De Commer
2/4/2013

Reciprocal random walks

1) Statement of the result

<u>Def.</u> Let G be a compact group and A a unital C^*-algebra. Then a continuous action α of G on A is ergodic if: if $a \in A$ and $\alpha_g(a) = a$ for all $g \in G$, then $a \in \mathbb{C}1_A$

Quantisation of homogeneous space:

<u>Ex</u>: ① $H \leq G$ a compact subgroup: $A = C(G/H)$.

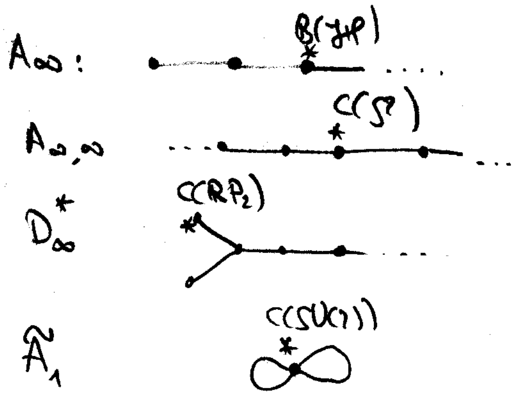
② If π is an irred. projective representation of G on $\mathcal{H}$, then G acts on $B(\mathcal{H})$ by $\alpha_g(x) = \pi(g) x \pi(g)^*$.

③ Combination of ① and ② by "induction"

Theorem (A. Wassermann '88) For $G = SU(2)$, <u>all</u> ergodic actions are induced from an irreducible representation of a subgroup

By Mackey correspondence, there is a 1-1 correspondence between ergodic actions of $SU(2)$ and extended A-D-E graphs with a distinguished vertex (up to isomorphism)

Examples: vertices = rep^s of the subgroup, edges = ...



$\dfrac{H}{SU(2)}$ subgroup. $H \subseteq SU(2)$, $\pi_{1/2}$ rep. decompose ⋮

S^1

D_∞^*

$\{e\}$.

<u>Theorem</u> (DC-Yamashita '12). Fix $q \in \mathbb{R}$, $0 < |q| \leq 1$. There is a 1-1 correspondence between ergodic actions of $SU_q(2)$ (on unital C^*-algebras) and $(q+q^{-1})$-reciprocal random walks on connected graphs with a distinguished vertex (up to iso.)

De Commer
2/4/2013
(2)

Inspired and motivated by: Ocneanu '98, Hayashi 1999, Ostrik '03, Etingof-Ostrik 04, Bichon-De Rijdt-Vaes '06, Pinzari-Roberts '08, Tomatsu '08

II Compact q groups and actions

Def: A cqg (C,Δ) with C a unital C^*algebra, $\Delta: C \to C \underset{\min}{\otimes} C$ a unital $*$-homomorphism s.t. ① $(\Delta\otimes id)\circ\Delta = (id\otimes\Delta)\circ\Delta$ as maps $C \to C\otimes C\otimes C$

② $[(C\otimes 1)\Delta(C)] = C\otimes C = [(1\otimes C)\Delta(C)]$

Examples: ① If G is a compact group, then $C(G)$ is a cqg : $\Delta(f)(g,h)=f(gh)$.

② for Γ a discrete group, then $C^*\Gamma$ is a CQG with $\Delta(\lambda_g)=\lambda_g\otimes\lambda_g$

"$C^*(\Gamma)=C(\hat{\Gamma})$"

③ For $q\in\mathbb{R}$ and $0<|q|\leq 1$, "$C(SU_q(2))$" is a C^*-algebra generated by elements a, b, a^*, b^* with the defining condition that $U=\begin{pmatrix} a & -qb^* \\ b & a^* \end{pmatrix}$ is unitary and $\Delta(a) = a\otimes a - qb^*\otimes b$

$\Delta(b) = a^*\otimes b + b\otimes a$

In general, I will write a CQG (C,Δ) as $C=$ "$C(G)$"

Def: If $\mathbb{G}$ is a CQG and A a unital C^*-algebra, an ergodic action of $\mathbb{G}$ on A consists of an injective $*$-homomorphism $A \longrightarrow A\otimes C(\mathbb{G})$ s.t.

