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The AP of Haagerup and Kraus in semisimple Lie groups.

w. Uffe Haagerup.

All groups are second countable. They are Lie groups. We can use harmonic analysis.

<u>Consider</u> property P for locally compact groups with C^* and W^*-analogues (C^*P, W^*P) such that (A) if G is a l.c. group and $\Gamma < G$ a lattice, then G has P <u>iff</u> Γ has P; (B) If Γ is a discrete group, Γ has P <u>iff</u> $C^*_\lambda\Gamma$ has C^*P <u>iff</u> $L\Gamma$ has W^*P.

<u>Recall</u>: Let G be a locally compact group. $A(G) = \{\langle\lambda(\cdot)\xi,\eta\rangle : \xi,\eta\in L^2(G)\}$ is the Fourier algebra of G. $A(G) = C(G)$ and $A(G)^* = L(G)$.

Let $\varphi : G \to \mathbb{C}$. φ is a multiplier ($\varphi \in MA(G)$) if $\varphi\psi \in A(G)$ for all $\psi \in A(G)$. φ defines $m_\varphi : A(G) \to A(G)$; we denote $\|\varphi\|_{MA(G)} = \|m_\varphi\|$.

Then m_φ induces $M_\varphi : L(G) \to L(G)$. If M_φ is cb, then φ is a c.b. multiplier.
We write $\varphi \in M_0A(G)$ [Herz-Schur multiplier].

<u>Fact</u>: $A(G) \subset M_0A(G) \subset MA(G)$. We won't be interested in $MA(G)$.

Def: Let G be a l.c. group. Then G is <u>amenable</u> if $A(G)$ has a bounded approximate identity: there is a net (φ_α) in the unit ball of $A(G)$ s.t. $\|\varphi_\alpha\psi - \psi\| \to 0$ for every $\psi \in A(G)$.

Def: Let G be a l.c. group. Then G is <u>weakly amenable</u> (w.a.) if there is a net (φ_α) such that $\sup_\alpha \|\varphi_\alpha\|_{M_0A(G)} \leq C$ and $\varphi_\alpha \to 1$ uniformly on compact sets.
This def is very natural for Lie groups! Then $\inf(C)$ is the Haagerup Cowling constant $\Lambda(G)$.

<u>We know</u>: for Γ discrete,

$$
\begin{array}{ccccccc}
\Gamma \text{ amenable} & \Rightarrow & \text{w.a.} & \Rightarrow & \text{AP} & \Rightarrow & \text{exact} \\
\Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow \text{ by def.} \\
C^*_\lambda\Gamma \text{ nuclear} & \Rightarrow & \text{CBAP for } C^*_\lambda\Gamma & \Rightarrow & \text{OAP for } C^*_\lambda(G) & \Rightarrow & \text{exact for } C^*_\lambda(G)
\end{array}
$$

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Let G be a connected Lie group. We have $G = KAK$ (A abelian) 2/4/2013

Let $\mathfrak{a}$ be the Lie algebra of A. Denote by $\mathrm{rank}_{\mathbb{R}}(G) = \dim \mathfrak{a}$ ②

Then if $\mathrm{rank}_{\mathbb{R}}(G) = 0$, G is amenable [folklore].

If $\mathrm{rank}_{\mathbb{R}}(G) = 1$, then G is locally isomorphic to
- $SO_0(n,1)$ and $\Lambda(G) = 1$
- $SU(n,1)$ and $\Lambda(G) = 1$
- $Sp(n,1)$ and $\Lambda(G) = 2n-1$
- $F_{4(-20)}$ and $\Lambda(G) = 21$

[Cowling Haagerup: finite center case]

[Hansen: infinite center case: covering gp of $SU(n,1)$]

If $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, then G is not weakly amenable. [Haagerup, Dorofaeff]

This finishes the question for connected simple Lie groups.

Proof w.a.: It does not go to extensions.

<u>Def</u>: G has <u>AP</u> (of Haagerup and Kraus) if there is a net $(\varphi_\alpha) \subset A(G)$ s.t. $\varphi_\alpha \to 1$ in $\sigma(M_0A(G), M_0A(G)_*)$-topology. [on bdd sets, this is just the w^*-topology on $L^\infty(G)$]

Ex: . $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ has the AP but is not w.a.

. If N is a closed normal subgroup of G s.t. N and G/N have AP, then G has AP.

<u>Conjecture</u> [H-K]: $SL(3,\mathbb{Z})$ does not have the AP.

<u>Theorem</u>: (de la Salle - Lafforgue) $SL(n,\mathbb{R})$ has not AP if $n \geq 3$.

Q: Can we do more? a classification?

