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Entire solutions of explicit and implicit l.d.e. in Banach space.

E Banach space
A closed linear operator on E with $\mathcal{D}(A)$ as domain.
A should have a bounded inverse.

example: $E = \mathcal{C}[0,1]$, $A = \frac{d^2}{dx^2}$, $\mathcal{D}(A) = \{u \in C^2[0,1] : u(0)=u(1)=0\}$

We will study (1) $w' = Aw + f(z)$, where $f(z) : \mathbb{C} \to E$
with $w(z) \in \mathcal{D}(A)$, holomorphic in the neighbourhood of 0.
$w(0) = w_0$.

Ex
Let E be a complex vector space, f be a formal power series $f(z) = \sum_{n=0}^{\infty} c_n z^n$
$f(z) \in E[[z]]$

$f = f_0 + f_1 z + f_2 z^2$
$w = w_0 + w_1 z + w_2 z^2 + \dots$

: the method of unknown coefficients gives

$$w = \sum_{n=0}^{\infty} \frac{1}{n!}(A^n w_0 + A^{n-1} f_0 + \dots + A f_{n-2} + f_{n-1}) z^n$$

"formal power series").

E Banach space
A bounded operator
$f(z)$ entire function

} The formula $w(z) = e^{zA} w_0 + \int_0^z e^{-(z-\xi)A} f(\xi) d\xi$
$\longrightarrow$ semi-group approach: Krein, Hille, Yoshida, ...

Here we follow another approach.

$\begin{cases} u' = Au + f(t) \\ u(0) = u_0 \end{cases}$, with $E = X$ reflexive space, f Lipschitz : Then $u(t) = T(t)u_0 + \int_0^t T(t-s) f(s) ds$ defines the unique solution of

Da Prato: Def: A is a Hille Yosida if $\exists w \in \mathbb{R}, M \geq 0$. $(w,\infty) \cap \sigma(A^{-1}) = \emptyset$
and $\| (\lambda - w)^n R(\lambda, A)^n \| \leq M$.

Under this condition + natural restrictions of f there is a ! solution.

Consider $Tw' + g(z) = w$ with $T = A^{-1}$ and $g(z) = -A^{-1} f(z)$
This is an equivalent form of the problem

Write it $\left(I - T\frac{d}{dz}\right)w = g$: $w = \left(1 - T\frac{d}{dz}\right)^{-1} g = \sum T^n g^{(n)}(z)$ (*)

Now consider E a Banach space and Q a bounded linear operator.

If $Qu + b = u$ and $\rho(Q) < 1$ (spectral radius), then there is a unique solution $u = \sum_{n=0}^{\infty} Q^n f$

(3) will be an analogue of this in terms of T and g: (T, g) should be small.
exponential type of g.

Proposal: $\rho(T)\,\sigma(g) < 1$.

ρ: T nilpotent and no restriction on g holomorphic.

σ: g is a polynomial and T is any bounded operator.

Otherwise, in general, (*) diverges.

Ex: If $T = 1$ and $g(z) = e^z$

Motivation: Krein-Beletskii: Stability of l.d.e. in B-spaces.
$\to f$ 0-exponential type

Let E be a Banach space and $\sigma > 0$. E_σ = space of entire f^ns for which $\underbrace{\sup_{z\in\mathbb{C}} \|f(z)\| e^{-\sigma|z|}}_{\|f\|_\sigma} < \infty$

Let $\tilde{E}_\sigma = \bigcup_{\tau<\sigma} E_\tau$ be the space of f^ns with exp type $< \sigma$.

$\tilde{E}_\infty$ be the space of all entire f^ns

Th 1: (implicit o.d.e. $Tw' + g(z) = w(z)$). If $T: E \to E$ and $\sigma_0 = \frac{1}{\rho(T)}$, $\sigma < \sigma_0$, $g(z)$ an entire function of exp. type σ. then there is a unique $w = \sum_{n=0}^{\infty} T^n g^{(n)}(z)$ which converges in $\tilde{E}_{\sigma_0}$.