1) $(\alpha\otimes id)\circ\alpha = (id\otimes\Delta)\circ\alpha$ " $\alpha_{gh}(a)=\alpha_g\circ\alpha_h(a)$

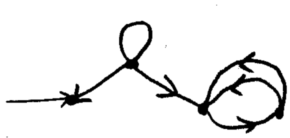
2) $[\alpha(A)(1\otimes C(\mathbb{G}))] = A\otimes C(\mathbb{G})$

3) If $a\in A$ and $\alpha(a)=a\otimes 1$, then $a\in\mathbb{C}1_A$ (ergodicity)

III T-reciprocal random walks.

Def: A graph Γ is a vertex set V, an edge set E, two maps $E \underset{t}{\overset{s}{\rightrightarrows}} V$

ex:



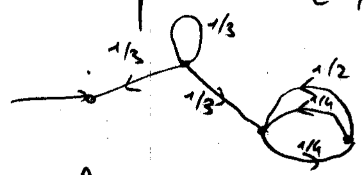
The adjacency matrix is $\mathcal{M}(\Gamma)_{v,w} = \#\left(s^{-1}(v)\cap t^{-1}(w)\right)$

Γ is symmetric if $\mathcal{M}(\Gamma)$ is.

$\|\Gamma\| = \|\mathcal{M}(\Gamma)\|$, $\mathcal{M}(\Gamma)$ viewed as unbounded operator on $\ell^2\Gamma$

N.B.: If Γ is symmetric, then $\|\Gamma\|<\infty \Leftrightarrow \underset{\sup_v s^{-1}(v)}{\overset{\parallel}{\deg\Gamma}}<\infty$

Def: A random walk on Γ is an assignment $p: E\to[0,1]$ such that $\sum'_{s(e)=v} p(e) = 1$ for all $v\in V$



Def: Let $T\in\mathbb{R}_0$. A random walk on Γ is called T-reciprocal if E can be equipped with an involution $e\mapsto\bar{e}$ s.t.

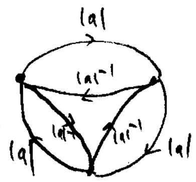
1) $s(\bar{e})=t(e)$, $t(\bar{e})=s(e)$
2) $p(e)p(\bar{e})=T^{-2}$
3) If $T>0$, the number of loops at each vertex should be even.

Trivial remarks: • If Γ is T-reciprocal, it is also $(-|T|)$-reciprocal.

• If Γ admits a T-reciprocal random walk, then Γ is symmetric.

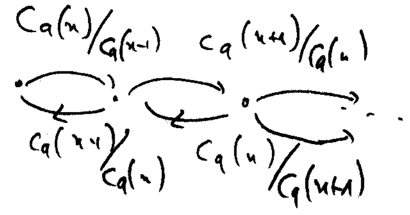
Examples: let: $w(e)=|T|\,p(e)$. Then $w(e)\,w(\bar{e})=1$ and $\sum_{s(e)=v} w(e) = |T|$ for all $v\in V$.

1) $|q|>0$



$C(SU_q(2)/\mathbb{Z}_3)$. Here $T=q+q^{-1}$.

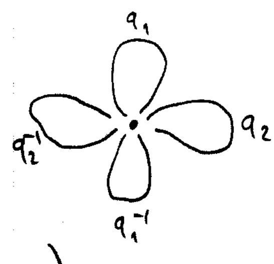
2) $|q|>0$, $n\in\mathbb{R}$. $\quad |q|>0$, $n\in\mathbb{R}$.



$T=q+q^{-1}$, $\quad C_q(n)=|q|^n+|q|^{-n}$

→ Podleś sphere

3) $q, q_1, q_2>0$, $q+q^{-1}=q_1+q_1^{-1}+q_2+q_2^{-1}$



$q+q^{-1}$ − reciprocal random walk.