<u>Fact</u>: Let G be a connected simple Lie group with $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$. Then G contains a closed subgroup H s.t. $H \overset{\text{locally}}{\approx} SL(3,\mathbb{R})$ or $Sp(2,\mathbb{R})$.

[cf Borel, root systems]

$\rightarrow$ take into account covering aspects.

three cases:
- $SL(3,\mathbb{R})$ (the covering group is finite)
- $Sp(2,\mathbb{R})$
- $\widetilde{Sp}(2,\mathbb{R})$ manifold that is a covering. Here it is infinite!

③

N.B.: $Sp(2,\mathbb{R}) = \{g \in GL(4,\mathbb{R}) : g\,{}^tJ g = J\}$: $J = \begin{pmatrix} 0 & +I_2 \\ -I_2 & 0 \end{pmatrix}$ de Laat.

Th: (Haagerup – dL) $Sp(2,\mathbb{R})$ does not have AP (2011)
$\widetilde{Sp}(2,\mathbb{R})$ ———————— (2012)

Let G be a connected simple Lie group. Then G has AP iff $\mathrm{rank}_{\mathbb{R}} \in \{0,1\}$.

Strategy of the proof: Let G be a connected simple Lie group. If G has the AP, then there is a net in $A(G)$, and we can in fact ① restrict the approximating class to other functions: $Sp(2,\mathbb{R}) = K A K$. Consider $A(K\backslash G/K)$: bi-invariant functions.

$K \cong U(2)$, A diagonal matrices

② We can prove that every $f \in A(K\backslash G/K)$ satisfies a certain behaviour

- Let $\varphi : G \xrightarrow{\text{continuous}} \mathbb{C}$, $[G = Sp(2,\mathbb{R}),\ K = U(2)]$. Then φ is K bi-invariant if $\varphi(k_1 g k_2) = \varphi(g)$ for $g \in G$, $k_1 \in K$, $k_2 \in K$.
- Recall $A(K\backslash G/K) \subset M_0A(G) \cap C_0(K\backslash G/K)$
- Lemma: $M_0A(G) \cap C_0(K\backslash G/K)$ is closed in $\sigma(M_0A(G), M_0A(G)_*)$

Then we cannot approximate 1!! as it is not in $C_0(K\backslash G/K)$.

- Proposition. There are C_1, C_2 s.t. for all $\varphi \in M_0A(G) \cap C_0(K\backslash G/K)$, $\beta, \gamma \geq 0$

$$|\varphi(D(\beta,\gamma))| \leq C_1 e^{-C_2\sqrt{\beta^2+\gamma^2}} \|\varphi\|_{M_0A(G)}$$ [Here: $G = KAK$, $A = \left\{ D(\beta,\gamma) = \begin{pmatrix} e^{\beta} & & & \\ & e^{\gamma} & & \\ & & e^{-\beta} & \\ & & & e^{-\gamma} \end{pmatrix} \right\}$]

Applications to nc L^p spaces.

N.B.: The proof of dS-Lafforgue also gives results here.
Our approach is more direct, but you can do it in the same setting:

th [Lafforgue, de la Salle]: for $p \in [1, \tfrac{4}{3}[\cup]4, \infty]$, the nc L^p-space $L^p(L(\Gamma))$, $\Gamma = SL(3,\mathbb{R})$, does not have the OAP.

Theorem: For $p \in [1, \tfrac{12}{11}[\cup]12, \infty]$ and Γ lattice in any connected simple Lie group with real rank ≥ 2, $L^p(L(\Gamma))$ does not have OAP.

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Something about the techniques:

- Harmonic analysis on Gelfand pairs. Recall: let G be a l.c. group and $K<G$ compact. Then (G,K) is a Gelfand pair if $C_c(K\backslash G/K)$ is commutative. (or equivalently, $L^1(K\backslash G/K)$ is : then use Plancherel th, Banach algebra techniques).
- Assume G is compact and let $\varphi \in M_0A(G) \cap C_0(K\backslash G/K)$: then $\varphi = \sum_{\pi \in \hat{G}_K} c_\pi \dim \mathcal{H}_\pi h_\pi$

... Hölder continuity condition: $|h^0_{p,q}(\frac{e^{i\theta_1}}{\sqrt{2}}) - h^0_{p,q}(\frac{e^{i\theta_2}}{\sqrt{2}})| \leq C|\theta_1-\theta_2|^{\frac{1}{4}}$

$Sp(2,\mathbb{R})$: $(\mathcal{U}(2),\mathcal{U}(1))$ is a Gelfand pair, and $(SU(2),SO(2))$ also

$\mathcal{U}(2) \supset \mathcal{U}(1)$

Q: Is there a group for which the L^p space have OAP but itself does not?
Is this really easier?

Q: Do they have the Banach space AP?!