\mathcal{B}^a\big(\Gamma(L^2(\mathbb{R}_+, K) \otimes \mathcal{B}(H))\big)$ leaves invariant $\mathcal{B}^a\big(\Gamma(L^2(\mathbb{R}_+, K) \otimes \mathcal{B})\big)$.
  - MS 2000. Markov semigroups on $\mathcal{B}$ with CE-generator.
    Calculus on $\mathcal{B}^a\big(\mathcal{F}(L^2(\mathbb{R}_+, F))\big)$. (No embedding into $\mathcal{H} \otimes \mathcal{B}$.)
  - Köstler 2000. Abstract calculus. (Faithful invariant state!)
  - Many more .... (Sorry!) Also Evans-Hudson flows ....
- Is it possible to write down the cocycle from MS 2000?
- What about the filtration $\mathcal{A}_t = C^*\{u_s : 0 \leq s \leq t\}$?

Nonspatial Markov semigroups

- Until recently, in the case $\mathcal{B}(H)$ only type III E_0–semigroup or simple derivations of them.
  Then Floricel 2008, using my construction of an E_0–semigroup for every Arveson system, discovered a Markov semigroup. In MS 2010 (preprint) I showed it is a proper Markov semigroup.
- Fagnola-Liebscher-MS (in preparation): Classical Markov semigroups of Brownian motion and Ornstein-Uhlenbeck are nonspatial. **Even vN!!**
  Cipriani-Fagnola-Lindsay 2000: It admits a (completely!) spatial quantum extension to $\mathcal{B}(H)$.

This raises questions about classification of PS.

- Embedding may change the type.
- Morita equivalence may not chance the type.
- Morita equivalence may change (in the vN-case) the strong type: In fact, the one-dimensional PS of every E_0–semigroup on $\mathcal{B}(H)$ has a strongly continuous unit, but a type III Arveson system has no unit.

Thank you!