Strong Banach property (T) for simple algebraic groups with higher rank (arXiv:1301.1861)

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Definition of strong Banach property (T)

A class of Banach spaces $\mathcal{E}$ is of type > 1 if there exist $p > 1$ and $T \in \mathbb{R}_+$ such that for any $E \in \mathcal{E}$, $n \in \mathbb{N}^*$ and $x_1, ..., x_n \in E$, we have

$$\Big(\mathop{\mathbb{E}}_{\varepsilon_i=\pm 1} \|\sum_{i=1}^{n} \varepsilon_i x_i\|_E^2\Big)^{\frac{1}{2}} \leq T\Big(\sum_{i=1}^{n} \|x_i\|_E^p\Big)^{\frac{1}{p}}.$$

- $G =$ locally compact group
- $\ell : G \to \mathbb{R}_+$ a length function on G
- $\mathcal{E} =$ a class of Banach spaces
- $\mathcal{E}_{G,\ell} =$ the set of isomorphic classes of representations (E, π) of G such that $E \in \mathcal{E}$ and for any $g \in G$ we have

$$\|\pi(g)\| \leq e^{\ell(g)}.$$

- $\mathcal{C}_\ell^{\mathcal{E}}(G) =$ the completion of $C_c(G)$ with the norm

$$\|f\|_{\mathcal{C}_\ell^{\mathcal{E}}(G)} = \sup_{(E,\pi)\in\mathcal{E}_{G,\ell}} \|\int f(g)\pi(g)dg\|_{\mathcal{L}(E)}.$$

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G is said to have strong Banach property (T), if for any length function ℓ, any class of Banach spaces $\mathcal{E}$ of type > 1, stable under complex conjugation and duality, there exists $s_0 > 0$ such that the following holds. For any $C > 0$, and for any s with $s_0 > s > 0$, there exists a real self-adjoint idempotent element $\mathrm{p} \in \mathcal{C}^{\mathcal{E}}_{C+s\ell}(G)$ such that for any $(E, \pi) \in \mathcal{E}_{G,C+s\ell}$ we have

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Higher rank algebraic groups have strong Banach property (T)

Theorem

Let F be a non archimedian local field of any characteristic. Let G be an almost F-simple algebraic group of F-split rank ≥ 2. Then $G(F)$ has strong Banach property (T).

- Examples of non archimedian local field: $\mathbb{Q}_p$ p-adic numbers, $\mathbb{F}_p((T)) = \{\sum_{i \geq i_0}^{\infty} a_i T^i : i_0 \in \mathbb{Z}, a_i \in \mathbb{F}_p\}$ the Laurent series of finite field $\mathbb{F}_p$.
- Almost simple means any normal subgroup is a finite group, and F-split rank is the dimension of the maximal F-split torus.
- Examples of such G: SL_n when $n \geq 3$, Sp_{2n} when $n \geq 2$.

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Application to expanders

Let G be as in the theorem. Let $\Gamma \subset G$ be a lattice, $\Gamma_i \subset \Gamma$ a family of subgoups with $|\Gamma/\Gamma_i| \to \infty$, and $S \subset \Gamma$ a finite set of generators, which associates Γ/Γ_i with a metric d_i.

Theorem

For any Banach space E of type > 1, the family of graphes $(\Gamma/\Gamma_i, d_i)$ does not admit a uniform embedding in E.

A family of graphes (X_i, d_i) admits a uniform embedding in a Banach space E if there exist 1-Lipschitz maps $f_i : X_i \to E$, and a map $\rho : \mathbb{N} \to \mathbb{R}_+$ with $\lim_{n\to\infty} \rho(n) = +\infty$, such that

$$\|f_i(x) - f_i(y)\| \geq \rho(d_i(x,y)), \forall (x,y) \in X_i^2.$$

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Application to fixed-point property

Proposition

Let E be a Banach space of type > 1. Then any affine isometric action of G on E has a fixed point.

Reduction of the theorem to SL_3 and Sp_4

- Higher rank algebraic group contains SL_3 or Sp_4 as a subgroup (up to finite index).
- By a similar argument as the proof of Mautner's lemma, the theorem is reduced to SL_3 and Sp_4.
  - SL_3 : V. Lafforgue 2009.
  - Sp_4 : this paper.
- In this talk, I will present the proof for $Sp_4(\mathbb{F}_2((T)))$, which also works for any non archimedian local field of characteristic 2 (i.e. finite extension of $\mathbb{F}_2((T))$). A similar argument also works for charateristic different from 2.

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The theorem of Sp_4

- $G = Sp_4(\mathbb{F}_2((T))) = \{M \in M_4(\mathbb{F}_2((T))) | {}^tMJM = J\}$ where
$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} (-1 = 1 \in \mathbb{F}_2).$$
- $K = Sp_4(\mathcal{O})$ ($\mathcal{O} = \mathbb{F}_2[[T]] = \{\sum_{i\geq 0}^{\infty} a_i T^i : a_i \in \mathbb{F}_p\}$ formal power series)
- $D(i,j) = \begin{pmatrix} T^{-i} & & & \\ & T^{-j} & & \\ & & T^j & \\ & & & T^i \end{pmatrix}.$
- $\Lambda = \{(i,j) \in \mathbb{N}^2 | i \geq j \geq 0\}$, which is in bijection with $K\backslash G/K$ by the map $(i,j) \to KD(i,j)K$.
- $\ell : G \to \mathbb{R}_+$ the length function defined by $\ell(kD(i,j)k') = i + j$ for $(i,j) \in \Lambda$, $k, k' \in K$.