(examples by Bichon − De Rijdt − Vaes)

 with $T=-2$. ($T=2$ does not work because odd # of loops.)

<u>Proposition</u>: (1) Any symmetric graph Γ with degree $\deg\Gamma<\infty$ admits a T-reciprocal random walk with $T=-\|\Gamma\|$

(2) If Γ admits a T-reciprocal random walk, then $\|\Gamma\|\leq|T|$.

[follows from Perron - Frobenius theory]

- This yields an easy classification of 2-reciprocal random walks and hence we recuperate Wassermann's result $(q=1)$.
- But for $q\to 0$ an abundance of ergodic actions appear.

IV The proof: (principal ingredients)

Idea: Develop a Tannaka-Krein theory for ergodic actions of compact qg.

$\boxed{\text{T-K for CQG}}$ [Woronowicz]: associated to $\mathbb{G}$, we have a "tensor C^*-category with duals". Its <u>objects</u> are finite dimensional unitary rep^s of $\mathbb{G}$ $(\mathcal{H},\pi)$. Its <u>morphisms</u> are intertwiners $T\in \mathrm{Mor}((\mathcal{H}_1,\pi_1),(\mathcal{H}_2,\pi_2))$ $\subset B(\mathcal{H}_1,\mathcal{H}_2)$ s.t. $\pi_2(g)T=T\pi_1(g)$ "for all $g\in\mathbb{G}$".

→ we can take tensor products, duals = contragredients

→ and an appropriate abstract axiom.

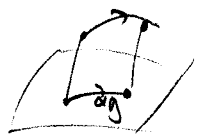
Moreover we get $\mathcal{F}: \mathrm{Rep}(\mathbb{G}) \hookrightarrow \mathrm{Hilb}$, $(\mathcal{H},\pi)\mapsto\mathcal{H}$

<u>Th</u>: $\mathbb{G}$ can be reconstructed from $(\mathrm{Rep}\,\mathbb{G},\mathcal{F})$ and all couples $(\mathcal{C},\mathcal{F})$ with $\mathcal{C}$ an abstract C^*-tensor category with duals are of this form.

$\boxed{\text{T-K for ergodic actions}}$ Let $\mathbb{G}\curvearrowright A$ an ergodic action. We have a C^*-category $\mathcal{D}(A)$ with 1) objects: $\mathbb{G}$-equivariant fin. gen^d projective right Hilbert A-modules $\mathcal{E}$ and $\langle\cdot,\cdot\rangle_A: \mathcal{E}\times\mathcal{E}\to A$, $\alpha: \mathbb{G}\times\mathcal{E}\to\mathcal{E}$ s.t. $\langle\alpha_g(\xi),\alpha_g(\eta)\rangle_A=\alpha_g(\langle\xi,\eta\rangle_A)$

if "A

(If "$A = \mathcal{C}(X)$", then these ξ are equivariant bundles of Hilbert spaces over X)



Morphisms are intertwiners $\xi \to \eta$. Moreover we have

3 $\otimes$-product $\quad \operatorname{Rep} G \times \mathcal{D}_A \longrightarrow \mathcal{D}_A$

$$(\mathcal{H}, \pi) \times (\xi, \alpha) \longrightarrow (\mathcal{H}_\pi \otimes \xi) \text{ with } \alpha_g(\xi \otimes \eta) = \alpha_g(\xi) \otimes \pi(g)\eta$$

and an appropriate axiom system: an "indecomposable module C^*-category for $\operatorname{Rep}(G)$"

Th (DC-Yamashita '12): $G \curvearrowright A$ can be reconstructed from $\mathcal{D}_A$ with its module structure and all indecomposable module C^*-categories arise in this way.

Q: If $q \to 0$: more and more actions.
If q close to 1, only deformations.

Also description for $A_o(F)$
Or as soon as the cqg comes from a universal category.

Adjoint: $\quad \rho\,\rho^* = (-\operatorname{sgn} q)$