Sketch: $\triangleright$, $\sigma > 0$, $\sigma < \sigma_1 < \sigma_0$, $g \in E_{\sigma_1}$.
Recall the Cauchy formula for the n^{th} derivative: you get $\|g^{(n)}(z)\|_{\sigma_1} \le n!\, e^n n^{-n} \sigma_1^n \|g(z)\|$,

$\to$ If $D = \frac{d}{dz}$, $\|D^n\| \le n!\, e^n n^{-n} \sigma_1^n$
$Q = \tilde{T}D$: $(\tilde{T}f)(z) = T(f(z))$. Then consider $Qw + g = w$: Q satisfies $\rho(Q) < 1$,
so that $w = \sum Q^n g = \sum_{n=0}^{\infty} T^n g^{(n)}$ is a formula for the solution

The homogeneous problem only has the 0 solution.

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Extreme cases:

Th2: Let G be an entire f^{n} of zero exponential type $[\forall \varepsilon > 0\ \exists C_\varepsilon\ \forall z \in \mathbb{C}\ \|f(z)\| \le C_\varepsilon e^{\varepsilon|z|}]$,
T a bounded operator: then there is a unique $w = \sum_{n=0}^{\infty} T^n g^{(n)}(t)$
$\to \tilde{E}_0$ space.

Th3: Let T be quasinilpotent: $\rho(T)=0$ and g be of exponential type, also!

Th4: If T is quasinilpotent and $(1-\lambda T)^{-1}$ is of exponential type and g is any entire function.

Let us come back to explicit linear eq^{n} $w' = Aw + f(z)$

th 5: Let A be a closed linear operator with an inverse, f of exp type $\sigma < \sigma_0$, $\sigma_0 = \frac{1}{\rho(A^{-1})}$. Then the unique solution is $w = -\sum_{n=0}^{\infty} A^{-(n+1)} f^{(n)}(z)$

The Cauchy problem $\begin{cases} w' = Aw + f(t) \\ w(0) = w_0 \end{cases}$ has solution iff $w_0 + \sum n!\, A^{-(n+1)} c_n = 0$ (c_n = coeffs of f.)

Example (1) $E = \mathbb{C}$, $T = 1$: $w' = w + f(z)$, f of exponential type $\sigma < 1$.
Then there's a solution $w = -\sum_{n=0}^{\infty} f^{(n)}(z)$ $\to$ consider $\tilde{E}_1$

another example (2) $\ddot{x} + \omega x = f(t)$ with $\omega > 0$, f of exp type ... σ ...
If $\sigma < \omega$, $x(t) = \sum \frac{(-1)^{k+1}}{\omega^{2k+2}} f^{(2k)}(t)$

another example (3) $E = C[0,1]$, $A = \frac{d^2}{dx^2}$ with $\mathcal{D}(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}$
Then $\rho(A^{-1}) = \frac{1}{\pi^2}$
We can write our equation as a boundary problem: $\begin{cases} w' = Aw + f(t) \\ \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + f(t,x) & t>0,\ x \in [0,1] \\ w(t,0) = w(t,1) = 0 \end{cases}$
the solution is $w(t,x) = -\sum_{n=0}^{\infty} \int_0^1 \underbrace{G_{n+1}(x,y)}_{\text{Green } f^{n} \text{ associated to } A!} \frac{\partial^n f}{\partial t^n}(t,y)\,dy$

Example (4): $E = C[0,1]$, $A = \frac{d}{dx}$, $\mathcal{D}(A) = \{u \in C^1[0,1] : u(0) = 0\}$
$\rho(A^{-1}) = 0$ corresponds to the boundary problem $\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} + f(t,x) & t \in \mathbb{R},\ x \in [0,1] \\ w(t,0) = 0 \end{cases}$
Solution $w(t,x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^x (x-y)^n \frac{\partial^n f}{\partial t^n}(t,y)\,dy$.

Ex : with a $f^{=}$ not of exponential type.

Hilbert space

weighted shift operator $T e_{n+1} = \frac{e_n}{\sqrt{n+1}}$, $f(z) = e^{z^2} e_0$

Solution : $\begin{cases} e^{t^2} = w_0(t) \\ \frac{1}{\sqrt{n+1}} w_n'(t) = w_{n+1}(t). \end{cases}$

We find the sequence $w_n(t) = \frac{1}{\sqrt{n!}} \left(e^{t^2}\right)^{n}$; $w_{2n}(0) = \frac{\sqrt{(2n)!}}{n!}$

then $\sum |w_n(0)|^2 = \infty$ so that w cannot be an entire solution.

Q : existence without uniqueness ? No!