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- *(i) for any representation $(E, \pi) \in \mathcal{E}_{G,C+\beta\ell}$, we have $\pi(\mathrm{p})E = E^G$, and moreover,*
- *(ii)*

$$\|\mathrm{p} - \mathrm{p}_n\|_{\mathcal{C}^{\mathcal{E}}_{C+\beta\ell}(G)} \leq C' e^{2C-tn},$$

with $\int_G |\mathrm{p}_n(g)| dg \leq 1$, and $Supp(\mathrm{p}_n) \subset \{g \in G, \ell(g) \leq n\}$.

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The use of type condition

Lemma

For any class of Banach spaces $\mathcal{E}$ of type > 1, there exists $h \in \mathbb{N}^*, \alpha > 0$ such that the following holds. For any $n \in \mathbb{N}^*, k \in \{0, \ldots, \lfloor n/2 \rfloor\}$, any $E \in \mathcal{E}$, and any family of vectors $(\xi_{x,y})_{x \in T^k\mathcal{O}/T^n\mathcal{O}, y \in T^{2k}\mathcal{O}/T^n\mathcal{O}}$ in E, we have

$$\mathop{\mathbb{E}}_{a \in T^k\mathcal{O}/T^n\mathcal{O}, b \in T^{2k}\mathcal{O}/T^n\mathcal{O}} \left\| \mathop{\mathbb{E}}_{x \in T^k\mathcal{O}/T^n\mathcal{O}} \left(\xi_{x, ax+b+T^{n-1}} - \xi_{x, ax+b} \right) \right\|^2$$

$$\leq 2^{2h-2} e^{-2(\frac{n-2k}{h}-1)\alpha} \mathop{\mathbb{E}}_{x \in T^k\mathcal{O}/T^n\mathcal{O}, y \in T^{2k}\mathcal{O}/T^n\mathcal{O}} \|\xi_{x,y}\|^2.$$

- This is the only place where the condition of type is used.
- The proof follows directly from a consequence of a proposition (V. Lafforgue 2009) on a variant of fast Fourier transform.

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Proof of the theorem by two propositions on matrix coefficients

If (E, π) is a representation of G, (V, τ) is a finite dimensional representation of K, $\xi \in E^K$ and $\eta \in (V \otimes E^*)^K$ (i.e. K-invariant vectors), we set

$$c(g) = \langle \eta, \pi(g)\xi \rangle \in V,$$

and

$$c(i, j) = \langle \eta, \pi(D(i, j))\xi \rangle$$

$(\Lambda \simeq K \backslash G / K)$.

Proof of the theorem by two propositions on matrix coefficients

Proposition

(Spherical) Let $\beta \in [0, \frac{\alpha}{4h})$. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, $(E, \pi) \in \mathcal{E}_{G, C+\beta\ell}$, and $\xi \in E^K$, $\eta \in (E^*)^K$ with norm 1. There exists $c_0, c_1 \in \mathbb{C}$ such that

$$|c(i,j) - c_l| \leq C' e^{2C - (\frac{\alpha}{2h} - 2\beta)i},$$

for any $i \geq j \geq 0$ satisfying $i + j \in 2\mathbb{N} + l, l = 0, 1$.

Proof of the theorem by two propositions on matrix coefficients

Proposition

(Non spherical) Let $\beta \in [0, \frac{\alpha}{4h})$, and (V, τ) a non trivial irreducible unitary representation of K. There exists $C' > 0$, such that the following holds. Let $C \in \mathbb{R}_+^*$, $(E, \pi) \in \mathcal{E}_{G, C+\beta\ell}$, and $\xi \in E^K$, $\eta \in (V \otimes E^*)^K$ (i.e. K-invariant vectors) with norm 1. Then we have

$$\|c(i,j)\|_V \leq C' e^{2C - (\frac{\alpha}{2h} - 2\beta)i}.$$

Proof of the theorem:

- Let $\mathrm{p}_g = e_K e_g \chi_K \in C_c(G)$, where e_g is the left translation by g and $e_K = \int_K e_k dk$ (note that $K \leq G$ is a compact open subgroup).
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- We set $\mathrm{p} = \frac{1}{2}(T_0 + T_1)$, therefore

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Proof of the propositions by two estimates

Let V be an irreducible representation of K (trivial or non trivial).

Lemma

$$\|c(i,j)-c(i,j+2)\|_V \leq C'e^{2C-(\frac{\alpha}{h}-2\beta)i+\frac{\alpha}{h}j},$$

for any $i \geq j$.

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Proof of the spherical proposition (V= trivial rep. of K):

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Proof of the first estimate using the lemma of FFT:

First step: construct maps $\alpha, \beta : (\mathcal{O}/T^{n_1}\mathcal{O})^2 \to G$, and $k_1 : (\mathcal{O}/T^{n_1}\mathcal{O})^3 \times \mathbb{F}_2 \to K$, where $n_1 = \lfloor \frac{i+j}{2} \rfloor - j - 1$, such that

- when $a, b, x, y \in \mathcal{O}/T^{n_1}\mathcal{O}$ satisfying $y = ax + b + T^{n_1 - 1}\varepsilon$, where $\varepsilon \in \mathbb{F}_2$, we have
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$$\begin{aligned} &\|c(i,j) - c(i,j+2)\|_V \\ = &\Big\| \mathop{\mathbb{E}}_{a,x \in T^k\mathcal{O}/T^{n_1}\mathcal{O}, b \in T^{2k}\mathcal{O}/T^{n_1}\mathcal{O}} \langle \Big(1_V \otimes \pi^*\big(\beta(a,b)\big)\Big)\eta, \\ &\pi\big(\alpha(x, ax+b)\big)\xi - \pi\big(\alpha(x, ax+b+T^{n_1-1})\big)\xi\rangle \Big\|_V . \end{aligned}$$

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$\square$

Thank you!